

A Comparative Study On Initial Value Problems (IVP) for Ordinary Differential Equations (ODE) with Euler's, Euler's Modified, Picard's and Runge Kutta Methods

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ABSTRACT :

This paper presents different numerical method for the solution of ordinary differential equations. We consider the first order differential equation $dy/dx = f(x, y)$ with the initial condition $y(x_0) = y_0$. We can solve the ordinary differential equations by using the following methods Euler's Method, Euler's Modified Method, Picard's Method and Runge-Kutta Method. In order to verify the accuracy, we compare numerical solutions with exact solutions. The numerical solutions are in good agreement with the exact solutions. Numerical comparisons between these methods have been presented. Also we compare the performance and the computational effort of such methods.

KEYWORDS : Initial Value Problem (IVP), Ordinary Differential Equation (ODE), Euler's Method, Euler's Modified Method, Picard's Method, Runge-Kutta Method.

INTRODUCTION :

Differential equations are commonly used for mathematical modeling in science and engineering. Many problems of mathematical physics can be started in the form of differential equations. These equations also occur as reformulations of other mathematical problems such as ordinary differential equations and partial differential equations. In most real life situations, the differential equation models the problem that are too complicated to solve exactly, and one of two approaches is taken to approximate the solution. The first approach is to simplify the

differential equation to one that can be solved exactly and use the solution of the simplified equation to approximate the solution to the original equation. The other approach uses methods for approximating the solution of original problem. This is the approach that is most commonly taken as the approximation methods give more accurate results and realistic error information. Numerical methods are generally used for solving mathematical problems that are formulated in science and engineering where it is difficult and sometime impossible for calculated exact solutions. There are a few differential equations that can be solved analytically. There are many analytical methods for finding the solution of ordinary differential equations. But there are large number of ordinary differential equations whose solutions cannot be obtained by using well-known analytical methods, in such cases we use the numerical methods for finding the approximate solution of a differential equation under the given initial conditions. There are many types of practical numerical methods for solving initial value problems for ordinary differential equations.

FORMATION OF PROBLEM :

In this section we consider numerical methods for finding the approximate solutions of the initial value problem (IVP) of the first-order ordinary differential equation has the form

$$y' = f(x, y(x)), x \in (x_0, x_n) \text{ with initial condition } y(x_0) = y_0 \dots\dots\dots(i)$$

Where $y' = dy/dx$ and $f(x, y(x))$ is a given function and $y(x)$ is the solution of the equation (i). In this section paper we determine the solution of this equation in a interval (x_0, x_n) , starting with the initial point x_0 . A continuous approximation to the solution $y(x)$ will not be obtained in the interval (x_0, x_n) . Numerical methods apply the equation (i) to obtain approximations to the values of the solution corresponding to various selected values of $x = x_n = x_0 + nh, n = 1, 2, 3, \dots$. The constant h is called the steps size. The numerical solutions of equation (i) is given by a set of points $\{(x_n, y_n) : n = 0, 1, 2, \dots, n\}$ and each point (x_n, y_n) is an approximation to the corresponding point $(x_n, y(x_n))$ on the solution curve.

Euler's Method -

The general formula for Euler's approximation is

$$y_{n+1} = y_n + h f(x_n, y_n), n = 0, 1, 2, 3, \dots$$

Modified Euler's Method -

The general formula for Modified Euler's approximation is

$$y_{n+1} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_{n+1}, y_{n+1}^{(1)})]$$

$$\text{Where } y_{n+1}^{(1)} = y_n + h f(x_n, y_n)$$

Picard's method -

The general formula for Picard's approximation is

$$y_n = y_0 + \int_{x_0}^x f(x, y_{n-1}) dx$$

Runge-Kutta Method -

Four types of Runge-Kutta method.

