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ANALYSIS OF CONVERGENCE RATE OF GAUSSIAN BELIEF PROPAGATION USING WALK SUMMABILITY AND LAPLACIAN APPROACHES

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Abstract: Gaussian belief propagation algorithm (GaBP) is one of the most important distributed algorithms in signal processing and statistical learning involving Markov networks. It is known that the algorithm correctly computes marginal density functions from a high dimensional joint density function over a Markov network in a finite number of iterations when the underlying Gaussian graph is acyclic. Analysis of convergence rate is an important factor. GaBP algorithm is shown to converge faster than classical iterative methods like Jacobi method,successive over relaxation.It is more recently known that walk summability approach extends for better convergence result.Convergence rate analysis of GaBP for markov network using walk summability approach and theoretical study of convergence rate analysis using laplacian operator are considered in this work.

***Index Terms*— Gaussian belief propagation,Markov network,Convergence rate,Walk summability,Laplacian operator**

I. INTRODUCTION

Belief propagation algorithm is a well-celebrated distributed algorithm for Markov networks that has been widely utilized in many disciplines, ranging from statistical learning and artificial intelligence to distributed estimation,distributed optimisation, networked control and digital communications. It is designed compute the marginal probability densities of random variables from the joint probability density function over a large Markov network with sparse connections among individual random variables. The Gaussian BP algorithm (GaBP), a special version of the BP algorithm for Markov networks with Gaussian distributions (also known as Gaussian graphical model), has received special attention for the study of its convergence properties.Markov network or undirected graphical model is a set of random variables having a Markov property described by an undirected graph. In other words, a random field is said to be a Markov random field if it satisfies Markov properties.

A Markov network or MRF is similar to a Bayesian network in its representation of dependencies; the differences being that Bayesian networks are directed and acyclic, whereas Markov networks are undirected and may be cyclic. Thus, a Markov network can represent certain dependencies that a Bayesian network cannot (such as cyclic dependencies); on the other hand, it can't represent certain dependencies that a Bayesian network can (such as induced dependencies). The underlying graph of a Markov random field may be finite or infinite.

In numerical analysis,the order of convergence and the rate of convergence of a convergent sequence are quantities that represent how quickly the sequence approaches its limit. In practice ,the rate and order of convergence provide useful insights when using iterative methods for calculating numerical approximations.Many methods exist to increase the rate of convergence of a given sequence,that is to transform a given sequence into one converging faster to the same limit.The goal of the transformed sequence is to reduce the computational cost of calculation.

Different methods or approaches can be used for the purpose of analysing convergence of markov network.Walk summability approach is the best among the existing one.This paper discusses about laplacian solvers for determing the convergence rate for cyclic networks.

II. LITERATURE REVIEW

A novel statistical manifold algorithm for position estimation in wireless sensor networks is developed in [1]. It ables to find distance information among unknown and anchor nodes, in following steps:First ,a ranging model about distance information is established.Then a solution problem to this established model,it is then transformed into a parameter estimation of curved exponential family.Next a solution to this estimation problem is obtained by natural gradient method.To ensure convergence of the proposed algorithm,a particle swarm optimization method is utilized to obtain initial values of the unknown nodes.PSO is a computational method that optimizes a problem iteratively.

In [2],convergence rate is analysing using Gaussian Belief Propagation Solver(GaBP).This paper describes an undirected graphical models corresponding to the linear systems of equations.Applying Belief Propagation in graphical model and updating the BP equation using performed. In ordinary BP, convergence does not guarantee exactness of the inferred probabilities, unless the graph has no cycles. Its underlying Gaussian nature yields a direct connection between convergence and exact inference. The following two theorems establish sufficient conditions under which GaBP is guaranteed to converge to the exact marginal means.

i.If the matrix A is strictly diagonally dominant (i.e , $|A_{ii}| > \sum_{j\neq i} |Aij|, \forall i$),.

then GaBP converges and the marginal means converge to the true means. This sufficient condition was recently relaxed to include a wider group of matrices.

ii.If the spectral radius (i.e. , the maximum of the absolute values of the eigen values) ρ of the matrix $|I_n - A|$ satisfies $\rho(|I_n - A|) < 1$, then GaBP converges.

