E-Cordial Families Related To Cycle and Path

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Abstract: The two copies of graph G(p,q) are joined by t paths on n-points each. We represent the family by G(tPn). The paths are attached at the same fixed point on G. We discuss E-cordiality of C4(Pn), C5(Pn), w4(Pn), C3P5W4, C4P5W4. We show that under certain conditions these graphs are E-cordial.

Keywords: graph, E-cordial, shel graph, S5, C4.

Introduction:

In 1997 Yilmaz and Cahit [4] introduced weaker version of edge graceful labeling E-cordial labeling. Let G be a (p,q) graph. f: E → {0,1} Define f on V by f(v) = ∑(f(uv))(mod 2). The function f is called as E-cordial labeling if |v(f(0)) - v(f(1))| ≤ 1 and |e(c(0)) - e(c(1))| ≤ 1 where v(f(i)) is the number of vertices labeled with i = 0,1. And e(c(i)) is the number of edges labeled with i = 0,1. We follow the convention that v(0) = a and v(1) = b further e(0) = x and e(1) = y. A graph that admits E-cordial labeling is called as E-cordial graph. Yilmaz and Cahit prove that trees Tn are E-cordial iff for n not congruent to 2 (mod 4), Kn are E-cordial if n not congruent to 2 (mod 4), Fans Fn are E-cordial iff for n not congruent to 1 (mod 4). Yilmaz and Cahit observe that A graph on n vertices cannot be E-cordial if n is congruent to 2 (mod 4). One should refer Dynamic survey of graph labeling by Joe Gallian [2] for more results on E-cordial graphs.

The graphs we consider are finite, undirected, simple and connected. For terminology and definitions we refer Harary [3] and Dynamic survey of graph labeling by Joe Gallian [2]. The families we discuss are obtained by taking two copies of graph G and join them by t paths of equal length. The paths are attached at the same fixed point on G. We represent these families by G(tPn). We take t = 1 and choose G from C3, C4, C5 and W4.

3. Preliminaries:

3.1 G1(Pn)G2 is graph obtained by joining a vertex of G1 with vertex of G2. It has p1+p2+n-2 vertices and q1+q2+n-1 edges where G1 is (p1,q1) and G2 is (p2,q2) graph. When there are t paths from G1 to G2 starting at one vertex and ending at one fixed vertex we denote this family of graphs as G(t(Pn)G2).

4. Main Results proved:

Theorem 4.1: G = C4(P4) is E-cordial for n is not congruent to 0 (mod 4)

Proof: We define G as V1 = {v1, v2, v3, v4}. These are vertices on path Pn and end points are on respective cycle. V2 = {u1, u2, u3, u4, u5, u6}, these are vertices on two copies of C4. It does not include the vertex common with path namely v1 and v6. Thus we have V(G) = V1UV2, E(G) = {e = (vi, vj) | i = 1,2,..,n-1} U {c1 = (u1,u2), c2 = (u2,u3), c3 = (u3,u4), c4 = (u4,u5), c5 = (u5,u6)}

Note that |E(G)| = q = n+7, |V(G)| = p = n+6

Define f: E(G) → {0,1} as follows:

f(v1u1) = f(v1u2) = 1;

f(v1u6) = 1;

f(u1u3) = 1;

f(u2u4) = 0,

f(u2u5) = 0,

f(u3u5) = 0,

f(u4u6) = 0.

f(ei) = 0 for i is odd number and i < 2k where k = [n/4]
\[ f(e_i) = 1 \text{ for } i \text{ is even and } i \leq 2k \]

\[ f(e_{2kp+j}) = 1 \text{ for } j = 1, \ldots, p-k \text{ where } p = \left\lfloor \frac{n}{2} \right\rfloor \]

\[ f(e_i) = 0 \text{ for } i = k+p+1, \ldots, n-1. \]