(i) Runge-Kutta Method of First Order -

$$y_{n+1} = y_n + h f(x_n, y_n)$$

(ii) Runge-Kutta Method of Second Order -

$$y_{n+1} = y_n + \frac{1}{2} (k_1 + k_2)$$

$$\text{where } k_1 = h f(x_n, y_n) \text{ and } k_2 = h f(x_n + h, y_n + k_1)$$

(iii) Runge-Kutta Method of third Order -

$$y_{n+1} = y_n + \frac{1}{6} (k_1 + 4 k_2 + k_3)$$

$$\text{where } k_1 = h f(x_n, y_n), k_2 = h f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}), k_3 = h f(x_n + h, y_n + k'), k' = h f(x_n + h, y_n + k_1)$$

(iv) Runge-Kutta Method of Fourth Order -

$$y_{n+1} = y_n + \frac{1}{6} (k_1 + 2 k_2 + 2 k_3 + k_4)$$

where $k_1 = h f(x_n, y_n)$, $k_2 = h f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$, $k_3 = h f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$, $k_4 = h f(x_n + h, y_n + k)$

NUMERICAL APPROACH :

Problem1. Solve the equation $\frac{dy}{dx} = 1 - y$, $y(0) = 0$ and find $y(0.1)$, $y(0.2)$, $y(0.3)$.

Solution : The given differential equation is

$$\frac{dy}{dx} = 1 - y$$

$$f(x, y) = 1 - y$$

Taking $h = 0.1$, we have

$$x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$$

The solution of this problem-

Table 1.1 (Exact Solution)

x	Exact Solution
0.1	0.095163
0.2	0.181269
0.3	0.259182

Table 1.2 (Euler's Method)

x	Euler's Method
0.1	0.1
0.2	0.19
0.3	0.271

Table 1.3 (Modified Euler's Method)

x	Euler's Modified Method
0.1	0.095238
0.2	0.181405
0.3	0.259366

Table 1.4 (Picard's Method)

x	Picard's Method
0.1	0.095167
0.2	0.181333
0.3	0.2595

Runge – Kutta Method -**R-K Method of 1st Order-**

Euler method is the Runge – Kutta method of first order.

Table 1.5 (R-K Method of 2nd Order)

x	R-K Method of 2 nd Order
0.1	0.095
0.2	0.180975
0.3	0.258783

Table 1.6 (R-K Method of 3rd Order)

x	R-K Method of 3 rd Order
0.1	0.095167
0.2	0.181277
0.3	0.259192

Table 1.7 (R-K Method of 4th Order)

x	R-K Method of 4 th Order
0.1	0.095163
0.2	0.181269
0.3	0.259182

The values tabulated -**Table 1.8**

x	Exact Solution	Euler's/R-K 1 st Order Method	Euler's Modified Method	Picard's Method	R-K 2 nd Order Method	R-K 3 rd Order Method	R-K 4 th Order Method
0.1	0.095163	0.1	0.095238	0.1	0.095	0.095167	0.095163
0.2	0.181269	0.19	0.181405	0.18	0.180975	0.181277	0.181269
0.3	0.259182	0.271	0.259366	0.2595	0.258783	0.259192	0.259182

Difference Table :

$$\text{Difference} = | \text{Exact Solution} - \text{Numerical Solution} |$$

$$= | E_n - N_n |, n = 1, 2, 3$$

where numerical solution are Euler's method, Modified Euler's method, Picard's method respectively.

Table 1.9

x	Exact Solution	Euler's/ R-K 1 st Order Method	Difference	Euler's Modified Method	Difference	Picard's Method	Difference	R-K 2 nd order method	Difference	R-K 3 rd order method	Difference	R-K 4 th order method	Differe nce
0.1	0.095163	0.1	0.004837	0.095238	0.000077	0.1	0.000004	0.095	0.000163	0.095167	0.000004	0.095163	0
0.2	0.181269	0.19	0.008731	0.181405	0.000141	0.18	0.000064	0.180975	0.000294	0.181277	0.000008	0.181269	0
0.3	0.259182	0.271	0.011818	0.259366	0.000188	0.2595	0.000318	0.258783	0.000399	0.259192	0.000001	0.259182	0

Problem 2. Solve the equation $\frac{dy}{dx} + \frac{y}{10} = e^{\frac{-x}{10}} \cos x$, $y(0) = 0$ and find $y(0.5)$, $y(1)$, $y(1.5)$.