Different convergence conditions for Gaussian Belief Propagation is proposed in [3].The complexity of directly computing the marginal PDF will be very high.So by passing messages from neighbouring nodes in factor graph ,BP provides an efficient way to compute the approximate marginal PDFs upon convergence.This paper deals with describing the message passing process of GaBP on pairwise factor graph as a set of updating function. The convergence conditions of beliefs for synchronous Gaussian BP,damped Gaussian BP and asynchronous Gaussian BP are derived:

i.In synchronous Gaussian BP, belief parameters($\sigma_i^2(t),\mu_i(t)$) converge to the same point for all choices of $v^a(0)\in \mathcal{A}$ and $\beta^a(0)\in \mathbb{R}^{|\varepsilon|}$ if and only if $S_1 \neq \emptyset, \rho(G^*) < 1$ and $p_{ii} + \sum_{k\in\mathcal{N}(i)} w_{ki}^* \neq 0$ where v^a , β^a are parameters,p is a column vector.

$S_1 \triangleq \{w|w \leq g(w) \text{and } w\epsilon\mathcal{W}\}$,

$\mathcal{A} \triangleq \{w \geq 0\} \cup \{w \geq w_0 | w_0 \epsilon \text{ int } (S_1)\} \cup \{w \geq w_0 | w_0 \in S_1 \text{ and } w_0 = \lim_{t\to\infty} g^{(t)}(0)\}$

ii.Damped GaBP, belief parameters ($\sigma_i^2(t),\mu_i(t)$) converge to the same point for all choices of $v^a(0)\in \mathcal{A}$ and $\beta^a(0)\in \mathbb{R}^{|\varepsilon|}$ under a nonzero damping factor d if and only if the three conditions hold:

1) $S_1 \neq \emptyset$,

2) $\max_{\lambda(G^*)} \Re\left(\lambda(G^*)\right) < 1$ or $\min_{\lambda(G^*)} \Re(\lambda(G^*) > 1$;

3) $p_{ii} + \sum_{k\in\mathcal{N}(i)} w_{ki}^* \neq 0$

iii.AsynchronousGaBP,if $S_1 \neq \emptyset, \rho(|G^*|) < 1$ and $p_{ii} + \sum_{k\in\mathcal{N}(i)} w_{ki}^* \neq 0$,belief parameters($\sigma_i^2(t),\mu_i(t)$) converge to the same point for all choices of $v^a(0)\in \mathcal{A}$and$\beta^a(0) \in \mathbb{R}^{|\varepsilon|}$ .Convergence condition for asynchronous GaBP is more stringent than that of synchronous GaBP.

In[4], investigates the behaviour of the min-sum message passing scheme to solve Laplacian matrices of graphs and to compute systems of linear equations in the electric flow.

In [5],analyzes belief propagation in network with arbitrary topologies when the nodes in the graph describes jointly Gaussian random variables .Then giving an analytical formula relating the true posterior probabilities. Sufficient condition for convergence is given and shows that when belief propagation converges it gives correct posterior means for all graph topologies.The performance of belief propagation in general networks with multiple loops is carried out. The sum-product and max-product belief propagation algorithms are appealing, fast and easily parallelizable algorithms. The results give a theoretical justification for applying belief propagation in certain networks with multiple loops. This may enable fast, approximate probabilistic inference in a range of new applications. In [6],the paper discusses about the theoretical framework for analyzing graph laplacians and operator.Analysis of graph laplacians including KNN graph.The framework reduces the problem of graph laplacian analysis to the calculation of a mean and varience for any graph construction method with positive weights and shrinking neighbourhood. It extends existing strong operator convergence results to non smooth kernels. Graphical models provide a powerful formalism for statistical signal processing. Due to their sophisticated modeling capabilities, they have found applications in a variety of fields such as computer vision, image processing, and distributed sensor networks.

