BAYES ESTIMATION OF WEIGHTED WEIBULL LENGTH-BIASED

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ABSTRACT

A sequence of independent lifetimes $X_1, X_2,\ldots, X_m, X_{m+1},\ldots, X_n$ where observed from the weighted Weibull length biased distribution but later it was found that there was a change in the system at some point of time m and it is reflected in the sequence X_m. Bayes estimators of change point m, θ_1 and θ_2 are derived under Linex and general entropy loss functions.

Key Words: Bayes Estimation, Change Point m, Linex Loss Function, General Entropy Loss Function

Introduction

The concept of length-biased distribution find various applications in biomedical area such as family history and disease, survival and intermediate events and latency period of AIDS due to blood transfusion (Gupta and Akman 1995). The study of human families and wildlife populations was the subject of an article developed by Patill and Rao (1978). Patill, et al. (1986) presented a list of the most common forms of the weight function useful in scientific and statistical literature as well as some basic theorems for weighted distributions and size-biased as special case they arrived at the conclusion that the length-biased version of some mixture of discrete distributions arises as a mixture of the length-biased version of these distributions. Gupta and Tripathi (1996) studied the weighted version of the bi-variate three-parameter logarithmic series distribution, which has applications in many fields such as: ecology, social and behavioral sciences and species abundance studies. The effects of correct and wrong prior information on the Bayes estimates was studied by Mayuri Pandya, Smita Pandya and and Paresh Andharia (2013). Mayuri Pandya and Ami Mehta (2017) studied Weighted length biased Weibull distribution with change point and derived the Bayes estimators of change point. Bayes estimation of change point m in two phase linear regression model was described by Mayuri Pandya and Paras Sheth (2016). Mayuri Pandya and Paras Sheth (2017) derived Bayes estimation of change point m and autoregressive coefficient using MHRW (Metropolis Hasting Random Walk) algorithm and Gibbs sampling.

Proposed Change Point Model

Let $X_1, X_2, X_3 \ldots, X_n$ (n ≥ 3) be a sequence of observed life time data. Let first m observations $X_1, X_2, \ldots, X_m$ have come from the Weighted Length Biased Weibull, WLBW (β, θ_1),

$$f(x_i) = \frac{\beta}{\Gamma\frac{3}{2}}\, \theta_1^{\left(\frac{3}{2}\right)}\, x_i^{\frac{3}{2}\beta - 1}\, e^{-\theta_1 x^{\beta}} \;; \quad i = 1,2,\ldots..m \qquad \ldots (1)$$

and later (n-m) observations $X_{m+1}, X_{m+2}, \ldots, X_n$ coming from the Weighted Length Biased Weibull, WLBW (β ,θ_2),

$$f(x_i) = \frac{\beta}{\Gamma\frac{3}{2}}\,\theta_2^{\left(\frac{3}{2}\right)}\; x_i^{\frac{3}{2}\beta-1}\; e^{-\theta_2 x_i^{\beta}} \quad ; \quad i = m+1, \ldots\ldots n \qquad \ldots (2)$$

Where $\beta\,, \theta_1, \theta_2 > 0$,

The likelihood function, given the sample information

$\underline{X} = (X_1, X_2, \ldots, X_m, X_{m+1}, X_{m+2}, \ldots, X_n)$ is,

$$L(\theta_1, \theta_2, \beta, m \mid \underline{X}) = \frac{\beta^n}{(\Gamma\frac{3}{2})^n}\,\theta_1^{\left(\frac{3}{2}\right)m} A_1^{\frac{3}{2}\beta-1} e^{-\theta_1 A_2}\, \theta_2^{\frac{3}{2}(n-m)} e^{-\theta_2 A_3} \qquad \ldots (3)$$

