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# The Existence Of Non-Axial Libration Points In The Kite Configuration.

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**Abstract: This paper deals with the existence of the libration points lying on the plane of motion of the four bodies forming kite but not lying on the axis of symmetry of the kite. Here the work of Hassan, M. R. (2023) and Khatun M.et al (2024) have been followed to establish the equation of motion of the infinitesimal mass moving in the gravitational field of the kite of the first kind and to discuss the existence of Libration points by using Python.**

**Keywords: Kite configuration, Cyclic kite configuration, Mass parameter, Mean motion, Rotating frame, Libration point.**

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**1.Definition:** Before to introduce the exact problem; let us define first the kite configuration with classification. Kite configuration means a quadrilateral with two pairs of equal adjacent sides and cyclic kite configuration is a kite whose vertices lie on the circumference of a common circle. From different articles of previous authors, it is to be noted that a quadrilateral is a kite if and only if their diagonals are perpendicular to each other and at least one of the two diagonals is a line of symmetry. In some cases, both the diagonals may be the lines of symmetry. Thus, the cyclic kite configuration can be classified into two classes.

(i) The kite formed by the combination of one equilateral and one isosceles triangle. (first kind)
(ii) The kite formed by the combination of two congruent isosceles right-angled triangles. (2nd kind)

**2.Introduction:** The specific examples of three-body, four-body, and five-body (configuration or Problem), were discussed by the researchers of celestial mechanics. Mac Millon et al. (1932) provided detailed proof of two theorems for the existence of quadrilateral configuration in the field of four body configuration. Brumberg (1957) invented permanent solution of four-body configuration. For the first time Albouy (1996) examined the symmetric central configuration of equal masses. According to Long et al (2002) a convex non-collinear planar four body central configuration with three equal masses must be a kite. Additionally, he examined the four-body central configuration using two equal-mass couples. According to Chavela et al. (2007), a convex four-body central configuration must be a kite-shaped quadrilateral if two equal masses are situated at opposite vertices of the quadrilateral and at most the mass of one of the remaining particles is more than the equal masses. Furthermore, it was demonstrated that the planar four-body configuration is a convex central configuration and is symmetric about its one diagonal if and only if the masses of two particles at the end of the other diagonal are equal. Albouy et al. (2008) also discussed some properties of a quadrilateral configuration and established some relations among the masses of the system. To construct a general four body problem Pina et al (2009) developed a new algorithm with the ratios of directed areas and the corresponding scalar areas and in (2010) they developed coordinates of four body particles of kite configuration in terms of principle moments of inertia and Eulerian angles independently. For the first time, Cors et al. (2012) examined the cyclic central configuration in the Newtonian four body problem. They used the six mutual distances of the particles as their coordinates and demonstrated that the four point masses constituting a kite lying on a two-dimensional plane. They further extended the kite configuration isosceles trapezoidal configuration. They have also demonstrated that a line of symmetry must exist in any central configuration that has two equal masses. By describing the masses of the central configuration in terms of angle coordinates. Balint Erdi et al. (2016) extended the work of Cors et al. (2012) in three cases (two concave cases and one convex case). They further asserted that the exact analytical solutions of the four-body configuration are represented by the obtained formulas. Deng et al. (2017) demonstrated that the diagonals of a cyclic central quadrilateral cannot be perpendicular unless the configuration is a kite by using mutual distances as the coordinates. Further they verified the same theorem in the four vertex convex central configuration Hassan(2023) studied cyclic kite configuration by three theorems and he expressed the masses of the four particles in terms of a mass parameter $\mu$ and the total mass M of the system as $m_1 = M(1-\mu)/2, m_2 = M\mu, m_3 = M(1-3\mu)/2, m_4 = M\mu$. Moreover, he established the coordinates of the four masses and he in terms of radius R of the common circular orbit $(R,0)$, $(-R/2, \sqrt{3}R/2), (-R,0), (-R/2, -\sqrt{3}R/2)$ used his findings to support all of the earlier findings. In addition, he calculated the mean motion of the rotating frame lying on the kite's plane. Manuara(2024) et al extended the work of Hassan (2023) by taking the first body of mass $m_1$ in an oblate spheroid and she showed that how the mean motion of the system is affected by the oblateness of the first body. It is found that the mean motion increases with the increase of oblateness. Further she showed that the axial libration points move away from the origin due to increase of oblateness parameter A. As the analytical existence of axial libration points was not possible hence presently, we proposed to investigate the existence of non-axial libration points by numerical method. Particularly by using Python we have shown the existence of non-axial libration points of the kite configuration of the first kind by drawing contour plots for different values of $\mu$ $(0 < \mu < 3^{-1})$.

## 3.The equations of motion of four bodies forming kite in rotating frame about z-axis:

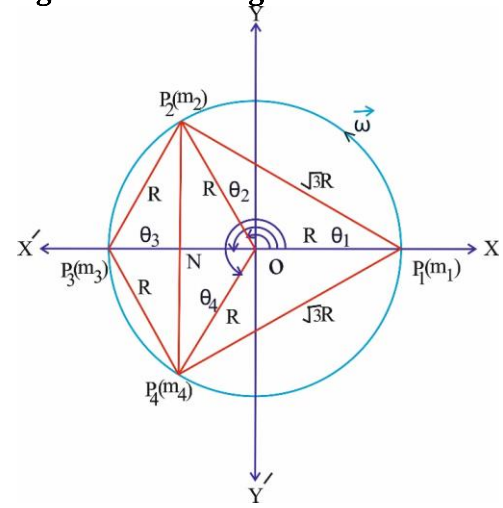


**Figure-1**
**Kite configuration of the first kind having one axis of symmetry $P_1OP_3$.**

Before to derive the equations of motion of four bodies forming cyclic kite configuration in synodic frame, let us derive first the gravitational force between two bodies (spherical both).

Let $\vec{r}_i = \overrightarrow{OP_i} (i = 1,2,3,4)$ be the position vector of the i-th particle and $V_{ij}$ *be* the potential between two spherical bodies of mass $m_i$ *and* $m_j (i \neq j)$ separated by a distance $r_{ij}$ is given by (Mccuskey-1967.pp166)

$$V_{ij} = -\frac{Gm_i m_j}{r_{ij}} (i \neq j = 1,2,3,4) \quad (1)$$

$$\text{where } \vec{r}_{ij} = \overrightarrow{P_iP_j} = \overrightarrow{OP_j} - \overrightarrow{OP_i} = \vec{r}_j - \vec{r}_i \quad (2)$$

Let the plane of motion of the four point masses (all spherical) forming kite configuration rotate with the angular velocity $\vec{\omega} = n\hat{k}$ about the z-axis, $\hat{r}_{ij}$ be the unit vector along the line $P_iP_j$ joining the i-th & j-th point masses, then the equation of motion of the i-th point mass relative to the other three point masses is given by

$$-m_i n^2 \vec{r}_i = \sum_{j=1}^{4} \frac{\partial V_{ij}}{\partial r_{ij}} \hat{r}_{ij} \qquad (i \neq j) \quad (3)$$

$$\Rightarrow m_i n^2 \vec{r}_i + \frac{\partial V_{i1}}{\partial r_{i1}} \hat{r}_{i1} + \frac{\partial V_{i2}}{\partial r_{i2}} \hat{r}_{i2} + \frac{\partial V_{i3}}{\partial r_{i3}} \hat{r}_{i3} + \frac{\partial V_{i4}}{\partial r_{i4}} \hat{r}_{i4} \quad (4)$$

Putting i=1,2,3,4 in the equation (4) we get the equations of motion of four masses $m_1, m_2, m_3, m_4$ relative to the other three masses as

$$\left.\begin{aligned}
& m_1 n^2 \vec{r}_1 + \frac{\partial V_{12}}{\partial r_{12}} \hat{r}_{12} + \frac{\partial V_{13}}{\partial r_{13}} \hat{r}_{13} + \frac{\partial V_{14}}{\partial r_{14}} \hat{r}_{14} = \hat{0} \\
& m_2 n^2 \vec{r}_2 + \frac{\partial V_{21}}{\partial r_{21}} \hat{r}_{21} + \frac{\partial V_{23}}{\partial r_{23}} \hat{r}_{23} + \frac{\partial V_{24}}{\partial r_{24}} \hat{r}_{24} = \hat{0} \\
& m_3 n^2 \vec{r}_3 + \frac{\partial V_{31}}{\partial r_{31}} \hat{r}_{31} + \frac{\partial V_{32}}{\partial r_{32}} \hat{r}_{32} + \frac{\partial V_{34}}{\partial r_{34}} \hat{r}_{34} = \hat{0} \\
& \& \ m_4 n^2 \vec{r}_4 + \frac{\partial V_{41}}{\partial r_{41}} \hat{r}_{41} + \frac{\partial V_{42}}{\partial r_{42}} \hat{r}_{42} + \frac{\partial V_{43}}{\partial r_{43}} \hat{r}_{43} = \hat{0}
\end{aligned}\right\} \quad (5)$$