In[7], a general class of algorithms for estimation in Gaussian graphical models with arbitrary structure is presented.These algorithms involve a sequence of inference problems on tractable sub graphs over subsets of variables. Analysis of algorithms based on the recently developed walk-sum interpretation of Gaussian inference. This leads to efficient methods for optimizing the next iteration step to achieve maximum reduction in error.Walk Summability approach is used for analyzing convergence rate. If every edge is updated infinitely often, then computed converges to the correct means in walk-summable models for any initial guess.Also shows that walk- summability is a sufficient condition for all algorithms to converge for a very large and flexible set of sequences of tractable subgraphs or subsets of variables on which to perform successive updates. For any non-walk-summable model,there exists at least one sequence of iterative steps that is ill-posed.

The paper [8] presents a new framework based on walks in a graph for analysis and inference in Gaussian graphical models. The key idea is to decompose correlations between variables as a sum over all walks between those variables in the graph. The weight of each walk is given by a product of edgewise partial correlations and provided a walk-sum interpretation of Gaussian belief propagation in trees and of the approximate method of loopy belief propagation in graphs with cycles. This perspective leads to a better understanding of Gaussian belief propagation and its convergence in loopy graphs.

Let $\varrho(A)$ denote the spectral radius of a symmetric matrix A, defined to be the maximum of the absolute values of the eigen values of A. The geometric series (I +A+A2+. . .) converges if and only if $\varrho(A) < 1$. If it converges, it converges to $(I - A)^{-1}$. If $\varrho(R) < 1$, then we have a geometric series for the covariance matrix: $.\sum_{l=0}^{\infty} R^l = (I - A)^{-1} = J^{-1} = P$.Let $\bar{R} = (|rij|)$

denote the matrix of element-wise absolute values.The model is walk-summable if $\varrho(\bar{R}) < 1$. Walk-summability implies $\varrho(R) < 1$ and $J \succ 0$.

In [9],investigates the expected rate convergence to consensus in an asymmetric network of weighted directed graph.The initial state of the network is represented by a random vector and the expectation is taken with respect to the random initial condition.In terms of the eigen values 1of the Laplacian matrix of the network graph, the proposed convergence rate is described.The Laplacian matrix of the network is transformed to a new matrix .The convergence analysis of the centralized algorithm is performed within a prescribed upper bound on the approximation error of algorithm.A distributed version of the centralized algorithm is then developed using of consensus observer.

III. METHODOLOGY

3.1 OVERVIEW

Gaussian belief propagation algorithm (GaBP) is one of the most important distributed algorithms in signal processing and statistical learning involving Markov networks. It is well known that the algorithm correctly computes marginal density functions from a high dimensional joint density function over a Markov network in a finite number of iterations when the underlying Gaussian graph is acyclic. The Gaussian BP algorithm (GaBP), a special version of the BP algorithm for Markov networks with Gaussian distributions (also known as Gaussian graphical model), has received special attention for the study of its convergence properties. Rate of convergence is the measure of how fast the difference between the solution point and its estimates goes to zero.Analysis of convergence rate of GaBP for markov network/Bayesian network includes following steps:

- Consider a markov/bayesian network.
- Applying GaBP to the supposed network.
- Finding convergence rate using walk summability approach.

3.2 BLOCK DIAGRAM

Figure 3.1 shows the block diagram of the system .Consider a network of markov or bayesian .This is the first step involved in the analysis of convergence rate.Next step evolved of applying GaBP equations to the supposed network and converting it to a mathematical format.Then applying the conditions for the analysis of convergence rate to the supposed one.Different conditions can be used for the analysis of convergence, this paper includes about walk summability and laplacian operator.