Where $A_1 = \prod_{i=1}^{n} X_i$

$A_2 = A_2(m, \beta\,/X_i) = \sum_{i=1}^{m} X_i^{\beta}$

$$A_3 = A_3(m, \beta\,/X_i) = \sum_{i=m+1}^{n} X_i^{\beta} \qquad \ldots (4)$$

POSTERIOR DENSITIES USING INFORMATIVE PRIOR

As in Broemeling et al.(1987), we suppose the marginal prior distribution of m to be discrete uniform over the set { 1, 2, …..n – 1}.

$$g\,(\mathrm{m}) = \frac{1}{n-1} \qquad \ldots (5)$$

As in Calabria and Pulcini (1992), we suppose the marginal prior distribution on β to be uniform over the interval β_1, β_2;

$$g(\beta) = \frac{1}{\beta_2-\beta_1}\ ; \quad \beta_1 \le \beta \le \beta_2 \qquad \ldots (6)$$

As in N. Sanjari Farsipour and H. Zakerzadeh (2005), under the assumption that the scale parameters θ_1 and θ_2 are unknown, we can use the Inverted Gamma prior with probability density functions with respective means values μ_1, μ_2 and common standard deviation σ *viz.*

$$g(\theta_i \mid \beta) = \frac{a_i^{\beta}}{\Gamma\beta}\,\theta_i^{-(\beta+1)}\, e^{-a_i/\theta_i} \qquad \ldots (7)$$

Where, $\beta > 0, a_i, \theta_i > 0$, i = 1,2

We assume that θ_1, θ_2, β and m are priori independent. The joint prior density is say,

$$g(\theta_1, \theta_2, \beta, m) = K_1 \frac{a_1^{\beta}}{\Gamma\beta}\,\theta_1^{-(\beta+1)}\, e^{-a_1/\theta_1}\, \frac{a_2^{\beta}}{\Gamma\beta}\,\theta_2^{-(\beta+1)}\, e^{-a_2/\theta_2} \qquad \ldots (8)$$

Where, $K_1 = \frac{1}{(\beta_2-\beta_1)(n-1)}$ … (9)

The Joint posterior density of parameters $\theta_1, \theta_2, \beta,$ and m is obtained using the likelihood function and the joint prior density of the parameters,

$$g(\theta_1, \theta_2, \beta, m \mid \underline{X}) = \frac{L(\theta_1, \theta_2, \beta, m|\underline{x})g(\theta_1, \theta_2, \beta, m)}{h(\underline{x})}$$

$$= K_1 \frac{a_1{}^{\beta}}{\Gamma\beta}\frac{a_2{}^{\beta}}{\Gamma\beta} A_1{}^{\frac{3}{2}\beta-1}\beta^n \theta_1{}^{\frac{3}{2}m-\beta-1} e^{-(\theta_1 A_2 + {a_1}/{\theta_1})}\ \theta_2{}^{\frac{3}{2}(n-m)-\beta-1} e^{-(\theta_2 A_3 + {a_2}/{\theta_2})}\ h^{-1}(\underline{X}) \quad \textbf{… (10)}$$

where, $h(\underline{X})$ is the marginal posterior density of $\underline{X}$.

The marginal density of θ_1 say $g(\theta_1|\underline{X})$ and the marginal density of θ_2 say $g(\theta_2|\underline{X})$ as,

$$g(\theta_1|\underline{X}) = K_1 \sum_{m=1}^{n-1} \int_{\beta_1}^{\beta_2} \frac{a_1{}^{\beta}}{\Gamma\beta}\frac{a_2{}^{\beta}}{\Gamma\beta}\frac{\beta^n}{(\Gamma\frac{3}{2})^n} A_1{}^{\frac{3}{2}\beta-1}\ \theta_1{}^{\frac{3}{2}m-\beta-1} e^{-\left(\theta_1 A_2 + {a_1}/{\theta_1}\right)} I_2(\mathrm{m},\beta)\, \mathrm{d}\beta\, h^{-1}(\underline{X}) \quad \textbf{… (11)}$$