Introducing equations (1) in equation (4) one can find

$$\left.\begin{aligned}
&n^2\vec{r}_1+G\left(\frac{m_2}{r_{12}^2}\right)\frac{\vec{r}_{12}}{r_{12}}+G\left(\frac{m_3}{r_{13}^2}\right)\frac{\vec{r}_{13}}{r_{13}}+G\left(\frac{m_4}{r_{14}^2}\right)\frac{\vec{r}_{14}}{r_{14}}=\hat{0}\\
&n^2\vec{r}_2+G\left(\frac{m_1}{r_{21}^2}\right)\frac{\vec{r}_{21}}{r_{21}}+G\left(\frac{m_3}{r_{23}^2}\right)\frac{\vec{r}_{23}}{r_{23}}+G\left(\frac{m_4}{r_{24}^2}\right)\frac{\vec{r}_{24}}{r_{24}}=\hat{0}\\
&n^2\vec{r}_3+G\left(\frac{m_1}{r_{31}^2}\right)\frac{\vec{r}_{31}}{r_{31}}+G\left(\frac{m_2}{r_{32}^2}\right)\frac{\vec{r}_{32}}{r_{32}}+G\left(\frac{m_4}{r_{34}^2}\right)\frac{\vec{r}_{34}}{r_{34}}=\hat{0}\\
&n^2\vec{r}_4+G\left(\frac{m_1}{r_{41}^2}\right)\frac{\vec{r}_{41}}{r_{41}}+G\left(\frac{m_2}{r_{42}^2}\right)\frac{\vec{r}_{42}}{r_{42}}+G\left(\frac{m_3}{r_{43}^2}\right)\frac{\vec{r}_{43}}{r_{43}}=\hat{0}
\end{aligned}\right\} \quad (6)$$

Putting the values of $m_1, m_2, m_3, m_4$ in (6) we get

$$n^2\vec{r}_1+G\left[\left(\frac{M\mu}{3\sqrt{3}R^3}\right)\vec{r}_{12}+\left\{\frac{M(1-3\mu)}{16R^3}\right\}\vec{r}_{13}+\left(\frac{M\mu}{3\sqrt{3}R^3}\right)\vec{r}_{14}\right]=\hat{0}$$

$$n^2\vec{r}_2+G\left[\left\{\frac{M(1-\mu)}{6\sqrt{3}R^3}\right\}\vec{r}_{21}+\left\{\frac{M(1-3\mu)}{2R^3}\right\}\vec{r}_{23}+\left(\frac{M\mu}{3\sqrt{3}R^3}\right)\vec{r}_{24}\right]=\hat{0}$$

$$n^2\vec{r}_3+G\left[\left\{\frac{M(1-\mu)}{16R^3}\right\}\vec{r}_{31}+\left(\frac{M\mu}{R^3}\right)\vec{r}_{32}+\left(\frac{M\mu}{R^3}\right)\vec{r}_{34}\right]=\hat{0}$$

$$n^2\vec{r}_4+G\left[\left\{\frac{M(1-\mu)}{6\sqrt{3}R^3}\right\}\vec{r}_{41}+\left(\frac{M\mu}{3\sqrt{3}R^3}\right)\vec{r}_{42}+\left\{\frac{M(1-3\mu)}{2R^3}\right\}\vec{r}_{43}\right]=\hat{0}$$

Now choosing units of masses lengths and forces in such a way that $m_1+m_2+m_3+m_4=M=1, 2R=1, G=1$, then the four masses are reduced to

$$m_1=\frac{1-\mu}{2}, m_2=\mu, m_3=\frac{1-3\mu}{2}, m_4=\mu$$

and above four equations are reduced to

$$\left.\begin{aligned}
&n^2\vec{r}_1+\left[\left(\frac{8\mu}{3\sqrt{3}}\right)\vec{r}_{12}+\left\{\frac{(1-3\mu)}{2}\right\}\vec{r}_{13}+\left(\frac{8\mu}{3\sqrt{3}}\right)\vec{r}_{14}\right]=\hat{0}\\
&n^2\vec{r}_2+\left[\left\{\frac{4(1-\mu)}{3\sqrt{3}}\right\}\vec{r}_{21}+\{4(1-3\mu)\}\vec{r}_{23}+\left(\frac{8\mu}{3\sqrt{3}}\right)\vec{r}_{24}\right]=\hat{0}\\
&n^2\vec{r}_3+\left[\left\{\frac{(1-\mu)}{2}\right\}\vec{r}_{31}+(8\mu)\vec{r}_{32}+(8\mu)\vec{r}_{34}\right]=\hat{0}\\
&n^2\vec{r}_4+\left[\left\{\frac{4(1-\mu)}{3\sqrt{3}}\right\}\vec{r}_{41}+\left(\frac{8\mu}{3\sqrt{3}}\right)\vec{r}_{42}+\{4(1-3\mu)\}\vec{r}_{43}\right]=\hat{0}
\end{aligned}\right\} \quad (7)$$

Introduction of (2) in (7) yields

$$\left.\begin{aligned}
&n^2\vec{r}_1+\left[\left(\frac{8\mu}{3\sqrt{3}}\right)(\vec{r}_2-\vec{r}_1)+\left\{\frac{(1-3\mu)}{2}\right\}(\vec{r}_3-\vec{r}_1)+\left(\frac{8\mu}{3\sqrt{3}}\right)(\vec{r}_4-\vec{r}_1)\right]=\hat{0}\\
&n^2\vec{r}_2+\left[\left\{\frac{4(1-\mu)}{3\sqrt{3}}\right\}(\vec{r}_1-\vec{r}_2)+\{4(1-3\mu)\}(\vec{r}_3-\vec{r}_2)+\left(\frac{8\mu}{3\sqrt{3}}\right)(\vec{r}_4-\vec{r}_2)\right]=\hat{0}\\
&n^2\vec{r}_3+\left[\left\{\frac{(1-\mu)}{2}\right\}(\vec{r}_1-\vec{r}_3)+(8\mu)(\vec{r}_2-\vec{r}_3)+(8\mu)(\vec{r}_4-\vec{r}_3)\right]=\hat{0}\\
&n^2\vec{r}_4+\left[\left\{\frac{4(1-\mu)}{3\sqrt{3}}\right\}(\vec{r}_1-\vec{r}_4)+\left(\frac{8\mu}{3\sqrt{3}}\right)(\vec{r}_2-\vec{r}_4)+\{4(1-3\mu)\}(\vec{r}_3-\vec{r}_4)\right]=\hat{0}
\end{aligned}\right\} \quad (8)$$

Now by arranging the above four equations in the form of linear combinations of position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$, the equations of motion of four bodies forming kite are respectively

$$\left.\begin{aligned}
&\left[n^2-\frac{16\mu}{3\sqrt{3}}-\frac{1-3\mu}{2}\right]\vec{r}_1+\frac{8\mu}{3\sqrt{3}}\vec{r}_2+\frac{1-3\mu}{2}\vec{r}_3+\frac{8\mu}{3\sqrt{3}}\vec{r}_4=\hat{0}\\
&\frac{4(1-\mu)}{3\sqrt{3}}\vec{r}_1+\left[n^2-\frac{4(1-\mu)}{3\sqrt{3}}-4(1-3\mu)-\frac{8\mu}{3\sqrt{3}}\right]\vec{r}_2+4(1-3\mu)\vec{r}_3+\frac{8\mu}{3\sqrt{3}}\vec{r}_4=\hat{0}\\
&\frac{1-\mu}{2}\vec{r}_1+8\mu\vec{r}_2+\left[n^2-\frac{1-\mu}{2}-16\mu\right]\vec{r}_3+8\mu\vec{r}_4=\hat{0}\\
&\text{and } \frac{4(1-\mu)}{3\sqrt{3}}\vec{r}_1+\frac{8\mu}{3\sqrt{3}}\vec{r}_2+4(1-3\mu)\vec{r}_3+\left[n^2-\frac{4(1-\mu)}{3\sqrt{3}}-\frac{8\mu}{3\sqrt{3}}-4(1-3\mu)\right]\vec{r}_4=\hat{0}
\end{aligned}\right\} \quad (9)$$

The system of equations (9) represents the equations of motion of four particles forming kite in rotating frame.

## 4.Mean motion of the rotating frame containing kite configuration of the first kind:

To find the mean motion 'n' of the rotating frame we have to express n in terms of $\mu$. In the above four equations of (9) there are two parameters $n, \mu$ other than four position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$.

