Fibonacci Prime Labeling Of Graphs

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Abstract: In this paper, we introduced Fibonacci prime labeling of graphs. A Fibonacci prime labeling of a graph G = (V(G), E(G)) with |V(G)| = n is an injective function: V(G) → {f₁, f₂, ..., fₙ+₁} where fₙ is the nᵗʰ Fibonacci number, that induces a function g*: E(G) → N defined by g*(uv) = gcd(g(u), g(v)) = 1 ∀ uv ∈ E(G). If g*(uv) = 1 ∀ uv ∈ E(G), we say that the graph G admits a Fibonacci prime labeling and is called a Fibonacci prime graph. In this paper we prove that path, cycle, friendship graph, fan graph, star graph, dragon graph and an umbrella graph are Fibonacci prime graphs.

Keywords: Fibonacci prime labeling, Fibonacci prime graph, path, cycle, friendship graph, fan graph, star graph, dragon graph, umbrella graph.

I. INTRODUCTION

In this paper, only finite simple undirected connected graphs are considered. The graph G has vertex set V = V(G) and edge set E = E(G). The set of vertices adjacent to a vertex u of G is denoted by N(u). For notations and terminology we refer to Bondy and Murthy [1].

The notion of prime labeling was introduced by Roger Entringer and was discussed in a paper by Tout [6]. Two integers a and b are said to be relatively prime if their greatest common divisor is 1. Many researchers have studied prime graph. Fu.H[3] has proved that the path Pₙ on n vertices is a prime graph. Deretsky et al [2] have proved that the cycle Cₙ on n vertices is a prime graph. Around 1980 Roger Entringer conjectured that all trees have prime labeling which has not been settled so far.


Definition 1.1

The Fibonacci number fₙ is defined recursively by the equations f₁ = 1; f₂ = 1; fₙ₊₁ = fₙ + fₙ₋₁ (n ≥ 2). Then g.c.d (fₙ, fₙ₊₁) = 1, for all n ≥ 1.

Definition 1.2

A prime labeling of a graph G is an injective function: f: V(G) → {1, 2, ..., |V(G)|} such that for every pair of adjacent vertices u and v, gcd(f(u), f(v)) = 1. A graph which admits a prime labeling is called a prime graph.

Definition 1.3

A fan graph Pₙ∗ is obtained by joining all vertices of a path Pₙ, n ≥ 2 to a further vertex, called the centre.

Definition 1.4

The friendship graph Fₙ can be constructed by joining n copies of the cycle graph C₃ with a common vertex.

Definition 1.5

The Dragon graph Dₙ(m) is the graph obtained by joining a cycle Cₙ to a path Pₘ with a bridge.

Definition 1.6

An umbrella graph U_(m,n) is the graph obtained by joining a path Pₙ with the central vertex of a fan Pₘ∗.

II MAIN RESULTS

Definition 2.1

A Fibonacci prime labeling of a graph G = (V, E) with |V(G)| = n is an injective function
g: V(G) → \{f_2, f_3, \ldots \ f_{n+1}\} where f_n is the n\textsuperscript{th} Fibonacci number, that induces a function g^* : E(G) → N defined by g^*(uv) = \text{gcd}(g(u), g(v)) = 1 \ \forall \ uv \in E(G).

The graph which admits a fibonacci prime labeling is called Fibonacci prime graph.

**Theorem 2.2**

Path P_n is a Fibonacci prime graph.

**Proof:**

Let G be a path P_n with n vertices. Then |V (G)|= n.

Denote the vertices of P_n as v_1, v_2, \ldots \ldots \ v_n in that order.

Define g: V(G) → \{f_2, f_3, \ldots \ldots \ f_{n+1}\} as

\[ g(v_i) = f_{i+1}, \quad 1 \leq i \leq n \ \forall \ v_i \in V(G). \]

The induced function g^* : E(G) → N is defined by

\[ g^*(uv) = \text{gcd}(g(u), g(v)), \forall uv \in E(G). \]

Now gcd \{f(v_i), f(v_{i+1})\} = gcd(f_{i+1}, f_{i+2}) = 1, \quad 1 \leq i \leq n - 1 \ \forall v_iv_{i+1} \in E(G).

Thus G admits a Fibonacci prime labeling.

Hence G is a Fibonacci prime graph.

**Example 2.3**

![Figure: 1 Fibonacci prime labeling of path P_n](image-url)

**Theorem 2.4**

Cycle C_n is a Fibonacci prime graph for n ≥ 3.

