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FIXED POINT THEOREM FOR COMPATIBLE MAPPING SATISFYING A CONTRACTIVE CONDITION OF INTEGRAL TYPE

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Abstract : Fixed point theory in metric spaces using interpolative contractive mapping has greatly developed in recent times. In this article we prove fixed point results satisfying interpolative type integral contractive condition in metric spaces.

Keywords: Metric space, integral type contractive condition, integral type interpolative contraction conditions, fixed point.

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1. INTRODUCTION

In 1976, Jungck [5] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem, which states that, "let (X, d) be a complete metric space. If T satisfies $d(Tx, Ty) \leq kd(x, y)$ for each $x, y \in X$ where $0 \leq k < 1$, then T has a unique fixed point in X". This result was further generalized and extended in various ways by many authors. On the other hand Sessa [ ] defined weak commutativity as follows: The mappings f and g are said to be weakly commuting if $d(fgx, gfx) \leq d(gx, fx)$ for all $x \in X$. further, Jungck [] introduced more generalized commutativity, so called compatibility, which is more general than that of weak commutativity. Let f and g be self mappings of a metric space (X, d). The mapping f and g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. clearly commuting, weakly commuting mappings are compatible but neither implication is reversible (see[]). Many authors have obtained a lot of fixed point theorems for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of mappings.

In 2002, A. Branciari [1] analyzed the existence of fixed point for mapping f defined on a complete metric space (X, d) satisfying a general contractive condition of integral type.

***Theorem 1.1** Let (X, d) be a complete metric space, $c \in (0,1)$ and let $f: X \to X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \xi(t)\ dt \leq c \int_0^{d(x,y)} \xi(t)\ dt \qquad 1.1$$

Where $\xi: [0,+\infty) \to [0,+\infty)$ *is a Lesbesgue integrable mapping which is summable on each compact subset of* $[0,+\infty)$*, non negative, and such that for each* $\epsilon > o, \int_0^{\epsilon} \xi(t)\ dt$ *, then* f *has a unique fixed point* $a \in X$ *such that for each* $x \in X$, $\lim_{n\to\infty} f^n x = a$.

After the paper of Branciari, a lot of a research works have been carried out on generalizing contractive conditions of integral type for a different contractive mapping satisfying various known properties. A fine work has been done by Rhoades [9] extending the result of Branciari by replacing the condition [1.1] by the following

$$\int_0^{d(fx,fy)} \xi(t)\ dt \leq \int_0^{\max\left\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\right\}} \xi(t)\ dt \qquad 1.2$$

In 2018, Karapinar [9] published a new type of contraction obtained from the definition of the Kannan contraction by interpolation as follows.
Let (X, d) be a metric space. A self-mapping $T: X \to X$ is said to be an interpolative Kannan type contraction if there are two constants $\tau, \beta \in (0,1)$ such that

$$d(Tx, Ty) \leq \tau\left(d(x, Tx)^{\beta}\right)\left(d(y, Ty)\right)^{1-\beta}$$

for all $x, y \in X$ with $x \neq Tx$ and $y \neq Ty$.
Karapinar obtained the following result.
Theorem 1.2: Let (X, d) be a complete metric space and $T: X \to X$ be an interpolative Kannan-type contraction mapping. Then T has a unique fixed point.
Fixed point theory in metric spaces using interpolative contractive mapping has greatly developed in recent times. In this article we prove fixed point results satisfying interpolative type integral contractive condition in metric spaces.

2. *MAIN RESULTS*

The purpose of this paper is to prove fixed point theorem by using rational contraction, Rhoades fixed point theorem [10], and Branciari result[1] to compatible maps.

Theorem 2.1 :- Let f *and* g *be compatible self maps of a complete metric space* (X, d)*satisfying the following conditions:*

$$f(X) \subset g(X),\ g \text{ is continuous},$$

$$\int_0^{d(fx,fy)} \xi(t)\ dt \leq \beta \int_0^{\left(\max\left\{d(gx,fx),d(gy,fy),\frac{d(gx,fy)+d(gy,fx)}{2}\right\}\right)^{\alpha}.\left(d(gx,gy)\right)^{1-\alpha}} \xi(t)\ dt \qquad 2.1$$

For each $x, y \in X$ *with non negative reals* $\alpha, \beta \in (0,1)$ *and* $\xi : \mathcal{R}^+ \to \mathcal{R}^+$ *is a lesbesgue- integrable mapping which is summable on each compact subset of* $\mathcal{R}^+$*, non negative, and such that*

$$for\ each\ \epsilon > 0, \int_0^{\epsilon} \xi(t)\ dt \qquad 2.2$$

Then f *and* g *have a unique common fixed point* $\in X$.