$$g(\theta_2|\underline{X}) = K_1 \sum_{m=1}^{n-1} \int_{\beta_1}^{\beta_2} \frac{a_1{}^{\beta}}{\Gamma\beta}\frac{a_2{}^{\beta}}{\Gamma\beta}\frac{\beta^n}{\left(\Gamma\frac{3}{2}\right)^n} A_1{}^{\frac{3}{2}\beta-1}\ \theta_2{}^{\frac{3}{2}(n-m)-\beta-1} e^{-\left(\theta_2 A_3 + {a_2}/{\theta_2}\right)} I_1(\mathrm{m},\beta)\, \mathrm{d}\beta\, h^{-1}(\underline{X}) \quad \textbf{… (12)}$$

where,

$$h(\underline{X}) = \sum_{m=1}^{n-1} \int_{\beta_1}^{\beta_2} \frac{a_1{}^{\beta}}{\Gamma\beta}\frac{a_2{}^{\beta}}{\Gamma\beta}\frac{\beta^n}{(\Gamma\frac{3}{2})^n} A_1{}^{\frac{3}{2}\beta-1} I_1(\mathrm{m},\beta) I_2(\mathrm{m},\beta)\, \mathrm{d}\beta,$$

$$I_1(\mathrm{m},\beta) = 2A_2{}^{-\left[\frac{3}{2}m-\beta\right]/2}\left[\frac{1}{a_1}\right]^{-\left[\frac{3}{2}m-\beta\right]/2} Bessel\ K\left[\frac{3}{2}m-\beta, 2\sqrt{A_2 a_1}\right] \quad \textbf{… (13)}$$

and

$$I_2(\mathrm{m},\beta) = 2A_3{}^{\left[\beta-\frac{3}{2}(n-m)\right]/2}\left[\frac{1}{a_2}\right]^{\left[\beta-\frac{3}{2}(n-m)\right]/2} Bessel\ K\left[\frac{3}{2}(n-m)-\beta, 2\sqrt{A_3 a_2}\right] \quad \textbf{… (14)}$$

Where, $Bessel\ K[\frac{3}{2}m-\beta, 2\sqrt{A_2 a_1}]$ and $Bessel\ K[\frac{3}{2}(n-m)-\beta, 2\sqrt{A_3 a_2}]$ are defied as,

$$\int_0^\infty t^{-r} \exp\left[-\left(at + \frac{b}{t}\right)\right] dt = 2\left(\frac{a}{b}\right)^{-(r-1)/2} K_{r-1}(2\sqrt{ab}),$$

Marginal posterior density of m say, $g(m|\underline{X})$ as,

$$g(m|\underline{X}) = \frac{I_3(\mathrm{m})}{\sum_{\mathrm{m}=1}^{\mathrm{n}-1} I_3(\mathrm{m})} \quad \textbf{… (15)}$$

Where, $I_3(\mathrm{m}) = \int_{\beta_1}^{\beta_2} \frac{a_1{}^{\beta}}{\Gamma\beta}\frac{a_2{}^{\beta}}{\Gamma\beta}\frac{\beta^n}{(\Gamma\frac{3}{2})^n} A_1{}^{\frac{3}{2}\beta-1} I_1(\mathrm{m},\beta) I_2(\mathrm{m},\beta)\, \mathrm{d}\beta$ **… (16)**

Where, $I_1(\mathrm{m},\beta), I_2(\mathrm{m},\beta)$ same as in (13) and (14).

In this section, we derive Bayes estimator of change point m under asymmetric loss function prior considerations explained above. A useful asymmetric loss, known as the Linex loss function was introduced by Varian (1975). Under the assumption that the minimal loss at d, the Linex loss function can be expressed as,

$$L4\ (\alpha, d) = \exp.\ [q1\ (d-\alpha)] - q1\ (d-\alpha) - I,\ q1 \neq 0.\ \ldots$$

Minimizing expected loss function E_m [L_4 (m, d)] and using posterior density (15), we get the Bayes estimates of m , using Linex loss function as,

$$m_L^* = -\frac{1}{q_1}\ln[E(e^{-mq_1})] = -\frac{1}{q_1}\ln\left[\frac{\sum_{m=1}^{n-1} e^{-mq_1} I_3(m)}{\sum_{m=1}^{n-1} I_3(m)}\right] \quad \ldots (18)$$

Where, $I_3(m)$ is same as in (17).

