α - Graceful Labeling for a Binary Tree and Graceful Labeling for a Regular Tree

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Abstract

Labeled graph is the topics of current interest and here we have discussed α-graceful labeling for the regular binary tree. We have also discussed graceful labeling for banana tree, symmetric tree and regular tree.
1 Introduction

In 1996 Rosa defined graceful labeling of a simple graph $G$ and $\alpha$-labeling (here we call $\alpha$-graceful labeling) for a graph. Banana tree $B(n,k)$, to be the tree obtained by joining one leaf of each $n$ copies $K_{1,k-1}$ ($k - 1$ - star) i.e. it is one point union of $n$ copies of star graph $K_{1,k-1}$. A symmetric tree $T_{k+1}(d)$, to be a tree with diameter $d$, in which all vertices other than leaves and root have the same degree $k + 1$ and all leaves have same eccentricity, where root is the center for $T_{k+1}(d)$, with degree $k$ and eccentricity $\frac{d}{2}$. Here $d$ is the diameter for $T_{k+1}(d)$. $d_G(v)$ is denoted for the degree of vertex $v$ in $G$.

In this paper a graph $G$ we mean a simple, finite and undirected graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. We follow Harary[1] for basic notation and terminology of graphs.

2 Main Results

Theorem 2.1

Let $T$ be a graceful tree. Let $f$ be a graceful labeling for $T$ and there is $v \in V(T)$, $d_T(v) = 1$ and $f(v) = 0$. Then one point union of two copies of $T$ at $v$ is $\alpha$ - graceful tree.

Proof: Let $p = |V(T)|$ then $q = |E(T)| = p - 1$. Let $V(T^{(1)}) = \{v_0 = v, v_1, v_2, \ldots, v_q\}$ be vertices of first copy $T^{(1)}$ of $T$. Let $f$ be a graceful labeling for $T = T^{(1)}$ such that $f(v_0) = 0$. Let $T^{(2)}$ be another copy of $T$ and $V(T^{(2)}) = \{u_0, u_1, u_2, \ldots, u_q\}$. Let $G$ be a graph (tree) obtained by merging $u_0 = v_0 = v, v_1, v_2, \ldots, v_q, u_1, u_2, \ldots, u_q$ and $|E(G)| = 2q$. Since $T = T^{(1)}$ is bipartite graph, for each $e = (u, w) \in E(T)$, there is a partition $V_1 \cup V_2$ of $V(T)$ such that $u \in V_1$ and $w \in V_2$. Take $v \in V_1$.

Define $g : V(G) \rightarrow \{0, 1, 2, \ldots, 2q\}$ by $g/V_{1}^{(i)} = f/V_{1}^{(i)}$, $g/V_{2}^{(i)} = f/V_{2}^{(i)} + q$, $g/V_{1}^{(2)} - \{v\} = f/V_{1}^{(1)} - \{v\} + q$ and $g/V_2 = f/V_2^{(1)}$ where $V_1^{(i)} \cup V_2^{(i)}$ is the vertex partition of $V(T^{(i)})$, $i = 1, 2$. First we shall prove here $g$ is a bijection.

Let $w_1, w_2 \in V(G)$ be such that $g(w_1) = g(w_2)$ and $w_1 \neq w_2$.

$\Rightarrow f(w_1) = f(w_2)$ and $w_1 \in V_1$, $w_2 \in V_2$ which is impossible as $f$ is one-one.
Since, $|V(G)| = 2q + 1, g : V(G) \rightarrow \{0, 1, ..., 2q\}$ must be a bijection. The induced edge labeling $g^* : E(G) \rightarrow \{1, 2, \ldots, 2q\}$ defined by $g^*(e = (w_1, w_2)) = |g(w_1) - g(w_2)|$. Now we shall prove $g^*$ is bijective map.

Let $e_1 = (w_1, w_2), e_2 = (w_3, w_4) \in E(G)$ such that $g^*(e_1) = g^*(e_2)$ where $w_1, w_3 \in V_1$.