Prove: Let $x_0 \in X$. *since* $f(X) \subset g(X)$, *choose* $x_1 \in X$ *such that* $gx_1 = fx_0$. *In general, we construct a sequence*x_{n+1} *of element of* X *such that* $y_n = gx_{n+1} = fx_n$ *for* $n = 0,1,2,3, \ldots ..$

For each integer $n \geq 1$ *from 2.1*

$$\int_0^{d(y_n,y_{n+1})} \xi(t)\, dt = \int_0^{d(fx_n,fx_{n+1})} \xi(t)\, dt$$

$$\int_0^{d(fx_n,fx_{n+1})} \xi(t)\, dt \leq \beta \int_0^{\left(\max\left\{\substack{d(gx_n,fx_n),d(gx_{n+1},fx_{n+1}) \\ ,\frac{d(gx_n,fx_{n+1})+d(gx_{n+1},fx_n)}{2}}\right\}\right)^{\alpha}.(d(gx_n,gx_{n+1}),)^{1-\alpha}} \xi(t)\, dt$$

$$\int_0^{d(y_n,y_{n+1})} \xi(t)\, dt \leq \beta \int_0^{\left(\max\left\{\substack{d(y_{n-1},y_n),d(y_n,y_n) \\ ,\frac{d(y_{n-1},y_n)+d(y_n,y_n)}{2}}\right\}\right)^{\alpha}.(d(y_{n-1},y_n))^{1-\alpha}} \xi(t)\, dt$$

$$\int_0^{d(y_n,y_{n+1})} \xi(t)\, dt \leq \beta \int_0^{d(y_{n-1},y_n)^{\alpha}.d(y_{n-1},y_n)^{1-\alpha}} \xi(t)\, dt$$

$$\int_0^{d(y_n,y_{n+1})} \xi(t)\, dt \leq \beta \int_0^{d(y_{n-1},y_n)} \xi(t)\, dt \qquad 2.3$$

In this way we can write,

$$\int_0^{d(y_n,y_{n+1})} \xi(t)\, dt \leq \beta^n \int_0^{d(y_0,y_1)} \xi(t)\, dt \qquad 2.4$$

Since $\beta < 1$ *, and as* $n \to \infty$*, we have*

$$\lim_{n\to\infty} \int_0^{d(y_n,y_{n+1})} \xi(t)\, dt = 0 \qquad 2.5$$

We now show that $\{y_n\}$ *is a Cauchy sequence. Suppose that it is not. Then there exists an* $\epsilon > 0$ *and subsequence* $\{m(p)\}$ *and* $\{n(p)\}$ *such that* $m(p) < n(p) < m(p+1)$ *with*

$$d(y_{m(p)}, y_{n(p)}) \geq \epsilon, \quad d(y_{m(p)}, y_{n(p)-1}) < \epsilon \qquad 2.6$$

Now

$$d(y_{m(p)-1}, y_{n(p)-1}) < d(y_{m(p)-1}, y_{m(p)}) + d(y_{m(p)}, y_{n(p)-1})$$

$$d(y_{m(p)-1}, y_{n(p)-1}) < d(y_{m(p)-1}, y_{m(p)}) + \epsilon \qquad 2.7$$

Hence

$$\lim_{p\to\infty} \int_0^{d(y_{m(p)-1},y_{n(p)-1})} \xi(t)\, dt = \int_0^{\epsilon} \xi(t)\, dt \qquad 2.8$$

Using (2.3), (2.6), *and* (2.8) *we get*

$$\int_0^{\epsilon} \xi(t)\, dt \leq \int_0^{d(y_{m(p)},y_{n(p)})} \xi(t)\, dt \leq \beta \int_0^{d(y_{m(p)-1},y_{n(p)-1})} \xi(t)\, dt \leq \beta \int_0^{\epsilon} \xi(t)\, dt$$