Thus, by eliminating $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$ from the above four linear combination of equations (9) we get

$$\begin{vmatrix} a & \frac{8\mu}{3\sqrt{3}} & \frac{1-3\mu}{2} & \frac{8\mu}{3\sqrt{3}} \\ \frac{4(1-\mu)}{3\sqrt{3}} & b & 4(1-3\mu) & \frac{8\mu}{3\sqrt{3}} \\ \frac{1-\mu}{2} & 8\mu & c & 8\mu \\ \frac{4(1-\mu)}{3\sqrt{3}} & \frac{8\mu}{3\sqrt{3}} & 4(1-3\mu) & d \end{vmatrix} = 0 \qquad (10)$$

where $a = n^2 - \frac{16\mu}{3\sqrt{3}} - \frac{1-3\mu}{2}$ ,

$b = n^2 - \frac{4(1-\mu)}{3\sqrt{3}} - 4(1-3\mu) - \frac{8\mu}{3\sqrt{3}}$, $c = n^2 - \frac{1-\mu}{2} - 16\mu$

& $d = n^2 - \frac{4(1-\mu)}{3\sqrt{3}} - \frac{8\mu}{3\sqrt{3}} - 4(1-3\mu)$

The equation (10) gives the mean motion 'n' of the rotating (synodic) frame as a function of $\mu$ . *i.e.* $n = f(\mu)$

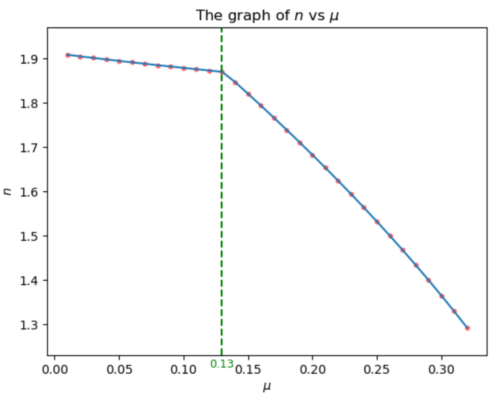


**Figure-2**

From the above graph of $\mu$ *versus* $n$, n decreases very showily for $0 < \mu \le 0.13$ and n decreases rapidly for $0.13 < \mu \le 0.32$.

## 5.The equations of motion of the satellite moving in the gravitational field of the kite:

To discuss the existence of libration points of the kite configuration of the first kind we need those points in the gravitational field of the kite at which all activities of the infinitesimal body must be stopped.

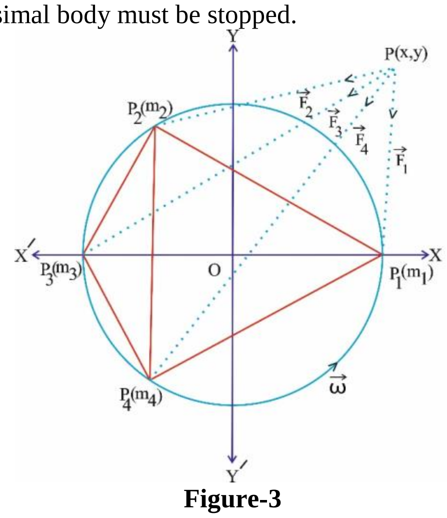


**Figure-3**

Position of the satellite at P(x,y) in the gravitational plane of the kite.

Let at any time t, $P(x, y)$ be the position of the infinitesimal body of mass m moving in the gravitational field of the four-point masses at $P_k(x_k, y_k), k = 1,2,3,4;$ be the positions of four bodies of the kite configuration, then

$$\left.\begin{aligned} &\overrightarrow{OP} = \vec{\rho}, \overrightarrow{P_kP} = \vec{\rho}_k \\ &\therefore \rho = x\hat{i} + y\hat{j}, \vec{\rho}_k = (x - x_k)\hat{i} + (y - y_k)\hat{j} \end{aligned}\right\} \qquad (11)$$

$$\left.\begin{aligned} &x_1 = \frac{1}{2}, y_1 = 0; x_2 = -\frac{1}{4}, y_2 = \frac{\sqrt{3}}{4}; x_3 = -\frac{1}{2}, \\ &y_3 = 0; x_4 = -\frac{1}{4}, y_4 = -\frac{\sqrt{3}}{4} \end{aligned}\right\} \qquad (12)$$

[ Hassan (2023)]

The forces attraction on the satellite of mass m due to the point masses $m_k$ are given by

$$F_k = \frac{Gmm_k}{\rho_k^2} \Rightarrow \vec{F}_k = -\frac{Gmm_k}{\rho_k^2}\hat{\rho}_k = -\frac{Gmm_k\vec{\rho}_k}{\rho_k^3},$$

Where $\hat{\rho}_k$ is the unit vector along $\overrightarrow{P_KP}$ .

Therefore, the total force of attraction on the satellite due to four-point masses at the vertices of a kite configuration of the first kind is given by

$$\begin{aligned} \vec{F} &= \sum_{k=1}^{4} \vec{F}_K = -\sum_{k=1}^{4} \frac{Gmm_k}{\rho_k^3}\vec{\rho}_k \\ &= -Gm\left[\frac{m_1\vec{\rho}_1}{\rho_1^3} + \frac{m_2\vec{\rho}_2}{\rho_2^3} + \frac{m_3\vec{\rho}_3}{\rho_3^3} + \frac{m_4\vec{\rho}_4}{\rho_4^3}\right] \\ &= -Gm\left[\begin{aligned} &\frac{(1-\mu)\vec{\rho}_1}{2\rho_1^3} + \frac{\mu\vec{\rho}_2}{\rho_2^3} \\ &+ \frac{(1-3\mu)\vec{\rho}_3}{2\rho_3^3} + \frac{\mu\vec{\rho}_4}{\rho_4^3} \end{aligned}\right] \end{aligned} \qquad (13)$$

The equation of motion of the satellite of mass 'm' at $P(\vec{\rho})$ in the gravitational field of the four-point masses $\frac{1-\mu}{2}, \mu, \frac{1-3\mu}{2}, \mu$ is given by

$$m\left[\frac{\partial^2\vec{\rho}}{\partial t^2} + 2\vec{\omega}\times\frac{\partial\vec{\rho}}{\partial t} + \frac{\partial\vec{\omega}}{\partial t}\times\vec{\rho} + \vec{\omega}\times(\vec{\omega}\times\vec{\rho})\right] =$$

$$-Gm\left[\frac{(1-\mu)\vec{\rho}_1}{2\rho_1^3} + \frac{\mu\vec{\rho}_2}{\rho_2^3} + \frac{(1-3\mu)\vec{\rho}_3}{2\rho_3^3} + \frac{\mu\vec{\rho}_4}{\rho_4^3}\right]$$

$$\Rightarrow \left[\frac{\partial^2\vec{\rho}}{\partial t^2} + 2\vec{\omega}\times\frac{\partial\vec{\rho}}{\partial t} + \frac{\partial\vec{\omega}}{\partial t}\times\vec{\rho} + \vec{\omega}\times(\vec{\omega}\times\vec{\rho})\right] =$$

$$-G\left[\frac{(1-\mu)\vec{\rho}_1}{2\rho_1^3} + \frac{\mu\vec{\rho}_2}{\rho_2^3} + \frac{(1-3\mu)\vec{\rho}_3}{2\rho_3^3} + \frac{\mu\vec{\rho}_4}{\rho_4^3}\right] \qquad (14)$$

Using (11) & (12) in (14) we get

$$\ddot{x}\hat{i} + \ddot{y}\hat{j} + 2(-n\dot{y}\hat{i} + n\dot{x}\hat{j}) + \hat{0} - n^2x\hat{i} - n^2y\hat{j} =$$

$$-G\left[\begin{aligned} &\frac{(1-\mu)}{2\rho_1^3}\left\{\left(x - \frac{1}{2}\right)\hat{i} + y\hat{j}\right\} + \frac{\mu}{\rho_2^3}\left\{\left(x + \frac{1}{4}\right)\hat{i} + \left(y - \frac{\sqrt{3}}{4}\right)\hat{j}\right\} \\ &+ \frac{(1-3\mu)}{2\rho_3^3}\left\{\left(x + \frac{1}{2}\right)\hat{i} + y\hat{j}\right\} + \frac{\mu}{\rho_4^3}\left\{\left(x + \frac{1}{4}\right)\hat{i} + \left(y + \frac{\sqrt{3}}{4}\right)\hat{j}\right\} \end{aligned}\right]$$

$$\Rightarrow\left(\ddot{x}-2n\dot{y}-n^2x\right)\hat{i}+\left(\ddot{y}+2n\dot{x}-n^2y\right)\hat{j}$$

$$=-G\left[\left\{\begin{array}{l}\left(\frac{1-\mu}{2\rho_1^3}\right)\left(x-\frac{1}{2}\right)+\frac{\mu}{\rho_2^3}\left(x+\frac{1}{4}\right)+\left(\frac{1-3\mu}{2\rho_3^3}\right)\\ \left(x+\frac{1}{2}\right)+\frac{\mu}{\rho_4^3}\left(x+\frac{1}{4}\right)\end{array}\right\}\hat{i}+\left\{\begin{array}{l}\left(\frac{1-\mu}{2\rho_1^3}\right)y+\frac{\mu}{\rho_2^3}\left(y-\frac{\sqrt{3}}{4}\right)+\left(\frac{1-3\mu}{2\rho_3^3}\right)y\\ +\frac{\mu}{\rho_4^3}\left(y+\frac{\sqrt{3}}{4}\right)\end{array}\right\}\hat{j}\right] \quad (15)$$

Choosing unit of force, so that G=1 & taking scalar products of $\hat{i}$ & $\hat{j}$ with (15) we get

$$\left.\begin{array}{l}\ddot{x}-2n\dot{y}=n^2x-\left(\frac{1-\mu}{2\rho_1^3}\right)\left(x-\frac{1}{2}\right)-\frac{\mu}{\rho_2^3}\left(x+\frac{1}{4}\right)-\left(\frac{1-3\mu}{2\rho_3^3}\right)\\ \left(x+\frac{1}{2}\right)-\frac{\mu}{\rho_4^3}\left(x+\frac{1}{4}\right)=\frac{\partial\Omega}{\partial x}=\Omega_x\\ \ddot{y}+2n\dot{x}=n^2y-\left(\frac{1-\mu}{2\rho_1^3}\right)y-\frac{\mu}{\rho_2^3}\left(y-\frac{\sqrt{3}}{4}\right)-\left(\frac{1-3\mu}{2\rho_3^3}\right)y\\ -\frac{\mu}{\rho_4^3}\left(y+\frac{\sqrt{3}}{4}\right)=\frac{\partial\Omega}{\partial y}=\Omega_y\end{array}\right\} \quad (16)$$

The equations of (16) represent the equations of motion of the infinitesimal mass in the gravitational field of a kite configuration of the first kind.