**Proof:**

Let v_1, v_2, \ldots \ldots v_n be the vertices of the cycle C_n. The edge set of C_n is E(C_n) = \{ v_i v_{i+1} | 1 \leq i \leq n - 1 \} \cup \{ v_n v_1 \}. Define g: V(C_n) → \{f_2, f_3, \ldots \ldots f_{n+1}\} as

\[ g(v_i) = f_{i+1}, 1 \leq i \leq n. \]

Then the induced function g^* : E(G) → N is defined by

\[ g^*(xy) = \text{gcd}(g(x), g(y)) \ \forall \ xy \in E(G). \]

Now gcd(g(v_i), g(v_{i+1})) = gcd(f_{i+1}, f_{i+2}) = 1, 1 \leq i \leq n - 1.

and gcd(g(v_n), g(v_1)) = gcd(f_{n+1}, f_{2}) = gcd(f_{n+1}, 1) = 1

Thus f^*(xy) = \text{gcd}(f(x), f(y)) = 1, \forall xy \in E(G). Hence C_n is a Fibonacci prime graph.
Example 2.5

Theorem 2.6

The Fan graph $P_n^*$, $n \geq 2$ is a Fibonacci Prime graph.

Proof:

Let $v_1, v_2, \ldots, v_n, v_{n+1}$ be the vertices of the fan graph $P_n^*$ with centre vertex $v_1$. Let $G = P_n^*$.

The edge set $E(G) = \{v_1v_i, 2 \leq i \leq n+1\}$ and $\{v_iv_{i+1}, 2 \leq i \leq n-1\}$.

Then $|V(G)| = n + 1$ and $|E(G)| = 2n - 1$.

Define $g: V(G) \to \{f_2, \ldots, \ldots, f_{n+2}\}$ by $g(v_i) = f_{i+1}$, $1 \leq i \leq n + 1$.

The induced function $g*: E(G) \to \mathbb{N}$ is defined by $g^*(uv) = g(u)g(v), \forall uv \in E(G)$

Now, $\text{g.c.d}(g(v_i), g(v_j)) = \text{g.c.d}(f_{2}, f_{n+2}) = \text{g.c.d}(1, f_{n+1}) = 1$ for $2 \leq i \leq n + 1$

$\text{g.c.d}(g(v_i), g(v_{i+1})) = \text{g.c.d}(f_{i+1}, f_{i+2}) = 1$ for $2 \leq i \leq n$.

Thus $g^*(uv) = \text{g.c.d}(g(u), g(v)) = 1$, $\forall uv \in E(G)$.

Hence the fan graph $P_n^*$ is a Fibonacci Prime graph.

Example 2.7

Theorem 2.8

Star graph $K_{1,n}$, $n \geq 1$ is a Fibonacci prime graph.

Proof:

Let $G$ be the star graph $K_{1,n}$. The vertex set of $G$ is $V(G) = \{v_1, v_2, \ldots, v_{n+1}\}$ where $v_1$ is the centre of the star graph.

Then $|V(G)| = n + 1$ and $|E(G)| = n$. 

Figure: 2  $C_6$ is a Fibonacci prime graph

Figure: 3  Fibonacci prime labeling of the fan graph $P_5^*$
Define \( g : V(G) \rightarrow \{ f, f_2, \ldots, f_{n+2} \} \) by
\[
g(v_i) = f_{i+1}, 1 \leq i \leq n + 1.
\]

The induced function \( g^* : V(G) \rightarrow N \) is defined by \( g^*(uv) = \gcd(g(u), g(v)) \forall uv \in E(G) \).

Now, \( \gcd(g(v_1), g(v_i)) = g.c.d(f_2, f_{i+1}) = g.c.d(1, f_{i+1}) = 1 \forall 2 \leq i \leq n + 1 \)

Thus all the vertices have distinct labels and \( g^*(uv) = \gcd(g(u), g(v)) = 1 \forall uv \in E(G) \)

Hence \( G \) is a Fibonacci prime graph.

**Example 2.9**

![Diagram of Fibonacci prime labeling of \( K_{1,6} \)](image)

**Theorem 2.10**

The friendship graph \( F_n, n \geq 2 \) is a Fibonacci prime graph.

**Proof**

Let \( G \) be the friendship graph \( F_n \).

Let \( v_1, v_2, \ldots, v_n, v_{n+1} \) be the vertices of \( G \) where \( v_1 \) is the centre vertex of \( G \).

The edge set \( E(G) = \{ v_1v_i | 2 \leq i \leq 2n + 1 \} \cup \{ v_2v_{2i+1} | 1 \leq i \leq n \} \).

Then \( |V(G)| = 2n + 1 \) and \( |E(G)| = 3n \).

Define a labeling \( g : V(G) \rightarrow \{ f_2, f_3, \ldots, f_{2n+2} \} \) by \( g(v_i) = f_{i+1}, 1 \leq i \leq 2n + 1 \).

The induced function \( g^* : E(G) \rightarrow N \) is defined by \( g^*(uv) = \gcd(f(u), f(v)), \forall uv \in E(G) \).

Now, \( \gcd(g(v_1), g(v_i)) = \gcd(f_2, f_{i+1}) \)
\[
= \gcd(1, f_{i+1})
= 1 \text{ for } 2 \leq i \leq 2n + 1
\]

\[
\gcd(f(v_2), f(v_{2i+1})) = \gcd(f_{2i+1}, f_{2i+2})
= 1, 1 \leq i \leq n.
\]

Thus \( g^*(uv) = \gcd(g(u), g(v)) = 1 \forall \ uv \in E(G) \).