The label number distribution is

\[ v_f(0,1) = \left( \frac{p+1}{2}, \frac{p-1}{2} \right) \]

\[ e_f(0,1) = \left( \frac{q}{2}, \frac{q}{2} \right) \text{ for } n \equiv 3 \pmod{4} \]

\[ v_f(0,1) = \left( \frac{p-1}{2}, \frac{p+1}{2} \right) \]

\[ e_f(0,1) = \left( \frac{q-1}{2}, \frac{q+1}{2} \right) \text{ for } n \equiv 1 \pmod{4} \]

\[ v_f(0,1) = \left( \frac{q}{2}, \frac{q}{2} \right) \text{ for } n \equiv 2 \pmod{4} \]

\[ e_f(0,1) = \left( \frac{q-1}{2}, \frac{q+1}{2} \right) \text{ for } n \text{ is divisible by } 4 \text{ the desired labeling does not exists.} \]

**Theorem 4.2.** \( G = C_5 \circ \mathcal{P}_n \) is e-cordial for \( n \) is not congruent to 2(mod 4)

Proof: We define \( G \) as \( V(G) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\} \) and \( E(G) = \{e_i=(v_i \ v_{i+1})/i = 1, 2, \ldots, n-1\} \cup \{c_i=(v_i \ u_1), c_2=(u_1 \ u_2), c_3=(u_2 \ u_3), c_4=(u_3 \ u_4), c_5=(u_4 \ v_1), c_6=(v_n \ u_6), c_7=(u_6 \ u_7), c_8=(u_7 \ u_8), c_9=(u_8 \ v_n)\} \)

Define \( f: E(G) \to \{0,1\} \) as follows:

\[ f(c_1) = 0; \]

\[ f(c_2) = 1; \]

\[ f(c_3) = 0; \]

\[ f(c_4) = 1; \]

\[ f(c_5) = 0; \]

\[ f(e_i) = 0 \text{ for } i = 2x-1, x = 1, 2, \ldots, k; \text{ where } k = 1 + \frac{n-3}{2} \text{ if } n-3 \text{ is divisible by } 4 \text{ otherwise } k \text{ is integer part of } \frac{n}{4}; \]

\[ f(e_i) = 1 \text{ for } i = 2x, x = 1, 2, k; \text{ where } k = 1 + \frac{n-3}{2} \text{ if } n-3 \text{ is divisible by } 4 \text{ otherwise } k \text{ is integer part of } \frac{n}{4}; \]

\[ f(e_{2kp+j}) = 1 \text{ for } j = 1, 2, \ldots \ (q_2-2-k), \text{ where } f \text{ is e- cordial labeling we have } e_f(0,1) = \left( q_1, q_2 \right); k \text{ as above.} \]

\[ f(e_j) = 0 \text{ for all others on } C_5. \]

Thus the graph is e-cordial.
**Theorem 4.3.** $G=W_d(P_n)$ is e-cordial for $n$ is not congruent to 2(mod 4).

Proof: We define $G$ as follows: The vertices on two copies of $w_4$ are $V_1 = \{w_1, w_2, u_1, u_2, u_3, u_4, u_5, v_1, v_2\}$. The path vertices are $V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $V_4$ are common vertices respectively with first and second copy of $w_4$. The edge set $E_1 = \{(v_1u_1), (u_1u_2), (u_2u_3), (u_4v_1), (u_6v_2), \}$ are cycle edges on first copy of $w_4$ and pokes on the same copy given by $(w_1u_1)/i=1,2,3,4$ where $u_4 = v_1$, $E_2 = \{(v_1u_2), (u_2u_3), (u_3u_4), (u_4v_1), (u_6v_2)\}$ these as cycle edges on second copy of $w_4$ and pokes given by $(w_2u_1)/i=4, 5, 6, 7$ where $u_7 = v_6$. $E_3$ are edges on path $P_n$ given by $E_3 = \{e_i=(v_{i+1}v_i)/i=1,2,..(n-1)\}$. Thus we have $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$ and $E(G) = E_1 \cup E_2 \cup E_3$.

Define $f: E(G) \rightarrow \{0,1\}$ as follows: $f(w_1u_1) = 1$ for $i = 1,2,3,4$; $f(w_2u_1) = 0$ for $i = 5, 6, 7, 8$; $f(u_au_1) = 0$, $i = 1, 2, 3, 4$ and $i+1$ taken (modulo 4).