The kinetic potential $\Omega$ can be defined as

$$\Omega(x,y)=\frac{1}{2}n^2\left(x^2+y^2\right)+\frac{(1-\mu)}{2\rho_1}+\frac{\mu}{\rho_2}+\frac{(1-3\mu)}{2\rho_3}+\frac{\mu}{\rho_4}=\frac{C}{2}$$

(17)

where C is the Jacobian constant.

If the satellite stops anywhere at $P(x,y)$ then $2\Omega(x,y)=C$.

Thus, from the equation (17) we have

$$n^2\left(x^2+y^2\right)+\frac{(1-\mu)}{\rho_1}+\frac{2\mu}{\rho_2}+\frac{(1-3\mu)}{\rho_3}+\frac{2\mu}{\rho_4}=C \quad (18)$$

The equation (18) represents the surface of zero velocity curves of the satellite. The motion of the satellite can occur in that region only for which $2\Omega(x,y)>C$ & $2\Omega(x,y)=C$ gives the curves of zero velocity of the satellite.

**6. Existence of libration points in the first kind kite configuration:**

For libration points we have $\dot{x}=\ddot{x}=0, \dot{y}=\ddot{y}=0$ then from the equation of (16) we have,

$$\frac{\partial\Omega}{\partial x}=\frac{\partial\Omega}{\partial y}=0$$

$$\left.\begin{array}{l}\Rightarrow n^2x-\frac{1-\mu}{2\rho_1^3}\left(x-\frac{1}{2}\right)-\frac{\mu}{\rho_2^3}\left(x+\frac{1}{4}\right)-\frac{1-3\mu}{2\rho_3^3}\\ \left(x+\frac{1}{2}\right)-\frac{\mu}{\rho_4^3}\left(x+\frac{1}{4}\right)=0\\ \& \; n^2y-\frac{1-\mu}{2\rho_1^3}y-\frac{\mu}{\rho_2^3}\left(y-\frac{\sqrt{3}}{4}\right)-\frac{1-3\mu}{2\rho_3^3}y\\ -\frac{\mu}{\rho_4^3}\left(y+\frac{\sqrt{3}}{4}\right)=0\end{array}\right\} \quad (19)$$

The points of intersection of the equations of (19) give the position of libration points of the kite.

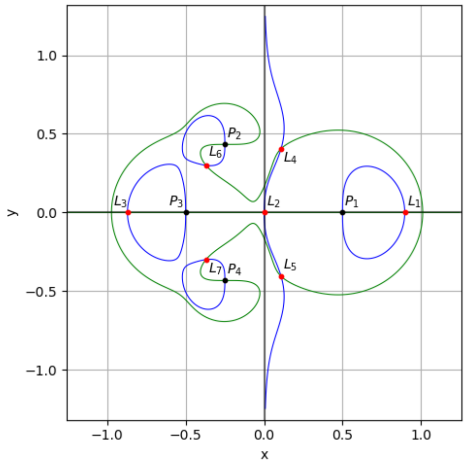


**Figure-4**

The positions of seven libration points indicated through the intersection of $\Omega_x=0$ &$\Omega_y=0$ for $\mu=0.10$ &$n=1.879308$ .Here black dots ( $P_i, i=1,2,3,4$ ) represent four bodies forming kite and four pink spots $L_j(j=4,5,...,7)$ represent non axial libration points.

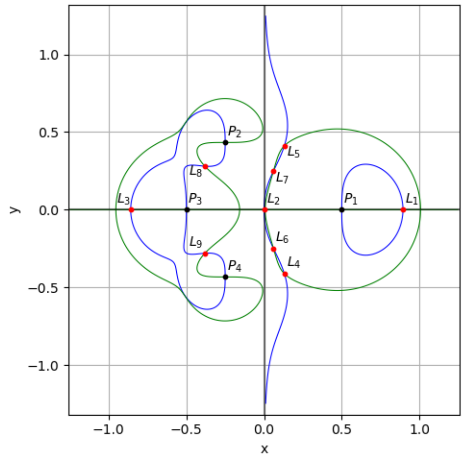


**Figure-5**

The position of nine libration points indicated through the intersection of $\Omega_x=0$ &$\Omega_y=0$ for $\mu=0.13$ &$n=1.8703261$ .Here black dots ( $P_i, i=1,2,3,4$ ) represent four bodies forming kite and six pink spots $L_j(j=4,5,...,9)$ represent non axial libration points.

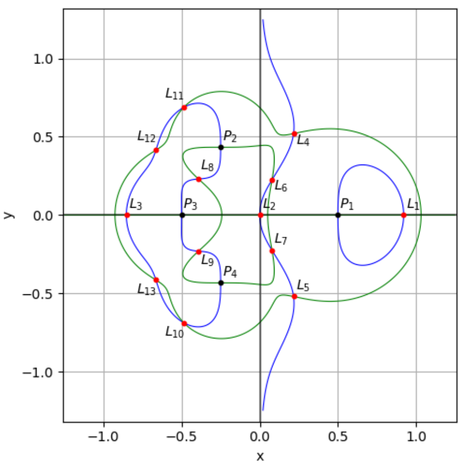


**Figure-6**

The position of thirteen libration points indicated through the intersection of $\Omega_x = 0$ &$\Omega_y = 0$ for $\mu = 0.20$ & $n = 1.682760$. Here black dots $(P_i, i = 1,2,3,4)$ represent four bodies forming kite and ten pink spots $L_j(j = 4,5,...,13)$ represent non axial libration points.

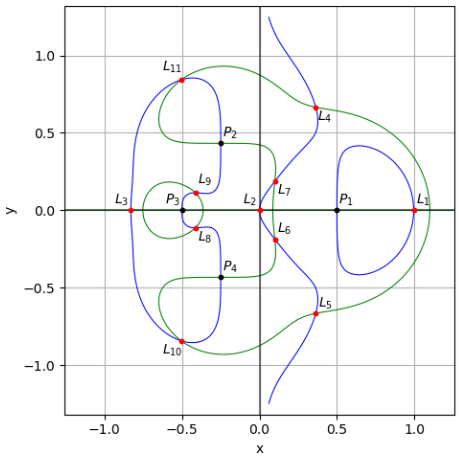


**Figure-7**

The position of eleven libration points indicated through the intersection of $\Omega_x = 0$ & $\Omega_y = 0$ for $\mu = 0.31$& $n = 1.328802$ .Here black dots ( $P_i, i = 1,2,3,4$ ) represent four bodies forming kite and eight pink spots $L_j(j = 4,5,...,11)$ represent non axial libration points.