Fig 3.1: Block diagram of the system

3.2.1 MARKOV NETWORK

A Markov random field, Markov network or undirected graphical model is a set of random variables having a Markov property described by an undirected graph. A random field is said to be a Markov random field if it satisfies Markov properties.Markov networks are undirected and may be cyclic. The underlying graph of a Markov random field may be finite or infinite.In the domain of artificial intelligence, a Markov random field is used to model various low-to mid-level tasks in image processing and computer vision.An example of a markov network is shown in figure 3.2. Each edge represents dependency,here A depends on B and D.B depends on A,B and E.E on D and C.C depends on E.

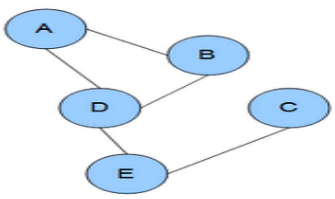

Fig 3.2.An example of a Markov random field.

3.2.2 GAUSSIAN BELIEF PROPAGATION:

Belief propagation, also known as sum-product message passing,is a message-passing algorithm for performing inference on graphical models,such as Bayesian networks and Markov random fields. It calculates the marginal distribution for each unobserved node (or variable), conditional on any observed nodes (or variables). Belief propagation is commonly used in artificial intelligence and information theory and has demonstrated empirical success in numerous applications including low-density parity-check codes, turbo codes, free energy approximation, and satisfiability.

Gaussian belief propagation is a variant of the belief propagation algorithm when the underlying distributions are Gaussian. The first work analyzing this special model was the seminal work of Weiss and Freeman.

The GaBP algorithm solves the following marginalization problem:

$$p(x_i)=\frac{1}{z}\int_{j\neq i}\exp(-1/2x^TAx+b^Tx)dx_j \qquad (1)$$

where Z is a normalization constant, A is a symmetric positive definite matrix (inverse covariance matrix or precision matrix) and b is the shift vector. Convergence of the GaBP algorithm is easier to analyze and there are two known sufficient convergence conditions. The first one was when the information matrix A is diagonally dominant. The second convergence condition was when the spectral radius of the matrix $\rho(I-|D^{-1/2}AD^{-1/2}|) < 1$,where D = diag(A). The GaBP algorithm was linked to the linear algebra domain, and it was shown that the GaBP algorithm can be viewed as an iterative algorithm for solving the linear system of equations Ax = b where A is the information matrix and b is the shift Gaussian Belief Propagation Construction Steps:

- Find all basic clusters and its intersection.
- Then all intersections of the intersection
- Organize all the regions into a region graph :a hierarchy of regions and their "direct"sub regions
- Direct subgraph s of region r is R
- To construct message connecting all regions r
- Construct belief equations

3.2.3. THE GRAPH LAPLACIAN

Consider an undirected graph G=(V,E)with $n \stackrel{\text{def}}{=} |V|$ and $m \stackrel{\text{def}}{=} |E|$.Let G is unweighted.Two basic matrices associated with G,indexed by its vertices, are its adjacency matrix A and its degree matrix D .Let d_i denote the degree if vertex i.

$$A_{i,j} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{otherwise,} \end{cases}$$

And (2)

$$D_{i,j} \stackrel{\text{def}}{=} \begin{cases} d_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The graph laplacian of G is defined to be $L \stackrel{\text{def}}{=} D-A$.

For weighted a graph G=(V,E) with edge weights given by a weight function w_G:E⟼ $\mathbb{R}_{\geq 0}$,define

$$A_{i,j} \stackrel{\text{def}}{=} \begin{cases} w_G(ij) & \text{if } ij \in E \\ 0 & \text{otherwise,} \end{cases}$$

And (3)

$$D_{i,j} \stackrel{\text{def}}{=} \begin{cases} \sum_l w_G(ij) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the case when ,in a system of equations Ax=b,A=L is the graph laplacian of an undirected graph and this system is not invertible unless b ∈Im(L).Hence we can solve the system of equations Lx=b if $\langle b, 1 \rangle$=0.Such a system is referred to as a Laplacian system of linear equations or Laplacian system.