$f(e_i) = 0$ for $i = 2x-1$ where $x = 1,2,..t$. $t = integer part of \frac{n}{4} + 1$ for $n-3$ is divisible by 4 and $t = integer part of \frac{n}{4}$ otherwise. $f(e_i) = 1$ for $i = 2x/n=1,2,..2t$.

$f(e_{2i+1}) = 1$ for $i = 1, 2,..p$ where $p = q_2-4-t$ rest of $f(e_i) = 0$ for all $i > p$.

**Theorem 4.4.** $G=C_3(P_n)W_4$ is e-cordial for $n$ is not congruent to 0(mod 4).

Proof: We define $G$ as follows: $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$. Where the vertices of cycle $C_3$ are $V_1 = \{v_1, v_2, v_3\}$; The path vertices are $V_2 = \{u_1, u_2,..u_t\}$ here any $v_i = 1, 2, 3$ will be same as $u_t$ and it depends on which vertex we start path $P_n$. Further $w_4$ vertices are $V_3 = \{v_4, v_5, v_6, v_7\}$ and hub vertex $v_4$. The edge set $E(G)$ is defined as follows: $E_1 = \{e_i=(v_{i+1}v_i)/i=1,2,3,..(n-1)\}$, These are cycle edges on $C_3$, $E_2 = \{e_i=(u_{i+1}u_i)/i=1,2,..t\}$, These edges are on path $P_n$, $E_3 = \{e_i=(v_{i+1}v_i)/i=1,2,3,4\}$. These are pokes of $w_4$, $E_4 = \{e_i=(v_{i+1}v_i)/i=1,2,..(n-1)\}$. Thus we have $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$. We get $E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$.

Define $f: E(G) \rightarrow \{0,1\}$ as follows:

$f(c_1) = 1$;

$f(c_2) = 1$;

$f(c_3) = 0$;

$f(c_4) = 0$ for $i = 2x-1$, $x = 1, 2,..t$ and $t = k+1$..

$f(e_i) = 1$ for $i = 2x$; $x = 1, 2,..k$. Where $k = (integer part of \frac{n}{4})$.

If $f$ we defined is e-cordial labeling then have got say, $e_i(0,1) = (q_1, q_2)$, i.e. number of edges with label 1 are say, $q_2$.

Let $s = q_2-2-k$. Then $f(e_{2k+1}) = 1$ for $i = 1, 2,..s$. For $i > s, f(e_{2k+1}) = 0$. That completes e-cordial labeling of $C_3(P_n)W_4$. We showcase $f$ in following diagram taking $n = 5$.
Theorem 4.5. \(G = C_3(P_3)C_4\) is e-cordial for \(n\) is not congruent to 1 (mod 4)

Proof: We define \(G\) as follows: \(V_1 = \{v_1, v_2, \ldots, v_n\}\). These are vertices on path \(P_{2n}\) and end points are on respective cycle. \(V_2 = \{u_1, u_2, u_3, u_4, u_5, u_6\}\), these are vertices on two cycles \(C_3\) and \(C_5\). The end point on path namely \(v_1, u_1\) on \(C_3\) and \(v_n, u_6\) on \(C_4\). Thus we have \(V(G) = V_1 \cup V_2\). \(E(G) = \{c_1 = (v_1v_2), c_2 = (u_1u_2), \ldots, c_n = (u_6u_n)\}\) and \(c_6 = (u_5u_6), c_{n+1} = (u_1u_2)\).

Note that \(|E(V)| = q = n+6, |V(G)| = p = n+5\)

Define \(f(E(G)) \rightarrow \{0,1\}\) as follows:

\[f(u_{12}) = f(u_{23}) = 1;\]
\[f(u_{34}) = 0;\]
\[f(u_{45}) = 0;\]
\[f(u_{56}) = 0;\]
\[f(u_{61}) = 0;\]
\[f(c_{12}) = 0;\]
\[f(c_2) = 1;\]
\[f(c_{1}) = 1;\]
\[f(c_{i}) = 0;\]
\[f(c_{i+1}) = 0;\]

\[f(e_i) = 0\] for \(i = 2x+1, x = 0, 1, 2, \ldots, k\).

\[f(e_i) = 1\] for \(i = 2x, x = 1, 2, \ldots, k.

\[f(e_{2k+1}) = 1\] for \(i = 1, 2, t.

\[f(e_i) = 0\] for all \(i > 2k+1+t\).