$$\Rightarrow |g(w_1) - g(w_2)| = |g(w_3) - g(w_4)|$$

$$\Rightarrow |\pm q + (f(w_1) - f(w_2))| = |\pm q + (f(w_3) - f(w_4))|$$

$$\Rightarrow q \pm |(f(w_1) - f(w_2))| = q \pm |(f(w_3) - f(w_4))|$$

$$\Rightarrow f(w_1) - f(w_2), f(w_3) - f(w_4) \text{ either both are positive or both are negative.}$$

$$\Rightarrow |f(w_1) - f(w_2)| = |f(w_3) - f(w_4)|$$

$$\Rightarrow f^*(e_1) = f^*(e_2)$$

$$\Rightarrow e_1 = e_2, \text{ as } f^* \text{ is a bijection.}$$

Now for any $e = (w_5, w_6) \in E(G)$,

$$g^*(e) = |g(w_5) - g(w_6)|$$

$$= \text{either } q + |f(w_5) - f(w_6)| \text{ or } q - |f(w_5) - f(w_6)|$$

$$= \text{either } q - f^*(e) \text{ or } q + f^*(e)$$

$$\Rightarrow g^*(E(G)) = \{1, 2, \ldots, q, q + 1, q + 2, \ldots, 2q\}$$

Thus, above labeling pattern $g$ gives rise to a graceful labeling to the graph (tree) $G$. 
Let $w_7, w_8 \in V(T)$ be such that $f(w_7) = q$ and $f(w_8) = 1$. Since $f(v) = 0$ and $d_T(v) = 1$, $v$ is adjacent only with the vertex $w_7$ in $T$. To produce the edge label $q - 1$ in $T$, $w_8$ should be adjacent with $w_7$.

Now $g^*(w_7, w_8) = |g(w_7) - g(w_8)|$

$= q + |f(w_7) - f(w_8)|$ or $q - |f(w_7) - f(w_8)|$

$= q - (f(w_7) - f(w_8))$ in second copy $T^{(2)}$

$= 1$ in second copy $T^{(2)}$

Take $k = g(w_7) = q$. It is observed that for any $e = (w_9, w_{10}) \in E(G)$, $\min\{g(w_9), g(w_{10})\} \leq k = q < \max\{g(w_9), g(w_{10})\}$ and so, $g$ is an $\alpha$-graceful labeling for $G$.

### 2.2 Regular binary tree:

Regular binary tree $BT_n$, where $n = 1 + 2 + \ldots + 2^m$, for some $m \in N$ i.e. $BT_1 = K_1$, $BT_3 = P_3$ and $BT_7$ is obtained by taking one point union of two copies of $K_{1,3}$ as shown in figure - 1.

![Figure - 1](image)

one point union of two copies of $K_{1,3}$.

Next step, add one pendent vertex at the common vertex of $2K_{1,3}$ in $BT_7$ and to obtain $BT_{15}$, take one point union of two copies of above said tree by muring the added pendent vertex as shown in figure - 2.
one point union of two copies of graph obtained by adding a pendent vertex at the root of $BT_7$.

Continue this way, add one pendent vertex at the root of $BT_{2m-1}$ and to obtain $BT_{2m+1-1}$ take one point union of two copies of $BT_{2m-1}$ with pendent vertex by merging the added pendent vertex as shown in figure - 3.

Thus, $BT_{2m+1-1}$ is the symmetric tree $T_3(2m)$.

2.3 Algorithm to obtain $\alpha$-graceful labeling $BT_{2m-1}$:

Obviously, following graceful labeling (given in figure - 4) for $K_{1,3}$ is an $\alpha$-graceful labeling, where $k = 2$. 
Using this according to Theorem 2.1 obtain $\alpha$-graceful labeling for $BT_7$ as shown in figure - 5.

Add one pendent vertex with vertex label 7, which gives an $\alpha$-graceful labeling and take its complement $\alpha$-graceful labeling by subtracting each vertex label from 7 and according to Theorem 2.1 obtain $\alpha$-graceful labeling for $BT_{15}$ as shown in figure - 6.
\textbf{Lemma 2.4} 
Let $T$ be graceful tree. Let $f$ be a graceful labeling for $T$ and there is $v \in V(T)$, $d_T(v) = 1$ and $f(v) = 0$. Then one point union of three copies of $T$ at $v$ is graceful.

\textbf{Proof:} Let $p = |V(T)|$ then $q = |E(T)| = p - 1$. Let $V(T^{(1)}) = \{v_0 = v, v_1, v_2, \ldots, v_q\}$ be vertices of first copy $T^{(1)}$ of $T$. Let $f$ be an arbitrary graceful labeling for $T = T^{(1)}$ such that $f(v_0) = 0$. Let $T^{(2)}, T^{(3)}$ be another copies of $T$ and $V(T^{(2)}) = \{u_0, u_1, u_2, \ldots, u_q\}$, $V(T^{(3)}) = \{w_0, w_1, w_2, \ldots, w_q\}$. Let $G$ be a graph (tree) obtained by merging $u_0, v_0, w_0$ \{(one point union of $T^{(i)}, i = 1, 2, 3$).