Which is contradiction, since $q \in (0,1)$*. therefore* $\{y_n\}$ *is a Cauchy, hence converges to*

$z \in X$ *from* 2.1, *we get*

$$\int_0^{d(fz,fx_n)} \xi(t)\, dt \leq \beta \int_0^{\left(\max\left\{ d(gz,fz),d(gx_n,fx_n),\frac{d(gz,fx_n)+d(gx_n,fz)}{2}\right\}\right)^{\alpha}.(d(gz,gx_n))^{1-\alpha}} \xi(t)\, dt$$

Taking limit as $n \to \infty$*, we get*

$$\int_0^{d(fz,z)} \xi(t)\ dt \ \leq \beta \int_0^{\left(\max\left\{d(gz,z), d(z,z), \frac{d(gz,z)+d(z,fz)}{2}\right\}\right)^{\alpha} . (d(gz,z))^{1-\alpha}} \xi(t)\ dt$$

Which implies $fz = z$ *and* $gz = z$.

Now we show that z *is a common fixed point of* f *and* g*. since* f *and* g *are compatible, therefore,*

$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$*, which since* $\lim_{n\to\infty} gfx_n = gz$ *implies that* $\lim_{n\to\infty} fgx_n = gz$.

Now from 2.1,

$$\int_0^{d(fgx_n,gx_n)} \xi(t)\ dt \ \leq \beta \int_0^{\left(\max\left\{d(ggx_n,fgx_n), d(gx_n,gx_n), \frac{d(ggx_n,gx_n)+d(gx_n fgx_n)}{2}\right\}\right)^{\alpha} . (d(ggx_n,gx_n))^{1-\alpha}} \xi(t)\ dt$$

Taking lim *as* $n \to \infty$ *we obtain* $z = gz$.

Again from 2.1, we can show that, $z = fz$ *and hence* z *is common fixed point of* f *and* g *in* X.

Uniqueness,

Let us w *is another fixed point of* f *and* g *in* X *different from* z *i.e.* $z \neq w$*, then from 2.1 we have,*

$$\int_0^{d(fw,fz)} \xi(t)\ dt \ \leq \beta \int_0^{\left(\max\left\{d(gw,fw), d(gz,fz), \frac{d(gw,fz)+d(gz,fw)}{2}\right\}\right)^{\alpha} . (d(gw,gz))^{1-\alpha}} \xi(t)\ dt$$

$$\int_0^{d(fw,fz)} \xi(t)\ dt \ \leq \beta \int_0^{(d(gw,gz))^{\alpha} . (d(gw,gz))^{1-\alpha}} \xi(t)\ dt$$

$$\int_0^{d(w,z)} \xi(t)\ dt \ \leq \beta \int_0^{d(w,z)} \xi(t)\ dt$$

Which contradiction. So that, z *is unique common fixed point of* f *and* g.

We also extend and generalize the theorem of Branciari for a pair of compatible mappings. In a similar we can generalize order results related to contractive conditions of same kind.

We prove our next theorem by using rational contraction in integral type mapping. In fact our next result is as follows,

Theorem 2.2: Let f *and* g *be compatible self maps of a complete metric space* (X, d)*satisfying the following conditions:*

$$f(X) \subset g(X), \ g \text{ is continuous},$$

$$\int_0^{d(fx,fy)} \xi(t)\ dt \ \leq a \int_0^{\left(\max\left\{\begin{matrix}\frac{[d(gx,fx).d(gy,fy)]}{d(gx,gy)}, \frac{[d(gx,fy)d(gy,fx)]}{d(gx,gy)}, \\ \frac{[d(gx,fx).d(gx,fy)]}{d(gx,gy)}, \frac{[d(gy,fx).d(gy,fy)]}{d(gx,gy)},\end{matrix}\right\}\right)^{\alpha} . (d(gx,gy))^{1-\alpha}} \xi(t)\ dt \qquad 2.9$$

For each $x, y \in X$ *with non negative reals* $\alpha, \beta \in (0,1)$ *and* $\xi : \mathcal{R}^+ \to \mathcal{R}^+$ *is a lesbesgue- integrable mapping which is summable on each compact subset of* $\mathcal{R}^+$*, non negative, and such that*

$$for\ each\ \epsilon > 0, \int_0^{\epsilon} \xi(t)\ dt \qquad 2.10$$

Then f and g have a unique common fixed point $\in X$.