The observed label numbers are \(e_i(0,1) = (x,y)\); \(v_i(0,1) = (x-1,y)\) for \(n\) is even number and \(x = \frac{n+6}{2}\).

If \(n\) is odd number \(x' = \frac{n+5}{2}\) we have \(e_i(0,1) = (x'+1,x')\); \(v_i(0,1) = (x',x')\).

For \(n = 1 \pmod{4}\) E-cordial labeling does not exists.

Theorem 4.6. \(G = C_3(P_n)W_4\) is e-cordial for \(n\) is not congruent to 0,2 (mod 4).

Proof: We define \(G\) as follows: The vertices of cycle \(C_3\) are \(V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}\); The path vertices are \(V_2 = \{u_1, u_2, \ldots, u_n\}\) here any \(v_i, i = 1, 2, 3, 4\) will be same as \(u_i\) and it depends on at which vertex of \(C_3\) we start path \(P_n\). Further \(W_4\) vertices are \(V_3 = \{u_3 = v_1', v_2', v_3', v_4'\} \) and hub vertex \(v_1\). We have \(V(G) = V_1 \cup V_2 \cup V_3\). The edge set \(E(G)\) is defined as follows: \(E_1 = \{c_1 = (v_1v_2), c_2 = (v_2v_3), c_3 = (v_3v_4), c_4 = (v_4v_1)\}\), these are cycle edges on \(C_4\). \(E_2 = \{c_1 = (u_1u_2), c_2 = (u_2u_3), c_3 = (u_3u_4), c_4 = (u_4u_1)\}\). These edges are on path \(P_n\).

\[E_3 = \{(v_1v_i')\} i = 1, 2, 3, 4\) are spokes of \(W_4\). \(E_4 = \{c_1 = (u_1u_2'), c_2 = (u_2u_3'), c_3 = (u_3u_4'), c_4 = (u_4u_1)\}\) where \(u_1 = u_4, u_2 = u_3, u_3 = u_1\). These four edges on \(C_4\) and \(W_4\). We get \(E(G) = E_1 \cup E_2 \cup E_3 \cup E_4\).

Define \(f(E(G)) \rightarrow \{0,1\}\) as follows:

\[f(v_1) = 1;\]
\[f(v_2) = 1;\]
\[f(v_3) = 1;\]
\[f(v_4) = 0;\]
\[f(v_5) = 0;\]
\[f(v_6) = 0;\]
\[f(c_1) = 0;\]
\[f(c_2) = 0;\]
\[f(c_3) = 0;\]
\[f(c_4) = 0;\]
\[f(c_5) = 0;\]
\[f(c_6) = 0;\]
\[f(c_{n+1}) = 0;\]

\[f(e_i) = 1\] for \(i = 2x, x = 1, 2, \ldots, k\). Where \(k = \lfloor \frac{n}{4} \rfloor \).

If \(f\) we defined is e-cordial labeling then have got say, \(e_i(0,1) = (q_1, q_2)\). i.e. number of edges with label 1 are say \(q_1\).

Let \(s = q_2 - 2k\).Then \(f(e_{2k+1}) = 1\) for \(i = 1, 2, \ldots, s\). For \(i > s\) \(f(e_{2k+1+i}) = 0\).That completes e-cordial labeling of \(C_3(P_n)W_4\). We showcase \(f\) in following diagram taking \(n = 5\).

Conclusion: In this paper we discuss the families of graphs obtained by joining two graphs by a path \(P_n\). We denote these graph families by \(G_3(P_n)G_4\). We have taken \(G_1\) and \(G_2\) from \(C_3, C_4, C_5\) and \(W_4\).

References:
[3] Harary, Graph Theory, Narosa publishing , New Delhi