Minimizing expected loss function E_m [L_4 (m, d)] and using posterior density (11), we get the Bayes estimates of θ_1, using Linex loss function as,

$$\theta_{1L}^* = -\frac{1}{q_1}\ln[K_1 \sum_{m=1}^{n-1} \int_{\beta_1}^{\beta_2} \frac{a_1{}^{\beta}}{\Gamma\beta}\frac{a_2{}^{\beta}}{\Gamma\beta}\frac{\beta^n}{(\Gamma\frac{3}{2})^n} A_1{}^{\frac{3}{2}\beta-1} \{A_2+q_1\}^{-\left[\frac{3}{2}m-\beta\right]/2}$$
$$\left[\frac{1}{a_1}\right]^{-\left[\frac{3}{2}m-\beta\right]/2} Bessel\ K\left[\frac{3}{2}m-\beta, 2\sqrt{a_1}\sqrt{A_2+q_1}\right] 2A_3{}^{\left[\beta-\frac{3}{2}(n-m)\right]/2}$$
$$\left[\frac{1}{a_2}\right]^{\left[\beta-\frac{3}{2}(n-m)\right]/2} Bessel\ K\left[\frac{3}{2}(n-m)-\beta, 2\sqrt{A_3 a_2}\right] d\beta\, h^{-1}(\underline{X}) \quad \ldots (19)$$

Minimizing expected loss function E_m [L_4 (m, d)] and using posterior density (12), we get the Bayes estimates of θ_2, using Linex loss function as,

$$\theta_{2L}^* = -\frac{1}{q_1}\ln[K_1 \sum_{m=1}^{n-1} \int_{\beta_1}^{\beta_2} \frac{a_1{}^{\beta}}{\Gamma\beta}\frac{a_2{}^{\beta}}{\Gamma\beta}\frac{\beta^n}{\left(\Gamma\frac{3}{2}\right)^n} 2\{A_3+q_1\}^{-\left[\frac{3}{2}(n-m)-\beta\right]/2}$$
$$A_1{}^{\frac{3}{2}\beta-1}\left[\frac{1}{a_2}\right]^{-\left[\frac{3}{2}(n-m)-\beta\right]/2} Bessel\ K\left[\frac{3}{2}(n-m)-\beta, 2\sqrt{a_2}\sqrt{A_3+q_1}\right]$$
$$2A_2{}^{\left[\beta-\frac{3}{2}m\right]/2}\left[\frac{1}{a_1}\right]^{\left[\beta-\frac{3}{2}m\right]/2} Bessel\ K\left[\frac{3}{2}m-\beta, 2\sqrt{A_2 a_1}\right] d\beta\, h^{-1}(\underline{X})] \quad \ldots (20)$$

Numerical Study:

We have generated 20 random observations from proposed Weighted Length Biased Weibull change point model given in Proposed Change Point Model. The first eight observations are from WLBW with β =1.5 and $\theta_1 = 0.005$ and next twelve are from WLBW with $\beta = 1.5$ and $\theta_2 = 0.002$. θ_1 and θ_2 they were random observations from inverted gamma distributions with prior means μ_1 = 0.005, μ_2 = 0.002 and variance $\sigma_1{}^2 = 0.00006$, $\sigma_2{}^2 = 0.00002$ resulting in $a_1 = 0.0071$ and $a_2 = 0.0024$ these observations are given in table-1.