**Data for existence of non-axial libration point $L_4\left(x_4, y_4\right)$ by using Python.**

**Table-1**

| S.N. | $\mu$ | $n$ | $x_4$ | $y_4$ | $\Omega$ | $C$ | Remark |
|---|---|---|---|---|---|---|---|
| 1 | 0.01 | 1.908937 | 0.018715 | 0.409329 | 1.877695 | 3.75539 | Exists |
| 2 | 0.02 | 1.905315 | 0.033261 | -0.40628 | 1.887922 | 3.775843 | Exists |
| 3 | 0.03 | 1.901802 | 0.045526 | -0.40416 | 1.902616 | 3.805232 | Exists |
| 4 | 0.04 | 1.898384 | 0.056352 | 0.402755 | 1.988732 | 3.977465 | Exists |
| 5 | 0.05 | 1.89505 | 0.066202 | 0.401977 | 2.037472 | 4.074944 | Exists |
| 6 | 0.06 | 1.89179 | 0.075363 | -0.40176 | 1.941172 | 3.882344 | Exists |
| 7 | 0.07 | 1.888594 | 0.084026 | -0.40206 | 1.952633 | 3.905266 | Exists |
| 8 | 0.08 | 1.885454 | 0.092319 | 0.402822 | 2.223557 | 4.447115 | Exists |
| 9 | 0.09 | 1.882361 | 0.100333 | -0.40401 | 1.973929 | 3.947857 | Exists |
| 10 | 0.1 | 1.879308 | 0.108129 | 0.405551 | 2.389553 | 4.779107 | Exists |
| 11 | 0.11 | 1.876288 | 0.115746 | 0.407389 | 2.488908 | 4.977815 | Exists |
| 12 | 0.12 | 1.873296 | 0.123207 | -0.40945 | 2.002432 | 4.004864 | Exists |
| 13 | 0.13 | 1.870326 | 0.130525 | 0.411663 | 2.729091 | 5.458181 | Exists |
| 14 | 0.14 | 1.847462 | 0.1434 | 0.429 | 2.955133 | 5.910265 | Exists |
| 15 | 0.15 | 1.821047 | 0.156395 | 0.446782 | 3.212118 | 6.424236 | Exists |
| 16 | 0.16 | 1.794242 | 0.168838 | 0.463086 | 3.458718 | 6.917436 | Exists |
| 17 | 0.17 | 1.767031 | 0.180929 | 0.478337 | 3.650891 | 7.301782 | Exists |
| 18 | 0.18 | 1.739394 | 0.192808 | 0.492816 | 3.750814 | 7.501627 | Exists |
| 19 | 0.19 | 1.711311 | 0.204579 | 0.506725 | 3.751921 | 7.503842 | Exists |
| 20 | 0.2 | 1.68276 | 0.216329 | -0.52021 | 1.981659 | 3.963318 | Exists |
| 21 | 0.21 | 1.653715 | 0.228131 | -0.5334 | 1.972026 | 3.944052 | Exists |
| 22 | 0.22 | 1.624152 | 0.240051 | 0.546385 | 3.433067 | 6.866134 | Exists |
| 23 | 0.23 | 1.59404 | 0.252152 | -0.55925 | 1.94889 | 3.89778 | Exists |
| 24 | 0.24 | 1.563348 | 0.264494 | 0.572065 | 3.1755 | 6.351 | Exists |
| 25 | 0.25 | 1.532041 | 0.27714 | -0.5849 | 1.920492 | 3.840984 | Exists |
| 26 | 0.26 | 1.500082 | 0.290155 | 0.597822 | 2.947988 | 5.895976 | Exists |
| 27 | 0.27 | 1.467426 | 0.303607 | -0.61089 | 1.886607 | 3.773214 | Exists |
| 28 | 0.28 | 1.434027 | 0.317575 | 0.624162 | 2.750074 | 5.500148 | Exists |
| 29 | 0.29 | 1.399831 | 0.332143 | 0.63771 | 2.659635 | 5.319269 | Exists |
| 30 | 0.3 | 1.364779 | 0.347409 | -0.6516 | 1.824665 | 3.64933 | Exists |
| 31 | 0.31 | 1.328802 | 0.363486 | -0.66591 | 1.800781 | 3.601562 | Exists |
| 32 | 0.32 | 1.291824 | 0.380507 | 0.68071 | 2.410366 | 4.820731 | Exists |

**Data for existence of non-axial libration point $L_5\left(x_5, y_5\right)$ by using Python.**

**Table-2**

| S.N. | $\mu$ | $n$ | $x_5$ | $y_5$ | $\Omega$ | $C$ | Remark |
|---|---|---|---|---|---|---|---|
| 1 | 0.01 | 1.908937 | 0.018715 | -0.40933 | 1.871892 | 3.743784 | Exists |
| 2 | 0.02 | 1.905315 | 0.033261 | 0.406282 | 1.908717 | 3.817433 | Exists |
| 3 | 0.03 | 1.901802 | 0.045526 | 0.404156 | 1.945841 | 3.891681 | Exists |
| 4 | 0.04 | 1.898384 | 0.056352 | -0.40276 | 1.916274 | 3.832548 | Exists |
| 5 | 0.05 | 1.89505 | 0.066202 | -0.40198 | 1.929083 | 3.858166 | Exists |
| 6 | 0.06 | 1.89179 | 0.075363 | 0.40176 | 2.092421 | 4.184842 | Exists |
| 7 | 0.07 | 1.888594 | 0.084026 | 0.402056 | 2.154177 | 4.308354 | Exists |
| 8 | 0.08 | 1.885454 | 0.092319 | -0.40282 | 1.963535 | 3.927069 | Exists |
| 9 | 0.09 | 1.882361 | 0.100333 | 0.404006 | 2.301598 | 4.603196 | Exists |
| 10 | 0.1 | 1.879308 | 0.108129 | -0.40555 | 1.983856 | 3.967711 | Exists |
| 11 | 0.11 | 1.876288 | 0.115746 | -0.40739 | 1.993348 | 3.986696 | Exists |
| 12 | 0.12 | 1.873296 | 0.123207 | 0.409449 | 2.601403 | 5.202806 | Exists |
| 13 | 0.13 | 1.870326 | 0.130525 | -0.41166 | 2.011131 | 4.022262 | Exists |
| 14 | 0.14 | 1.847462 | 0.1434 | -0.429 | 2.012025 | 4.02405 | Exists |
| 15 | 0.15 | 1.821047 | 0.156395 | -0.44678 | 2.010342 | 4.020684 | Exists |
| 16 | 0.16 | 1.794242 | 0.168838 | -0.46309 | 2.007261 | 4.014522 | Exists |
| 17 | 0.17 | 1.767031 | 0.180929 | -0.47834 | 2.002825 | 4.005651 | Exists |
| 18 | 0.18 | 1.739394 | 0.192808 | -0.49282 | 1.997067 | 3.994134 | Exists |
| 19 | 0.19 | 1.711311 | 0.204579 | -0.50672 | 1.990007 | 3.980015 | Exists |
| 20 | 0.2 | 1.68276 | 0.216329 | 0.520214 | 3.678459 | 7.356918 | Exists |
| 21 | 0.21 | 1.653715 | 0.228131 | 0.533401 | 3.563644 | 7.127288 | Exists |
| 22 | 0.22 | 1.624152 | 0.240051 | -0.54639 | 1.961106 | 3.922213 | Exists |
| 23 | 0.23 | 1.59404 | 0.252152 | 0.559249 | 3.301424 | 6.602848 | Exists |
| 24 | 0.24 | 1.563348 | 0.264494 | -0.57207 | 1.93536 | 3.87072 | Exists |
| 25 | 0.25 | 1.532041 | 0.27714 | 0.584902 | 3.05763 | 6.115261 | Exists |
| 26 | 0.26 | 1.500082 | 0.290155 | -0.59782 | 1.904254 | 3.808508 | Exists |
| 27 | 0.27 | 1.467426 | 0.303607 | 0.610888 | 2.845824 | 5.691648 | Exists |
| 28 | 0.28 | 1.434027 | 0.317575 | -0.62416 | 1.867501 | 3.735002 | Exists |
| 29 | 0.29 | 1.399831 | 0.332143 | -0.63771 | 1.846877 | 3.693754 | Exists |
| 30 | 0.3 | 1.364779 | 0.347409 | 0.6516 | 2.573472 | 5.146943 | Exists |
| 31 | 0.31 | 1.328802 | 0.363486 | 0.665906 | 2.490656 | 4.981313 | Exists |
| 32 | 0.32 | 1.291824 | 0.380507 | -0.68071 | 1.775126 | 3.550252 | Exists |