Solution : The given differential equation is

$$\frac{dy}{dx} + \frac{y}{10} = e^{\frac{-x}{10}} \cos x$$

$$f(x, y) = e^{\frac{-x}{10}} \cos x - \frac{y}{10}$$

Taking $h = 0.5$, we have

$$x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5$$

The Solution of this problem-

Table 2.1 (Exact Solution)

x	Exact Solution
0.5	0.008301
1	0.015792
1.5	0.022531

Table 2.2 (Euler's Method)

x	Euler's Method
0.5	0.5
1	0.950597
1.5	1.355417

Table 2.3 (Euler's Modified Method)

x	Euler's Modified Method
0.5	0.475901
1	0.905343
1.5	1.291695

Table 2.4 (Picard's Method)

x	Picard's Method
0.5	0.009199
1	0.016648
1.5	0.023347

Runge – Kutta Method -

Runge - Kutta Method of First Order-

Euler method is the Runge – Kutta method of first order

Table 2.5 (R-K Method of 2nd Order)

x	R-K Method of 2 nd Order
0.5	0.475299
1	0.428913
1.5	0.837937

Table 2.6 (R-K Method of 3rd Order)

x	R-K Method of 3 rd Order
0.5	0.475611
1	0.904796
1.5	1.290921

Table 2.7 (R-K Method of 4th Order)

x	R-K Method of 4 th Order
0.5	0.475609
1	0.904792
1.5	1.290915

The values tabulated –

Table 2.8

x	Exact Solution	Euler's/R-K 1 st order method	Euler's Modified method	Picard's method	R-K 2 nd order method	R-K 3 rd order method	R-K 4 th order method
0.5	0.008301	0.5	0.927703	0.009199	0.475299	0.475611	0.475609
1	0.015792	0.950597	0.905343	0.016648	0.428913	0.904796	0.904792
1.5	0.022531	1.355417	1.291695	0.023347	0.837973	1.290921	1.290915

Difference Table :

$$\text{Difference} = | \text{Exact Solution} - \text{Numerical Solution} |$$

$$= | E_n - N_n |, n = 1, 2, 3$$

where numerical solution are Euler's method, Modified Euler's method, Picard's method respectively.

Table 2.9

x	Exact Solution	Euler's/ R-K 1 st Order Method	Difference	Euler's Modified Method	Difference	Picard's Method	Difference	R-K 2 nd order method	Difference	R-K 3 rd order method	Difference	R-K 4 th order method	Difference
0.5	0.008301	0.5	0.491699	0.475901	0.4676	0.009199	0.000898	0.450597	0.442296	0.475611	0.46731	0.475609	0.467308
1	0.015792	0.950597	0.934805	0.905357	0.889565	0.016648	0.000856	0.428913	0.413121	0.904796	0.889004	0.904792	0.889
1.5	0.022531	1.355417	1.332886	1.291695	1.269164	0.023347	0.000816	0.837973	0.815442	1.290921	1.26839	1.290915	1.268384

CONCLUSION :

This paper focused on comparative study on among various numerical method's like Euler's Method, Euler's Modified Method, Picard's and Runge-Kutta Method by numerical approach. In problem 1 we notice table 1.9 the difference is zero by fact Runge-Kutta Method of order fourth is more accurate. By problem 2 we notice table 2.9 the difference is very close to exact value. So Picard's Method is also more accurate method further more, the accurate numerical solution is also depend on the type of function. For example, problem 1 is an algebraic function and problem 2 has exponential and trigonometric function.

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