It is obvious that $V(G) = \{v, v_1, \ldots, v_q, u_1, \ldots, u_q, w_1, \ldots, w_q\}$ and $|E(G)| = 3q$. Since $T = T^{(1)}$ is bipartite graph, for each $e = (u, w) \in E(T)$, there is a partition $V_1 \cup V_2$ of $V(T)$ such that $u \in V_1$ and $w \in V_2$. Take $v \in V_1$.

Define $g : V(G) \rightarrow \{0, 1, 2, \ldots, 3q\}$ by $g/V_1^{(1)} = f/V_1^{(1)}$, $g/V_2^{(1)} = f/V_2^{(1)} + 2q, g/V_1^{(2)} - \{v\} = 2q + f/V_1^{(1)} - \{v\}$ and $g/V_2^{(2)} = f/V_2^{(1)}$ and $g/V(T^{(3)}) - \{v\} = q + f/V(T^{(3)}) - \{v\}$ where $V_1^{(i)} \cup V_2^{(i)}$ is the vertex partition of $V(T^{(i)})$, $i = 1, 2$. First we shall prove here $g$ is a bijective map.

Let $s_1, s_2 \in V(G)$ be such that $g(s_1) = g(s_2)$ and $s_1 \neq s_2$ if possible.

$\Rightarrow f(s_1) = f(s_2)$ which is not possible as $f$ is one-one.
Thus, \( g : V(G) \rightarrow \{0, 1, \ldots, 3q\} \) is a bijection, as \( g \) is one-one and \( |V(G)| = 3q + 1 \).

The induced edge labeling \( g^* : E(G) \rightarrow \{1, 2, \ldots, 3q\} \) defined by \( g^*(e = (s_3, s_4)) = |g(s_3) - g(s_4)| \). We have to prove \( g^* \) is also a bijective map. It is enough to prove \( g^* \) is one-one.

Let \( e_1 = (s_5, s_6), e_2 = (s_7, s_8) \in E(G) \) and \( g^*(e_1) = g^*(e_2) \).

\[
\Rightarrow |g(s_5) - g(s_6)| = |g(s_7) - g(s_8)|
\Rightarrow |f(s_5) - f(s_6)| = |f(s_7) - f(s_8)|, \text{ by definition of } g
\Rightarrow f^*(e_1) = f^*(e_2)
\Rightarrow e_1 = e_2, \text{ as } f^* \text{ is a bijection.}
\]

Now for any \( e = (s_9, s_{10}) \in E(G) \),

\[
g^*(e) = |g(s_9) - g(s_{10})|
= \text{ either } 2q + |f(s_9) - f(s_{10})| \text{ or } 2q - |f(s_9) - f(s_{10})| \text{ or } |f(s_9) - f(s_{10})|.
\Rightarrow g^*(E(G)) = \{1, 2, \ldots, 3q\}
\]

Thus, above labeling pattern give rise a graceful labeling \( g \) to the given graph (tree) \( G \).

**Theorem 2.5**

Let \( T \) be graceful tree. Let \( f \) be a graceful labeling for \( T \) and there is \( v \in V(T), d_T(v) = 1 \) and \( f(v) = 0 \). Then one point union of \( l \) copies of \( T \) at \( v \) is also a graceful tree.

**Proof:** Let \( V(T) = \{v_0 = v, v_1, \ldots, v_q\} \). Let \( f \) be a graceful labeling for \( T = T^{(1)} \) with \( f(v_0) = 0 \). Let \( T^{(2)}, T^{(3)}, \ldots, T^{(l)} \) be another copies of \( T \) and \( G \) be a tree obtained by merging vertex \( v \) of each copies \( T^{(1)}, T^{(2)}, \ldots, T^{(l)} \).
It is obvious that \( V(G) = lq + 1 \) and \(|E(G)| = lq\). Since, \( T^{(1)} \) is bipartite graph, for each \( e = (u, w) \in E(T) \), there is a vertex partition \( V_1 \cup V_2 \) of \( V(T) \) such that \( u \in V_1 \) and \( w \in V_2 \). Take \( v \in V_1 \).

To define \( g : V(G) \to \{0, 1, 2, \ldots, lq\} \) we consider following two cases.