Prove: Let $x_0 \in X$. since $f(X) \subset g(X)$, choose $x_1 \in X$ such that $gx_1 = fx_0$. In general, we construct a sequence x_{n+1} of element of X such that $y_n = gx_{n+1} = fx_n$ for $n = 0,1,2,3, \ldots$.

For each integer $n \geq 1$ from 2.9

$$\int_0^{d(y_n,y_{n+1})} \xi(t)\, dt = \int_0^{d(fx_n,fx_{n+1})} \xi(t)\, dt$$

$$\int_0^{d(fx_n,fx_{n+1})} \xi(t)\, dt \leq \beta \int_0^{\left(\max\left\{\frac{[d(gx_n,fx_n).d(gx_{n+1},fx_{n+1})]}{d(gx_n,gx_{n+1})}, \frac{[d(gx_n,fx_{n+1})d(gx_{n+1},fx_n)]}{d(gx_n,gx_{n+1})}, \frac{[d(gx_n,fx_n).d(gx_n,fx_{n+1})]}{d(gx_n,gx_{n+1})}, \frac{[d(gx_{n+1},fx_n).d(gx_{n+1},fx_{n+1})]}{d(gx_n,gx_{n+1})},\right\}\right)^{\alpha}.(d(gx_n,gx_{n+1}))^{1-\alpha}} \xi(t)\, dt$$

$$\int_0^{d(y_n,fx_{n+1})} \xi(t)\, dt \leq \beta \int_0^{\left(\max\left\{\frac{[d(y_{n-1},y_n).d(y_n,y_{n+1})]}{d(y_{n-1},y_n)}, \frac{[d(y_{n-1},y_{n+1})d(y_n,y_n)]}{d(y_{n-1},y_n)}, \frac{[d(y_{n-1},y_n).d(y_{n-1},y_{n+1})]}{d(y_{n-1},y_n)}, \frac{[d(y_n,y_n).d(y_n,y_{n+1})]}{d(y_{n-1},y_n)},\right\}\right)^{\alpha}.(d(y_{n-1},y_n))^{1-\alpha}} \xi(t)\, dt$$

$$\int_0^{d(y_n,fx_{n+1})} \xi(t)\, dt \leq \beta \int_0^{(d(y_{n-1},y_n))^{\alpha}.(d(y_{n-1},y_n))^{1-\alpha}} \xi(t)\, dt$$

$$\int_0^{d(y_n,fx_{n+1})} \xi(t)\, dt \leq \beta \int_0^{d(y_{n-1},y_n)} \xi(t)\, dt \qquad 2.11$$

In this way we can write,

$$\int_0^{d(y_n,y_{n+1})} \xi(t)\, dt \leq \beta^n \int_0^{d(y_0,y_1)} \xi(t)\, dt \qquad 2.12$$

as $n \to \infty$, we have

$$\lim_{n\to\infty} \int_0^{d(y_n,y_{n+1})} \xi(t)\, dt = 0 \qquad 2.13$$

We now show that $\{y_n\}$ is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ and subsequence $\{m(p)\}$ and $\{n(p)\}$ such that $m(p) < n(p) < m(p+1)$ with

$$d\big(y_{m(p)}, y_{n(p)}\big) \geq \epsilon, \quad d\big(y_{m(p)}, y_{n(p)-1}\big) < \epsilon \qquad 2.14$$

Now

$$d\big(y_{m(p)-1}, y_{n(p)-1}\big) < d\big(y_{m(p)-1}, y_{m(p)}\big) + d\big(y_{m(p)}, y_{n(p)-1}\big)$$

$$d\big(y_{m(p)-1}, y_{n(p)-1}\big) < d\big(y_{m(p)-1}, y_{m(p)}\big) + \epsilon \qquad 2.15$$