Table – 1: Generated observations from proposed model

i	1	2	3	4	5	6	7	8	9	10
X_i	0.022	0.971	0.116	0.550	0.848	0.667	0.200	0.306	0.052	0.158
i	11	12	13	14	15	16	17	18	19	20
X_i	0.291	0.227	0.466	0.398	0.709	0.992	0.002	0.105	0.879	0.963

We have calculated posterior mean of m, θ_1, θ_2, β and the posterior median and posterior mode of m. The results are shown in table-2.

Table – 2: The values of Bayes estimates of change point m and θ_1, θ_2

Prior Density	Bayes estimates of change point m			Bayes estimates of Posterior means of parameters θ_1 and θ_2	
	Posterior Median	Posterior Mean	Posterior mode	θ_1^*	θ_2^*
Inverted Gamma prior	8	8.23	8	0.005	0.002

We also compute the Bayes estimates m_L^* , m_E^* of m, θ_{1L}^* , θ_{1E}^* of θ_1, θ_{2L}^* , θ_{2E}^* of θ_2, β_L^*, β_E^* of β, Using the data given in table 1 and for different values of shape parameter q_1 and q_3 ,the results are shown in Tables 3 and 4.

TABLE 3: The Bayes Estimates Under Linex Loss Function

Prior Density	q_1	m_L^*	θ_{1L}^*	θ_{2L}^*
Inverted Gamma prior	0.09	8	0.005	0.0023
	0.10	8	0.005	0.0022
	0.20	8	0.005	0.0021
	1.2	7	0.003	0.0018
	1.5	6	0.002	0.0014
	-1.0	9	0.009	0.0027
	-2.0	10	0.010	0.0029

TABLE 4: The Bayes Estimates under General Entropy Loss Function

Prior Density	q_3	m_E^*	θ_{1E}^*	θ_{2E}^*
Inverted Gamma prior	0.09	8	0.005	0.0023
	0.10	8	0.005	0.0021
	0.20	8	0.005	0.0020
	1.2	6	0.003	0.0017
	1.5	5	0.002	0.0015
	-1.0	9	0.009	0.0025
	-2.0	10	0.010	0.0028

Conclusion:

It can be seen from the above Table-3 and Table-4 if we select negative value of shape parameter of Linex loss function and General entropy loss function we found over estimation, i.e. we can stop under estimation. And if we select positive more than 1 value of shape parameter of Linex loss function and General entropy loss function we found under estimation, i.e. we can stop over estimation by selecting more than 1 value of shape parameter of asymmetric loss function.

References:

- **Gupta, RC. and Akman, H.O. (1995)** "On the reliability studies of a weighted inverse Gaussian moder', JOllrnal of Statistical Planning and Inference 48,69-83.
- **Patill, G. P. and Rao, C. R. (1978)**. "Weighted distribution and size-biased sampling with application to wildlife populations and human families", Biometrics, 34, 179-189.
- **Gupta, R C. and Tripathi, RC. (1996).** "Weighted bivariate logarithmic series distributions", Commlln. Statist. - Theory Meth., 25(5), 1099- 1117.
- **Mayuri Pandya, Smita Pandya and Paresh Andharia (2013).** "Bayes Estimation Of Weibull Length Biased Distribution", Asian Journal of Current Engineering and Maths 2, 44 - 49.
- **M. Pandya & Paras Sheth (2016)**, "Study on Two Phase Linear Regression Model: Bayesian Approach", IJEMR, Volumn-6, Issue-2, page-number: 528-534.
- **Mayuri Pandya and Paras Sheth (2017).** "Auto Regressive Process (1) with Change Point: Bayesian Approch", International Journal of Research in Engineering and Science, Volume 5 Issue 1, PP. 27-31.
- **Mayuri Pandya and Ami Mehta (2017).** "Some Contribution of Bayesian Inference in Weighted Length Biased Weibull Distribution", International Journal for Scientific Research & Development, Vol. 5, Issue 01, 269-273.