**Data for existence of non-axial libration point $L_6\left(x_6, y_6\right)$ by using Python.**

**Table-3**

| S.N. | $\mu$ | $n$ | $x_6$ | $y_6$ | $\Omega$ | $C$ | Remark |
|---|---|---|---|---|---|---|---|
| 1 | 0.01 | 1.908937 | -0.30853 | 0.393918 | 2.143811 | 4.287623 | Exists |
| 2 | 0.02 | 1.905315 | -0.32414 | -0.37538 | 2.354511 | 4.709021 | Exists |
| 3 | 0.03 | 1.901802 | -0.33419 | 0.361271 | 2.204686 | 4.409373 | Exists |
| 4 | 0.04 | 1.898384 | -0.34178 | -0.34955 | 2.467245 | 4.934489 | Exists |
| 5 | 0.05 | 1.89505 | -0.348 | 0.33936 | 2.234072 | 4.468144 | Exists |
| 6 | 0.06 | 1.89179 | -0.35335 | 0.330221 | 2.242876 | 4.485752 | Exists |
| 7 | 0.07 | 1.888594 | -0.35811 | 0.321854 | 2.248839 | 4.497678 | Exists |
| 8 | 0.08 | 1.885454 | -0.36245 | 0.314069 | 2.252409 | 4.504819 | Exists |
| 9 | 0.09 | 1.882361 | -0.36647 | -0.30673 | 2.608051 | 5.216102 | Exists |
| 10 | 0.1 | 1.879308 | -0.37026 | -0.29975 | 2.622484 | 5.244969 | Exists |
| 11 | 0.11 | 1.876288 | -0.37387 | 0.293042 | 2.251437 | 4.502874 | Exists |
| 12 | 0.12 | 1.873296 | 0.04416 | 0.221381 | 2.07612 | 4.15224 | Exists |
| 13 | 0.13 | 1.870326 | 0.060263 | 0.248422 | 2.137961 | 4.275922 | Exists |
| 14 | 0.14 | 1.847462 | 0.064304 | -0.24742 | 2.02243 | 4.04486 | Exists |
| 15 | 0.15 | 1.821047 | 0.066922 | -0.24362 | 2.027086 | 4.054171 | Exists |
| 16 | 0.16 | 1.794242 | 0.06944 | 0.239884 | 2.193282 | 4.386563 | Exists |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 17 | 0.17 | 1.767031 | 0.071866 | -0.23621 | 2.035952 | 4.071904 | Exists |
| 18 | 0.18 | 1.739394 | 0.07421 | 0.232592 | 2.224788 | 4.449576 | Exists |
| 19 | 0.19 | 1.711311 | 0.076479 | 0.229014 | 2.239978 | 4.479957 | Exists |
| 20 | 0.2 | 1.68276 | 0.078678 | -0.22547 | 2.048295 | 4.096589 | Exists |
| 21 | 0.21 | 1.653715 | 0.080814 | 0.221964 | 2.269146 | 4.538292 | Exists |
| 22 | 0.22 | 1.624152 | 0.082891 | -0.21848 | 2.055989 | 4.111977 | Exists |
| 23 | 0.23 | 1.59404 | 0.084915 | -0.21501 | 2.059699 | 4.119398 | Exists |
| 24 | 0.24 | 1.563348 | 0.086889 | -0.21156 | 2.063328 | 4.126656 | Exists |
| 25 | 0.25 | 1.532041 | 0.088817 | 0.208119 | 2.322247 | 4.644493 | Exists |
| 26 | 0.26 | 1.500082 | 0.090702 | -0.20468 | 2.07037 | 4.140739 | Exists |
| 27 | 0.27 | 1.467426 | 0.092547 | 0.201245 | 2.346001 | 4.692002 | Exists |
| 28 | 0.28 | 1.434027 | 0.094355 | 0.197805 | 2.357146 | 4.714292 | Exists |
| 29 | 0.29 | 1.399831 | 0.096129 | -0.19436 | 2.080492 | 4.160984 | Exists |
| 30 | 0.3 | 1.364779 | 0.097871 | -0.1909 | 2.083775 | 4.167551 | Exists |
| 31 | 0.31 | 1.328802 | 0.099583 | 0.18742 | 2.387549 | 4.775099 | Exists |
| 32 | 0.32 | 1.291824 | 0.101267 | -0.18392 | 2.090248 | 4.180495 | Exists |

**Data for existence of non-axial libration point $L_7\left(x_7, y_7\right)$ by using Python.**

**Table-4**

| S.N. | $\mu$ | $n$ | $x_7$ | $y_7$ | $\Omega$ | $C$ | Remark |
|---|---|---|---|---|---|---|---|
| 1 | 0.01 | 1.908937 | -0.30853 | -0.39392 | 2.265977 | 4.531954 | Exists |
| 2 | 0.02 | 1.905315 | -0.32414 | 0.375377 | 2.180658 | 4.361317 | Exists |
| 3 | 0.03 | 1.901802 | -0.33419 | -0.36127 | 2.417807 | 4.835613 | Exists |
| 4 | 0.04 | 1.898384 | -0.34178 | 0.349555 | 2.221732 | 4.443464 | Exists |
| 5 | 0.05 | 1.89505 | -0.348 | -0.33936 | 2.507283 | 5.014565 | Exists |
| 6 | 0.06 | 1.89179 | -0.35335 | -0.33022 | 2.540227 | 5.080453 | Exists |
| 7 | 0.07 | 1.888594 | -0.35811 | -0.32185 | 2.567452 | 5.134904 | Exists |
| 8 | 0.08 | 1.885454 | -0.36245 | -0.31407 | 2.589854 | 5.179709 | Exists |
| 9 | 0.09 | 1.882361 | -0.36647 | 0.306735 | 2.253896 | 4.507791 | Exists |
| 10 | 0.1 | 1.879308 | -0.37026 | 0.299751 | 2.253518 | 4.507036 | Exists |
| 11 | 0.11 | 1.876288 | -0.37387 | -0.29304 | 2.633479 | 5.266957 | Exists |
| 12 | 0.12 | 1.873296 | 0.04416 | -0.22138 | 2.008468 | 4.016935 | Exists |
| 13 | 0.13 | 1.870326 | 0.060263 | -0.24842 | 2.017113 | 4.034227 | Exists |
| 14 | 0.14 | 1.847462 | 0.064304 | 0.24742 | 2.160421 | 4.320843 | Exists |
| 15 | 0.15 | 1.821047 | 0.066922 | 0.243616 | 2.177008 | 4.354016 | Exists |
| 16 | 0.16 | 1.794242 | 0.06944 | -0.23988 | 2.031589 | 4.063179 | Exists |
| 17 | 0.17 | 1.767031 | 0.071866 | 0.236212 | 2.209216 | 4.418432 | Exists |
| 18 | 0.18 | 1.739394 | 0.07421 | -0.23259 | 2.040184 | 4.080369 | Exists |
| 19 | 0.19 | 1.711311 | 0.076479 | -0.22901 | 2.044296 | 4.088591 | Exists |
| 20 | 0.2 | 1.68276 | 0.078678 | 0.225474 | 2.25477 | 4.509539 | Exists |
| 21 | 0.21 | 1.653715 | 0.080814 | -0.22196 | 2.05219 | 4.104379 | Exists |
| 22 | 0.22 | 1.624152 | 0.082891 | 0.218478 | 2.283094 | 4.566188 | Exists |
| 23 | 0.23 | 1.59404 | 0.084915 | 0.215012 | 2.296601 | 4.593202 | Exists |
| 24 | 0.24 | 1.563348 | 0.086889 | 0.211561 | 2.309655 | 4.61931 | Exists |
| 25 | 0.25 | 1.532041 | 0.088817 | -0.20812 | 2.066883 | 4.133765 | Exists |
| 26 | 0.26 | 1.500082 | 0.090702 | 0.204682 | 2.334365 | 4.66873 | Exists |
| 27 | 0.27 | 1.467426 | 0.092547 | -0.20125 | 2.073796 | 4.147592 | Exists |
| 28 | 0.28 | 1.434027 | 0.094355 | -0.19781 | 2.077168 | 4.154335 | Exists |
| 29 | 0.29 | 1.399831 | 0.096129 | 0.194357 | 2.367791 | 4.735583 | Exists |
| 30 | 0.3 | 1.364779 | 0.097871 | 0.190897 | 2.377929 | 4.755857 | Exists |
| 31 | 0.31 | 1.328802 | 0.099583 | -0.18742 | 2.087025 | 4.17405 | Exists |
| 32 | 0.32 | 1.291824 | 0.101267 | 0.183923 | 2.396645 | 4.793291 | Exists |

**Data for existence of non-axial libration point $L_8\left(x_8, y_8\right)$ by using Python.**

**Table-5**

| S.N. | $\mu$ | $n$ | $x_8$ | $y_8$ | $\Omega$ | $C$ | Remark |
|---|---|---|---|---|---|---|---|
| 1 | 0.12 | 1.873296 | 0.007779 | -0.10043 | 2.001893 | 4.003786 | Exists |
| 2 | 0.13 | 1.870326 | -0.38071 | 0.280212 | 2.242615 | 4.485231 | Exists |
| 3 | 0.14 | 1.847462 | -0.38256 | 0.273095 | 2.227449 | 4.454897 | Exists |
| 4 | 0.15 | 1.821047 | -0.38413 | 0.265947 | 2.209726 | 4.419453 | Exists |
| 5 | 0.16 | 1.794242 | -0.38565 | 0.258869 | 2.190872 | 4.381745 | Exists |
| 6 | 0.17 | 1.767031 | -0.38715 | 0.251818 | 2.170901 | 4.341802 | Exists |
| 7 | 0.18 | 1.739394 | -0.38864 | 0.24475 | 2.149813 | 4.299627 | Exists |
| 8 | 0.19 | 1.711311 | -0.39012 | -0.23762 | 2.561543 | 5.123087 | Exists |
| 9 | 0.2 | 1.68276 | -0.39162 | 0.23039 | 2.104244 | 4.208488 | Exists |
| 10 | 0.21 | 1.653715 | -0.39314 | 0.223001 | 2.079709 | 4.159418 | Exists |
| 11 | 0.22 | 1.624152 | -0.39469 | 0.215401 | 2.05395 | 4.107901 | Exists |
| 12 | 0.23 | 1.59404 | -0.39628 | 0.207521 | 2.026907 | 4.053814 | Exists |
| 13 | 0.24 | 1.563348 | -0.39794 | 0.19928 | 1.998498 | 3.996997 | Exists |
| 14 | 0.25 | 1.532041 | -0.39968 | 0.190574 | 1.968621 | 3.937242 | Exists |
| 15 | 0.26 | 1.500082 | -0.40152 | -0.18127 | 2.329185 | 4.65837 | Exists |
| 16 | 0.27 | 1.467426 | -0.40349 | 0.171163 | 1.903882 | 3.807764 | Exists |
| 17 | 0.28 | 1.434027 | -0.40564 | -0.15999 | 2.225339 | 4.450677 | Exists |
| 18 | 0.29 | 1.399831 | -0.40802 | 0.14729 | 1.83103 | 3.66206 | Exists |
| 19 | 0.3 | 1.364779 | -0.41074 | 0.132304 | 1.790714 | 3.581427 | Exists |
| 20 | 0.31 | 1.328802 | -0.41397 | -0.11344 | 2.006659 | 4.013318 | Exists |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 21 | 0.32 | 1.291824 | -0.41812 | 0.086355 | 1.700178 | 3.400357 | Exists |