Hence

$$\lim_{p\to\infty} \int_0^{d(y_{m(p)-1},y_{n(p)-1})} \xi(t)\, dt = \int_0^{\epsilon} \xi(t)\, dt \qquad 2.16$$

Using (2.9), (2.13), and (2.15) we get

$$\int_0^{\epsilon} \xi(t)\ dt \le \int_0^{d(y_{m(p)},y_{n(p)})} \xi(t)\ dt \le \beta \int_0^{d(y_{m(p)-1},y_{n(p)-1})} \xi(t)\ dt \le \beta \int_0^{\epsilon} \xi(t)\ dt$$

Which is contradiction, since $a \in (0,1)$. *therefore* $\{y_n\}$ *is a Cauchy, hence converges to*

$z \in X\ from\ 2.9,\ \ we\ get$

$$\int_0^{d(fz,fx_n)} \xi(t)\ dt \le \beta \int_0^{\left(\max\left\{\frac{[d(gz,fz).d(gx_n,fx_n)]}{d(gz,gx_n)},\frac{[d(gz,fx_n)d(gx_n,fz)]}{d(gz,gx_n)},\frac{[d(gx,fx).d(gx,fx_n)]}{d(gz,gx_n)},\frac{[d(gx_n,fz).d(gx_n,fx_n)]}{d(gz,gx_n)}\right\}\right)^{\alpha}.(d(gz,gx_n))^{1-\alpha}} \xi(t)\ dt$$

Taking limit as $n \to \infty$, *we get,* $fz = z$ *and* $gz = z$.

Now we show that z is a common fixed point of f *and* g. *since* f *and* g *are compatible, therefore,*

$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, *which since* $\lim_{n\to\infty} gfx_n = gz$ *implies that* $\lim_{n\to\infty} fgx_n = gz$.

Now from 2.9,

$$\int_0^{d(fgx_n,gx_n)} \xi(t)\ dt \le \beta \int_0^{\left(\max\left\{\frac{[d(ggx_n,fgx_n).d(gx_n,gx_n)]}{d(ggx_n,gx_n)},\frac{[d(gx,fy)d(gy,fx)]}{d(ggx_n,gx_n)},\frac{[d(ggx_n,fgx_n).d(ggx_n,gx_n)]}{d(ggx_n,gx_n)},\frac{[d(gx_n,fgx_n).d(gx_n,gx_n)]}{d(ggx_n,gx_n)}\right\}\right)^{\alpha}(d(ggx_n,gx_n))^{1-\alpha}} \xi(t)\ dt\ xce$$

Taking limit as $n \to \infty$, *we get*

$$\int_0^{d(fz,z)} \xi(t)\ dt \le \beta \int_0^{\left(\max\left\{\frac{[d(gz,fz).d(z,z)]}{d(gz,z)},\frac{[d(gz,z)d(z,fz)]}{d(gz,z)},\frac{[d(gz,fz).d(gz,z)]}{d(gz,z)},\frac{[d(z,fz).d(z,z)]}{d(gz,z)}\right\}\right)^{\alpha}.(d(gz,z))^{1-\alpha}} \xi(t)\ dt$$

Taking $\lim$ *as* $n \to \infty$ *we obtain* $z = gz$.

Again from 2.1, we can show that, $z = fz$ *and hence* z *is common fixed point of* f *and* g *in* X.

Uniqueness,

Let us w is another fixed point of f *and* g *in X different from z* $i.e.$ $z \neq w$, *then from 2.1 we have,*

$$\int_0^{d(fw,fz)} \xi(t)\ dt \le \beta \int_0^{\left(\max\left\{\frac{[d(gw,fw).d(gy,fz)]}{d(gw,gz)},\frac{[d(gw,fz)d(gz,fw)]}{d(gw,gz)},\frac{[d(gw,fw).d(gw,fz)]}{d(gw,gz)},\frac{[d(gz,fw).d(gz,fz)]}{d(gw,gz)}\right\}\right)^{\alpha}.(d(gw,gz))^{1-\alpha}} \xi(t)\ dt$$

$$\int_0^{d(w,z)} \xi(t)\ dt \le \beta \int_0^{d(w,z)} \xi(t)\ dt$$

Which contradiction. So that, z is unique common fixed point of f *and* g.

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