3.2.4 Laplace Operator

A linear differential operator,which associates to the function $\emptyset(x_1, x_2, \ldots . x_n)$ Of η variables $x_1, x_2, \ldots . x_n$ the function

$$\Delta\emptyset = \frac{\partial^2\emptyset}{\partial x_1^2} + \frac{\partial^2\emptyset}{\partial x_2^2} + \ldots\ldots\ldots\ldots\ldots + \frac{\partial^2\emptyset}{\partial x_n^2} \quad (4)$$

In particular,if $\emptyset = \emptyset(x, y)$is a function of two variables,x,y,then laplace operator has the form

$$\Delta\emptyset = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} \quad (5)$$

and if $\emptyset = \emptyset(x)$ is a function of one variables,then laplacian of $\emptyset$coincides with the second derivative,

$$\Delta\emptyset = \frac{\partial^2\emptyset}{\partial x^2} \quad (6)$$

The equation $\Delta\emptyset = 0$ is usually called the Laplace equation and hence the name Laplace operator.

IV. IMPLEMENTATION AND RESULTS

This paper includes analyzing of convergence rate of Gaussian Belief Propagation for markov network. The steps for convergence rate analysis using walk summability approach involves:recognition of a markov model state diagram and transition matrix .Initialization of the matrix and providing convergence rate condition. Theoretical study of rate of convergence of markov network using laplace operator is also included in this work.

4.1 Theoretical approach of Convergence rate analysis-walk summability

Walk-sum analysis is an elegant approach for studying the convergence of GaBP. Given a matrix R=$\{r_{ij}\} \epsilon \mathbb{R}^{n \times n}$ and its induced graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$,a walk w in the graph is a node sequence.w=$(w_0, w_1, \ldots . . w_l)$,$\forall w_i \in \mathcal{V}, (w_i, w_{i+1}) \in \mathcal{E}$ and its length is l . The weight of the walk is defined to be

$$\emptyset(w) = \prod_{i=0}^{l-1} r_{w_i w_{i+1}} \quad (7)$$

Walk summability approach can be used for convergence rate analysis. Step involves :

A. Unwrapped Tree Graph

1) Find all leaves of the tree (start with the root);

2) For each leaf, find all the nodes in the loopy graph that neighbour this leaf node, except its parent node in the tree, and add all these nodes as the children to this leaf node.

The variables and weights for each node in the unwrapped tree are copied from the corresponding nodes in the loopy graph. It is clear that taking each node as root node will generate a different unwrapped tree.

For a graphical model,the message $m_{i\to j}^{(k)}$ can be expressed as

$$m_{i\to j}^{(k)} \alpha \exp[-\tfrac{1}{2} a_{i\to j}(k) x_j^2 + b_{i\to j}(k) x_j] \qquad (8)$$

This results in GaBP ,

$$a_{i\to j}(k) = -\frac{a_{ij}a_{ji}}{a_{i\to j}(k)}, b_{i\to j}(k) - \frac{a_{ji}b_{i\to j}(k)}{a_{i\to j}(k)} \qquad (9)$$

with

$$a_{i\to j}(k) = a_{ii} + \sum_{\upsilon \epsilon N_{i\setminus j}} a_{v\to i}(k-1) \qquad (10)$$

$$b_{i\to j}(k) = b_i + \sum_{\upsilon \epsilon N_{i\setminus j}} b_{v\to i}(k-1) \qquad (11)$$

The initialization is done by taking $a_{i\to j}$ (0)= a_{ii} and $b_{i\to j}(0) = b_i$ .The marginal mean is given by,

$$\mu_i(k) = \frac{b_{i+\sum_{\upsilon \epsilon N_{i\setminus j}} b_{v\to i}(k-1)}}{a_{ii}+\sum_{\upsilon \epsilon N_{i\setminus j}} a_{v\to i}(k-1)} \qquad (12)$$