**Case - I :** \( l \) is even.
\[
g/V^{(1)}_1 = f/V^{(1)}_1, \quad g/V^{(1)}_2 = f/V^{(1)}_2 + (l - 1)q, \quad g/V^{(2)}_1 - \{v\} = (l - 1)q + f/V^{(1)}_1 - \{v\}, \quad g/V^{(2)}_2 = f/V^{(2)}_2 + (l - 1)q + f/V^{(1)}_1 - \{v\},
\]
\[
g/V^{(i+1)}_1 = (l - \frac{i+1}{2})q + f/V^{(i+1)}_1, \quad g/V^{(i+1)}_2 = (l - \frac{i+1}{2})q + f/V^{(i+1)}_2, \quad \forall i = 3, 5, \ldots, l - 1
\]

**Case - II :** \( l \) is odd.
\[
g/V^{(1)}_1 = f/V^{(1)}_1, \quad g/V^{(1)}_2 = (l - 1)q + f/V^{(1)}_2, \quad g/V^{(2)}_1 - \{v\} = (l - 1)q + f/V^{(1)}_1 - \{v\}, \quad g/V^{(2)}_2 = (l - 1)q + f/V^{(2)}_2 + (l - 1)q + f/V^{(1)}_1 - \{v\},
\]
\[
g/V^{(i+1)}_1 = (l - \frac{i+1}{2})q + f/V^{(i+1)}_1, \quad g/V^{(i+1)}_2 = (l - \frac{i+1}{2})q + f/V^{(i+1)}_2, \quad \forall i = 3, 5, \ldots, l - 2
\]

Where \( V^{(i)}_1 \cup V^{(i)}_2 \) is the vertex partition of \( V(T^{(i)}) \), \( i = 1, 2, \ldots, l - 1 \).

Thus, above labeling pattern give rise a graceful labeling \( g \) to the given graph (tree) \( G \) and so, \( G \) is a graceful graph.

**Corollary 2.6**

Every banana tree \( B(n, k) \) is graceful.

Since \( B(n, k) \) is one point union of \( n \) copies of the star graph \( K_{1,k-1} \) and \( K_{1,k-1} \) has required graceful labeling, by **Theorem 2.5**, we can obtain a graceful labeling for \( B(n, k) \)

**Corollary 2.7**

Symmetric tree \( T_{n+1} \) is a graceful graph.

\[
T_{(n+1)}(2) = K_{1,n}, \quad T_{n+1}(4) = B(n, n + 1), \quad T_{n+1}(6) \text{ is one point union of } n \text{ copies of a graph obtained by adding a pendent vertex to } T_{n+1}(4) \text{ at the root.}
\]

Similarly, \( T_{n+1}(d) \) is one point union of \( n \) copies of a graph obtained by adding a pendent vertex to \( T_{n+1}(d - 2) \) at the root.
To get graceful labeling for $T_{n+1}(d)$, use Algorithm 2.3 and Theorem 2.5 recursively, as $T_{n+1}(2) = K_{1,n}$, $T_{n+1}(4) = B(n, n+1)$ both are graceful trees.

2.8 Graceful Labeling and Regular Tree:

Regular tree is $R(t_1, t_2, \ldots, t_l)$, where $t_1, t_2, \ldots, t_l \in N$ and we define it as follows.

Take $t_3$ copies of a banana tree $B(n, k)$ and add one pendent edge to each copy of $B(n, k)$ at the root. Now take one point union of $t_3$ copies of the graph obtain from $B(n, k)$ at the pendent vertex, as shown in figure-7.

Such regular graph we denote by $R(k, n, t_3)$. By taking $n = t_2$ and $k = t_1$, we rewrite it by $R(t_1, t_2, t_3)$. Next take $t_4$ copies of $R(t_1, t_2, t_3)$ and add one pendent edge to each copy at root. Then take one point union of $t_4$ trees at the added pendent vertex to get the regular tree $R(t_1, t_2, t_3, t_4)$.

Continuing in this way, we get $R(t_1, t_2, t_3, \ldots, t_{(l-1)})$ with a pendent edge at the root. It is obvious that $BT_7 = B(2, 3)$, $BT_{15} = R(3, 2, 2)$ and $BT_{2m+1} - 1 = R(3, 2, \ldots, 2(m\text{-times}))$

To obtain graceful labeling for $R(t_1, t_2, t_3, \ldots, t_l)$ to get recursively way graceful labeling for $R(t_1, t_2, t_3, \ldots, t_{(l-1)})$, $R(t_1, t_2, t_3, \ldots, t_{(l-2)})$, $R(t_1, t_2, t_3)$ and $B(t_2, t_1)$ obtain by Corollary 2.6.
Illustration 2.9

$R(4, 5, 3)$ with its graceful labeling, to obtain this a graph obtain by adding a pendent edge to $B(5, 4)$ and $B(5, 4)$, $K_{1,3}$ with their graceful labeling are shown in figure - 10, 9, 8.

A graph obtained by adding a pendent edge to $B(5, 4)$ and its complement graceful labeling

$B(5, 4), K_{1,3}$ and their graceful labeling
$R(4, 5, 3)$ and its graceful labeling obtained by Theorem 2.5

References