**Data for existence of non-axial libration point $L_9\left(x_9, y_9\right)$ by using Python.**

**Table-6**

| S.N. | $\mu$ | $n$ | $x_9$ | $y_9$ | $\Omega$ | $C$ | Remark |
|---|---|---|---|---|---|---|---|
| 1 | 0.12 | 1.873296 | 0.007779 | 0.100428 | 2.006089 | 4.012179 | Exists |
| 2 | 0.13 | 1.870326 | -0.38071 | -0.28021 | 2.646051 | 5.292102 | Exists |
| 3 | 0.14 | 1.847462 | -0.38256 | -0.27309 | 2.63968 | 5.279361 | Exists |
| 4 | 0.15 | 1.821047 | -0.38413 | -0.26595 | 2.629302 | 5.258603 | Exists |
| 5 | 0.16 | 1.794242 | -0.38565 | -0.25887 | 2.616298 | 5.232596 | Exists |
| 6 | 0.17 | 1.767031 | -0.38715 | -0.25182 | 2.600684 | 5.201367 | Exists |
| 7 | 0.18 | 1.739394 | -0.38864 | -0.24475 | 2.582446 | 5.164892 | Exists |
| 8 | 0.19 | 1.711311 | -0.39012 | 0.237623 | 2.127601 | 4.255202 | Exists |
| 9 | 0.2 | 1.68276 | -0.39162 | -0.23039 | 2.537904 | 5.075808 | Exists |
| 10 | 0.21 | 1.653715 | -0.39314 | -0.223 | 2.511422 | 5.022844 | Exists |
| 11 | 0.22 | 1.624152 | -0.39469 | -0.2154 | 2.481951 | 4.963902 | Exists |
| 12 | 0.23 | 1.59404 | -0.39628 | -0.20752 | 2.449297 | 4.898594 | Exists |
| 13 | 0.24 | 1.563348 | -0.39794 | -0.19928 | 2.4132 | 4.8264 | Exists |
| 14 | 0.25 | 1.532041 | -0.39968 | -0.19057 | 2.373317 | 4.746634 | Exists |
| 15 | 0.26 | 1.500082 | -0.40152 | 0.181266 | 1.93714 | 3.874281 | Exists |
| 16 | 0.27 | 1.467426 | -0.40349 | -0.17116 | 2.280164 | 4.560328 | Exists |
| 17 | 0.28 | 1.434027 | -0.40564 | 0.159985 | 1.868615 | 3.737229 | Exists |
| 18 | 0.29 | 1.399831 | -0.40802 | -0.14729 | 2.163324 | 4.326647 | Exists |
| 19 | 0.3 | 1.364779 | -0.41074 | -0.1323 | 2.091847 | 4.183694 | Exists |
| 20 | 0.31 | 1.328802 | -0.41397 | 0.113442 | 1.747148 | 3.494295 | Exists |
| 21 | 0.32 | 1.291824 | -0.41812 | -0.08636 | 1.897576 | 3.795152 | Exists |

**Data for existence of non-axial libration point $L_{10}\left(x_{10}, y_{10}\right)$ by using Python.**

**Table-7**

| S.N. | $\mu$ | $n$ | $x_{10}$ | $y_{10}$ | $\Omega$ | $C$ | Remark |
|---|---|---|---|---|---|---|---|
| 1 | 0.12 | 1.873296 | -0.37734 | 0.286545 | 2.247773 | 4.495547 | Exists |
| 2 | 0.14 | 1.847462 | -0.49372 | -0.59153 | 2.4673 | 4.934601 | Exists |
| 3 | 0.15 | 1.821047 | -0.48877 | -0.61316 | 2.451833 | 4.903665 | Exists |
| 4 | 0.16 | 1.794242 | -0.48648 | 0.630625 | 2.148669 | 4.297338 | Exists |
| 5 | 0.17 | 1.767031 | -0.48539 | 0.646212 | 2.126909 | 4.253818 | Exists |
| 6 | 0.18 | 1.739394 | -0.48501 | 0.660737 | 2.104781 | 4.209562 | Exists |
| 7 | 0.19 | 1.711311 | -0.48512 | 0.674614 | 2.082178 | 4.164356 | Exists |
| 8 | 0.2 | 1.68276 | -0.48557 | -0.68809 | 2.352723 | 4.705445 | Exists |
| 9 | 0.21 | 1.653715 | -0.48631 | 0.701348 | 2.035294 | 4.070588 | Exists |
| 10 | 0.22 | 1.624152 | -0.48728 | 0.714505 | 2.010921 | 4.021841 | Exists |
| 11 | 0.23 | 1.59404 | -0.48844 | 0.727669 | 1.985875 | 3.97175 | Exists |
| 12 | 0.24 | 1.563348 | -0.4898 | -0.74093 | 2.248219 | 4.496439 | Exists |
| 13 | 0.25 | 1.532041 | -0.49132 | -0.75437 | 2.218547 | 4.437095 | Exists |
| 14 | 0.26 | 1.500082 | -0.49301 | -0.76807 | 2.187413 | 4.374826 | Exists |
| 15 | 0.27 | 1.467426 | -0.49486 | -0.7821 | 2.154781 | 4.309562 | Exists |
| 16 | 0.28 | 1.434027 | -0.49688 | 0.796563 | 1.849173 | 3.698345 | Exists |
| 17 | 0.29 | 1.399831 | -0.49908 | -0.81153 | 2.084847 | 4.169694 | Exists |
| 18 | 0.3 | 1.364779 | -0.50146 | -0.82711 | 2.04743 | 4.09486 | Exists |
| 19 | 0.31 | 1.328802 | -0.50404 | 0.843406 | 1.7562 | 3.5124 | Exists |
| 20 | 0.32 | 1.291824 | -0.50684 | -0.86055 | 1.967323 | 3.934646 | Exists |

**Data for existence of non-axial libration point $L_{11}\left(x_{11}, y_{11}\right)$ by using Python.**

**Table-8**

| S.N. | $\mu$ | $n$ | $x_{11}$ | $y_{11}$ | $\Omega$ | $C$ | Remark |
|---|---|---|---|---|---|---|---|
| 1 | 0.12 | 1.873296 | -0.37734 | -0.28655 | 2.641275 | 5.282549 | Exists |
| 2 | 0.14 | 1.847462 | -0.49372 | 0.591532 | 2.192224 | 4.384449 | Exists |
| 3 | 0.15 | 1.821047 | -0.48877 | 0.613157 | 2.170262 | 4.340525 | Exists |
| 4 | 0.16 | 1.794242 | -0.48648 | -0.63063 | 2.434855 | 4.86971 | Exists |
| 5 | 0.17 | 1.767031 | -0.48539 | -0.64621 | 2.416435 | 4.832871 | Exists |
| 6 | 0.18 | 1.739394 | -0.48501 | -0.66074 | 2.396598 | 4.793196 | Exists |
| 7 | 0.19 | 1.711311 | -0.48512 | -0.67461 | 2.375358 | 4.750715 | Exists |
| 8 | 0.2 | 1.68276 | -0.48557 | 0.688094 | 2.059032 | 4.118065 | Exists |
| 9 | 0.21 | 1.653715 | -0.48631 | -0.70135 | 2.328696 | 4.657393 | Exists |
| 10 | 0.22 | 1.624152 | -0.48728 | -0.71451 | 2.303277 | 4.606554 | Exists |
| 11 | 0.23 | 1.59404 | -0.48844 | -0.72767 | 2.276456 | 4.552911 | Exists |
| 12 | 0.24 | 1.563348 | -0.4898 | 0.74093 | 1.960119 | 3.920238 | Exists |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 13 | 0.25 | 1.532041 | -0.49132 | 0.75437 | 1.933614 | 3.867228 | Exists |
| 14 | 0.26 | 1.500082 | -0.49301 | 0.768068 | 1.906319 | 3.812638 | Exists |
| 15 | 0.27 | 1.467426 | -0.49486 | 0.782105 | 1.878189 | 3.756378 | Exists |
| 16 | 0.28 | 1.434027 | -0.49688 | -0.79656 | 2.12061 | 4.24122 | Exists |
| 17 | 0.29 | 1.399831 | -0.49908 | 0.811531 | 1.819214 | 3.638428 | Exists |
| 18 | 0.3 | 1.364779 | -0.50146 | 0.827108 | 1.788248 | 3.576496 | Exists |
| 19 | 0.31 | 1.328802 | -0.50404 | -0.84341 | 2.008285 | 4.01657 | Exists |
| 20 | 0.32 | 1.291824 | -0.50684 | 0.860552 | 1.722984 | 3.445968 | Exists |