Hence \( G \) admits a Fibonacci prime labeling. Hence the friendship graph \( F_n \) is a Fibonacci prime graph.
Example 2.11

![Diagram of a graph with vertices labeled 1, 2, 3, 4, 5, and 8, and edges connecting them.]

Figure: 5  Fibonacci prime labeling of the fan graph $F_4$

Theorem 2.12

The dragon graph $D_n(m)$ is a Fibonacci prime graph for $n \geq 3, m \geq 1$.

Proof:

Let $G$ be the dragon graph $D_n(m)$.

Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle $C_n$ and $v_1, v_2, \ldots, v_m$ be the vertices of the path $P_m$.

The edge set $E(G) = \{u_iu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{u_1u_n\} \cup \{v_iv_{i+1} \mid 1 \leq i \leq m-1\}$, where $u_1v_1$ is the bridge joining $C_n$ with $P_m$.

Then $|V(G)| = n + m$ and $|E(G)| = n + m$.

Define the mapping $g: V(G) \to \{f_2, f_3, \ldots, f_{n+m+1}\}$ as follows

$$g(u_i) = f_{i+1}, 1 \leq i \leq n$$
$$g(v_i) = f_{n+i+1}, 1 \leq i \leq m.$$ 

Then the induced function $g^*: E(G) \to N$ is defined by $g^*(xy) = g.c. d(g(x), g(y)) \forall xy \in E(G)$.

Now, $g.c. d(g(u_i), g(u_{i+1}) = g.c. d(f_{i+1}, f_{i+2}) = 1, 1 \leq i \leq n-1.$

$g.c. d(g(u_i), g(v_n)) = g.c. d(f_2, f_{n+1}) = g.c. d(1, f_{n+1}) = 1.$

$g.c. d(g(v_1), g(v_2)) = g.c. d(1, f_{n+2}) = 1.$

$g.c. d(g(v_i), g(v_{i+1})) = g.c. d(g_{n+i+1}, f_{n+i+2}) = 1, 1 \leq i \leq m-1.$

Thus $g^*(xy) = g.c. d(g(x), g(y)) = 1 \forall xy \in E(G)$.

Hence the dragon graph is the Fibonacci prime graph.

Example 2.13

![Diagram of a graph labeled with vertices $u_1, u_2, u_3, v_1, v_2, v_3$ and edges connecting them.]

Figure: 6  Fibonacci prime labeling of a dragon graph $D_4(3)$
Theorem 2.14
An umbrella graph $U_{m,n}$ is a Fibonacci prime graph for all $m$ and $n$.

Proof
Let $G = U_{m,n}$. The vertex set of $G$ is $V(G) = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}$. The edge set of $G$ is $E(G) = \{x_ix_{i+1} / 1 \leq i \leq m-1\} \cup \{y_iy_{i+1} / 1 \leq i \leq n-1\}$. Then $|V(G)| = m + n$ and $|E(G)| = 2m + n - 2$. Define a function $g: V(G) \to \{f_2, f_3, \ldots, f_{m+n+1}\}$ as follows.

$g(x_i) = f_{i+2}$ for $1 \leq i \leq m$  
$g(y_i) = f_{m+i+1}$ for $2 \leq i \leq n$

The induced function $g^*: E(G) \to N$ is defined by $g^*(uv) = \gcd(g(u), g(v)) \forall uv \in E(G)$.

Now, $g. c. d\{g(x_i), g(x_{i+1})\} = \gcd(f_{i+2}, f_{i+3}) = 1$ for $1 \leq i \leq m - 1$.

$g. c. d\{g(x_i), g(y_i)\} = \gcd(f_{i+2}, f_2) = g. c. d\{f_{i+2}, 1\} = 1$ for $1 \leq i \leq m$.

$g. c. d\{g(y_1), g(y_2)\} = g. c. d\{1, f_{n+3}\} = 1$.

$g. c. d\{g(y_i), g(y_{i+1})\} = g. c. d\{f_{m+i+1}, f_{m+i+2}\} = 1, \quad 2 \leq i \leq n - 1$.

Thus $g^*(uv) = \gcd(g(u), g(v)) = 1 \forall uv \in E(G)$. Hence $G$ admits a Fibonacci prime labeling and hence the umbrella graph $U_{m,n}$ is a Fibonacci prime graph.

Example 2.15

![Diagram of Fibonacci prime labeling of an Umbrella graph $U_{7,5}$]

Figure: 8  Fibonacci prime labeling of an Umbrella graph $U_{7,5}$
CONCLUSION

We have introduced a new labeling namely Fibonacci prime labeling of graphs. We prove that path, cycle, star, fan graph, friendship graph, dragon graph and an umbrella graph are all Fibonacci prime graphs. Extending the study to other families of graphs is an open area of research.

References