Let A be an information matrix and it is invertible, then $\mu = A^{-1}b$.Using walk summability condition,

$$\mu_i(k) = \sum_{j=1}^{n} \sum_{w:j\to i|\mathcal{G}_i^{(k)}} \phi(w) b_j \qquad (13)$$

where $w: j \to i|\mathcal{G}_i^{(k)}$ denotes a walk from j to i inside the unwrapped graph $\mathcal{G}_i^{(k)}$.

$$\mu_i = \sum_{j=1}^{n} \sum_{l=0}^{\infty} \sum_{w=j\xrightarrow{l} i} \phi(w) b_j \qquad (14)$$

Combining (8) and (9),we get

$$\mu_i(k) - \mu_i = \sum_{j=1}^{n} \sum_{l=0}^{\infty} \left(\sum_{w:j\to i|\mathcal{G}_i^{(k)}} \phi(w) b_j - \sum_{w=j\xrightarrow{l} i} \phi(w) b_j \right) \qquad (15)$$

$$\mu_i(k) - \mu_i = \sum_{j=1}^{n} \sum_{l=0}^{\infty} \sum_{w=j\xrightarrow{l} i|\widetilde{W}_i^{(k)}} \phi(w) b_j \qquad (16)$$

$$\mu_i(k) - \mu_i = \sum_{j=1}^{n} \sum_{l=k+1}^{\infty} \sum_{w=j\xrightarrow{l} i|\widetilde{W}_i^{(k)}} \phi(w) b_j \qquad (17)$$

The absolute value of above equation shows that it converges to a constant value.

4.2.Practical approach of Convergence rate analysis-walk summability

The simulation result shows that as the iteration number increases ,error decreases exponentially.And also shown that the gaussian mean and theoretical mean are same. The output of walk summability approach is shown in figure 4.1.It is clear from that ,Gaussian mean is equal to theoretical mean ,x=$A^{-}b$.

```
error_matrix =

    5.5141          0          0

Gaussian Propagation mean

ans =

    0.0379
   -0.0255
   -0.0483

Therotical mean=

ans =

    0.0379
   -0.0255
   -0.0483

iteration number

it_num =

     3
```

Fig 4.1 Output of walk summability –gaussian mean equal to theoretical mean

The figure below shows that as the iteration number increases ,convergence will be exponentially decreases.The graph is plotted between iteration number in the x- axis and log error in the y- axis.Different convergence plot for three different iterations is shown in figure below.Figure 4.2,4.3,4.4 shows the convergence plot for the iterations K=3,50 and 1000.