**Data for existence of non-axial libration point $L_{12}\left(x_{12}, y_{12}\right)$ by using Python.**

**Table-9**

| S.N. | $\mu$ | $n$ | $x_{12}$ | $y_{12}$ | $\Omega$ | $C$ | Remark |
|---|---|---|---|---|---|---|---|
| 1 | 0.14 | 1.847462 | -0.54522 | 0.514416 | 2.20581 | 4.411619 | Exists |
| 2 | 0.15 | 1.821047 | -0.56769 | 0.495212 | 2.188098 | 4.376196 | Exists |
| 3 | 0.16 | 1.794242 | -0.58778 | -0.47914 | 2.411264 | 4.822528 | Exists |
| 4 | 0.17 | 1.767031 | -0.607 | 0.463839 | 2.145905 | 4.291809 | Exists |
| 5 | 0.18 | 1.739394 | -0.6259 | -0.44836 | 2.352227 | 4.704454 | Exists |
| 6 | 0.19 | 1.711311 | -0.64478 | 0.432119 | 2.097087 | 4.194174 | Exists |
| 7 | 0.2 | 1.68276 | -0.66382 | 0.414628 | 2.070526 | 4.141051 | Exists |
| 8 | 0.21 | 1.653715 | -0.68316 | 0.395397 | 2.042631 | 4.085262 | Exists |
| 9 | 0.22 | 1.624152 | -0.70293 | -0.37385 | 2.203113 | 4.406225 | Exists |
| 10 | 0.23 | 1.59404 | -0.72323 | 0.349221 | 1.983133 | 3.966266 | Exists |
| 11 | 0.24 | 1.563348 | -0.74417 | 0.320443 | 1.95174 | 3.903481 | Exists |
| 12 | 0.25 | 1.532041 | -0.76585 | 0.285803 | 1.919524 | 3.839049 | Exists |
| 13 | 0.26 | 1.500082 | -0.78838 | 0.242186 | 1.886983 | 3.773966 | Exists |
| 14 | 0.27 | 1.467426 | -0.81185 | 0.182335 | 1.855489 | 3.710978 | Exists |
| 15 | 0.28 | 1.434027 | -0.83637 | -0.07098 | 1.866264 | 3.732527 | Exists |

**Data for existence of non-axial libration point $L_{13}\left(x_{13}, y_{13}\right)$ by using Python.**

**Table-10**

| S.N. | $\mu$ | $n$ | $x_{13}$ | $y_{13}$ | $\Omega$ | $C$ | Remark |
|---|---|---|---|---|---|---|---|
| 1 | 0.14 | 1.847462 | -0.54522 | -0.51442 | 2.459936 | 4.919872 | Exists |
| 2 | 0.15 | 1.821047 | -0.56769 | -0.49521 | 2.436913 | 4.873825 | Exists |
| 3 | 0.16 | 1.794242 | -0.58778 | 0.479141 | 2.167935 | 4.335871 | Exists |
| 4 | 0.17 | 1.767031 | -0.607 | -0.46384 | 2.383023 | 4.766046 | Exists |
| 5 | 0.18 | 1.739394 | -0.6259 | 0.44836 | 2.122244 | 4.244487 | Exists |
| 6 | 0.19 | 1.711311 | -0.64478 | -0.43212 | 2.318884 | 4.637767 | Exists |
| 7 | 0.2 | 1.68276 | -0.66382 | -0.41463 | 2.282968 | 4.565936 | Exists |
| 8 | 0.21 | 1.653715 | -0.68316 | -0.3954 | 2.244415 | 4.488831 | Exists |
| 9 | 0.22 | 1.624152 | -0.70293 | 0.373846 | 2.013472 | 4.026945 | Exists |
| 10 | 0.23 | 1.59404 | -0.72323 | -0.34922 | 2.15888 | 4.31776 | Exists |
| 11 | 0.24 | 1.563348 | -0.74417 | -0.32044 | 2.111437 | 4.222874 | Exists |
| 12 | 0.25 | 1.532041 | -0.76585 | -0.2858 | 2.060329 | 4.120657 | Exists |
| 13 | 0.26 | 1.500082 | -0.78838 | -0.24219 | 2.004739 | 4.009479 | Exists |
| 14 | 0.27 | 1.467426 | -0.81185 | -0.18233 | 1.94285 | 3.885701 | Exists |
| 15 | 0.28 | 1.434027 | -0.83637 | 0.070982 | 1.832798 | 3.665596 | Exists |

**Conclusion:** In section-1 of this paper kite has been defined and classified. In section-2 the previous works have been reviewed starting from Mac Millon (1932) to khatun (2024). In section-3 the equations of motion of each of the four-point masses forming kite relative to other three in synodic frame rotating about vertical z-axis have been established. Later on the equations have been reduced to the linear combination of the position vectors of the four bodies forming kite. By eliminating position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$ from the equations of motion, a functional relation between $\mu \,\&\, n$ was established in section-4 and hence $n$ is obtained in terms of $\mu$. In section-5 the equation of motion and energy integral of the satellite moving in the gravitational field of the kite have been established. In section-6 the existence of non-axial libration points have been discussed with the contour plots of $\Omega_x = 0 \,\&\, \Omega_y = 0$ .For $0.01 \le \mu \le 0.11$ four non-axial libration points exists whereas for $\mu = 0.13$ six libration points, for $0.29 \le \mu \le 0.32 \,\&\, \mu = 0.12$ eight libration points and for $0.14 \le \mu \le 0.28$ ten libration points exist respectively. It is to be noted that for only one value of $\mu = 0.13$ ,six libration points exists and for $\mu = 0.12$, eight libration points exist but for other numbers of libration points there exists a domain for $\mu$ .As the previous authors khatun & Hassan (2023) worked on the existence of libration points $L_1, L_2, L_3$ lying on the axis of symmetry (i.e.axial libration points).So we here are concerned with the non-axial libration points i.e. $L_j\left(j = 4,5,6,7,8,9,10\right)$.

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**References:**

1. Albouy,A,:The symmetric central configurations of four equal masses, Contemp, Math 198(1996) 131-136

2. Albouy, A., Fu, Y., Sun, S.: Symmetry of planar four-body Convex central configurations. Proc. R. Soc. A 464.1355-1375(2008).

3. Brumberg, V.A.: Permanent configuration in the problem of four bodies and their Stability, Translated from Russian. (1957)

4. BalintErdi, Zalan Czirjak,Central Configurations of four bodies with an axis of symmetry, Celestial Mechanics and Dynamical Astronomy.125(1)(2016)33-70.

5. Benhammouda, B., Mansur, A., Szucs-Csillik, I., D.: Central configuration and action minimizing orbits in four-body problem, Advance in Astronomy, volume 2020 Article I D 5263750, https//dot.org/101155(2020):

6. Cors, J.M., Roberts, G.E.: Four-body co-circular central configurations. Nonlinearity 25,343-370(2012)

7. Corbera, M., Cors, E., Roberts, G.E.: A Four-body Convex Central Configuration with perpendicular Diagonals is necessarily a kite, https://arxiV.org/abs/1610.08654.(2017).

8. Hassan, M.R., Ullah, M.S., Hassan, M.A., Prasad, U.: Application of planar Newtonion four-body problem to the central configuration, Application and Applied Mathematics. Texas USA, volume(12)2pp.1088-1108(2017).

9. Hassan, M.R.: Existence of cyclic kite configuration and its application to the Newtonion four-body problem for mean motion, Tuijin Jishu/Journal of propulsion Technology.Vol.44 No.6, pp-4041-4056. Dec.2023.

10. Isaac Newton.: Philosophi naturalis principia mathematica, Royal society of London (1687)

11. Long. Y., Sun, S.: Four-body central configurstions with some equal masses. ArchRation. MechAnal.

12. Li, B., Deng, Y., Zhang, S.: A property for four-body isosceles trapezoid central configuration, preprint. http://www.paper.edu.cn/download/downpaper/201606-641.

13. MacMillon, W.D., Bartky, W.: Permanent configurations in the problem of four bodies. Trans. Am. Math.Soc.34(1932)838-875.

14. Khatun; M.: The effect of oblateness on the mean motion of rotating cyclic kite configuration. Tuijin Jishu/journal of propulsion Technology.ISSN:1001-4055. Vol. 45 No.1(2024).

15. Perez-Chavela, E., Santoprete, M.: Convex four-body central configurations With some equal masses. Arch. Ration. Mech. Anal.185,481-494(2007).

16. Pina, E., Lonngi, P.: Central configurations for the planar Newtonian four-body problem. Celestial Mechanics and Dynamical Astronomy.108 73-93(2010).

17. Pina, E.: New coordinates for the four-body problem, submitted to publication (ArXiv) (2010).

18. Roberts, G.E.: Cyclic central configuration in the four-body problem, American Mathematical society, Spring Eastern sectional Meeting (2011)

19. Szebehely, V.: Theory of orbits, Academic Press, New York.