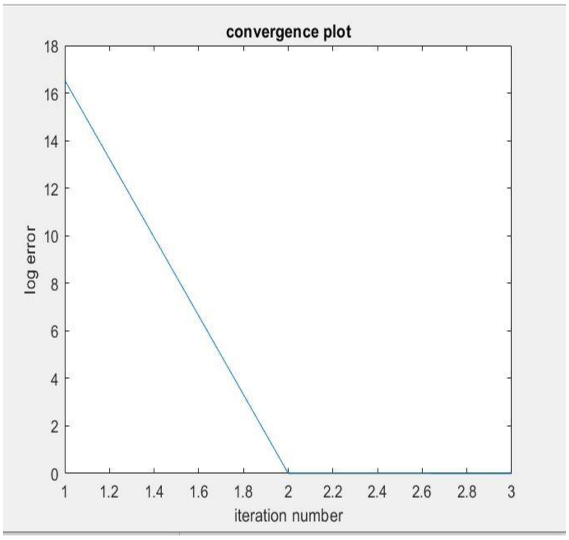

Fig 4.2convergence rate plot for iteration K=3

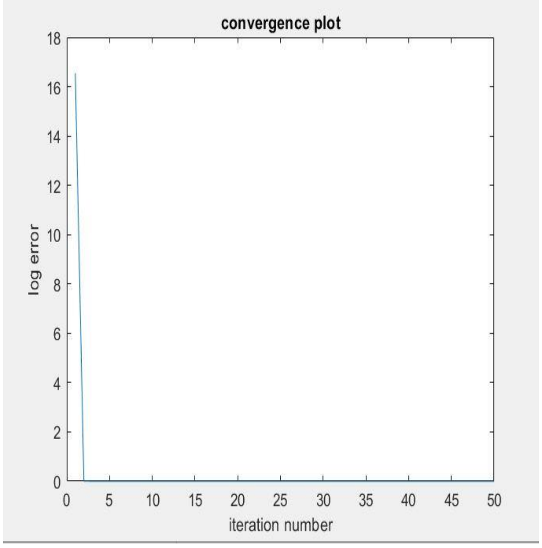

Fig 4.3 convergence plot for K=50

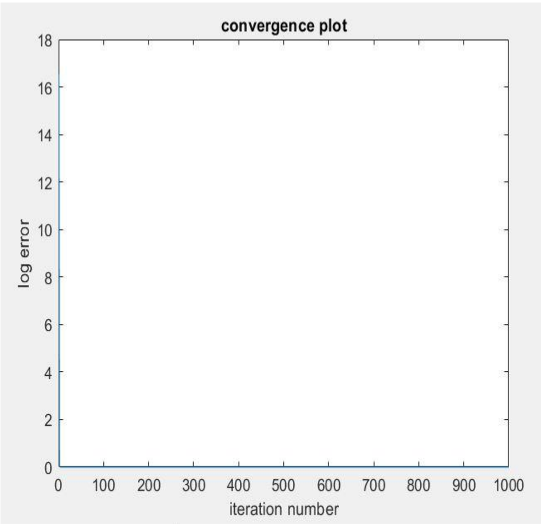

Fig 4.4 convergence plot for K =1000

4.2 Convergence rate analysis-Laplace operator

First a state diagram of a markov network is considered and its transition matrix is obtained.Adjacency matrix is calculated and initializing the matrix .Using adjacency matrix and degree matrix,laplacian matrix is calculated.Eigen values of the matrix are calculated and plotted.

As a review from certain work,it is noticed that eigen value of a laplacian matrix can be used for analysis of convergence rate .A steady state is an eigen vector of the Markov transition matrix corresponding to the eigen value1.For a markov chain which does converge to a unique steady state ,the speed of convergence is given by the size of the second eigen value :if this is one,the process has multiple steady state .If it is near one, convergence occurs very slowly .Also it may occur exponentially.

The figures below shows the adjacency and laplacian matrices ,from which convergence can be detected.And figure 4.6 shows the eigen value of the matrix.

```
adj_matrix =

     0     1     0
     0     0     1
     1     0     0


L =

     2    -1     0
     0     2    -1
    -1     0     2
```

```
P =

    0.4000    0.5000    0.1000
    0.2000    0.2000    0.4000
         0    0.6000    0.4000


eVals =

   -0.3150
    0.3650
    1.0000
```

Fig 4.5 Adjacency matrix and laplacian matrix

Fig 4.6 Eigen value

V. CONCLUSION AND FUTURE WORKS

In this paper ,it is analysed that the convergence property of GaBP for markov network and provided a simple bound on the convergence rate.It is analysed that as the iteration number increases,the error decreases exponentially using the walk summability approach.Laplacian matrix and its eigen values are plotted for determining the rate of convergence using laplacian operator. Theoretical study says that eigen value of a laplacian matrix can be used for determining the convergence speed.Future work of this project work includes the practical explanation for the rate of convergence of a markov network using laplacian operator .And it is also possible to extend the work using a comparative study of convergence rate analysis using walk summability and laplacian operator.

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