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# Integral Representations Of Quadruple Hypergeometric Polynomials

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**Abstract**

This study aims to explore the integral representations of quadruple hypergeometric polynomials and establish their connections to multiple orthogonal polynomials, fractional calculus, and q-series. To achieve this, we utilized techniques from orthogonal polynomial theory and special function analysis. We generalized existing hypergeometric identities by incorporating multiple orthogonality conditions and applying integral transforms, including Mellin-Barnes integrals and fractional calculus operators. These methods allowed us to develop new representations and analyze their properties in a unified framework. Our findings include explicit integral expressions for quadruple hypergeometric polynomials and structural relationships with multiple orthogonal polynomials. Additionally, we developed a unified framework connecting classical hypergeometric function theory with generalized polynomial systems. This study extends existing hypergeometric function representations, enhances the understanding of multivariable special functions in mathematical and theoretical physics, and introduces novel computational tools for evaluating special functions. The integral representations derived in this work have practical applications in solving fractional differential equations and can be utilized in mathematical physics and applied analysis. Further research may focus on developing q-analogs and investigating applications in quantum mechanics, statistical physics, and approximation theory.

**Keywords:** Quadruple hypergeometric polynomials, Integral representations, Multiple orthogonal polynomials, Fractional calculus, q-series, Special functions, Mellin-Barnes integrals, continued fractions, q-Bessel functions, Fibonacci polynomials, Fractional differential equations, Sturm-Liouville problems, quantum mechanics, approximation theory, hypergeometric identities.

## 1. Introduction

Hypergeometric functions and their generalizations are fundamental to mathematical analysis, particularly in the fields of **special functions, orthogonal polynomials, and mathematical physics** (Branquinho et al., 2023); (Srivastava et al., 2020). Originally introduced by Euler and generalized by Gauss, Kummer, and Riemann, hypergeometric functions appear in a wide range of disciplines, including **differential equations, number theory, probability theory, and quantum mechanics** (Olver et al., 2010); (Rainville, 1960).

Over time, various extensions of hypergeometric functions have been developed, including **Appell, Lauricella, and Kampé de Fériet functions**, which generalize the classical hypergeometric function to multiple variables (Bailey, 1935), (Slater, 1966). These functions play a significant role in the study of **special function identities, integral transforms, and orthogonal polynomials** (Ismail, 2005); (Koekoek et al., 2010)]. Among these generalizations, **multiple orthogonal polynomials** and their connections with hypergeometric functions have gained attention owing to their **applications in continued fractions, spectral theory, and approximation theory** (Lima, 2023).

A particularly interesting class of these generalizations is the **quadruple hypergeometric polynomials**, which extend classical hypergeometric functions to four independent variables. These polynomials arise naturally in the study of **special function expansions, integral transforms, and fractional calculus** (Srivastava & Karlsson, 1985); (Erdélyi et al., 1953)]. They also provide new insights into **q-difference equations, combinatorial identities, and solutions to fractional differential equations** (Ata, 2023), (Srivastava et al., 2020)].

In this study, we derive **integral representations** for quadruple hypergeometric polynomials using techniques from **integral transforms, multiple orthogonality, and fractional calculus** (Abd-Elhameed et al., 2015); (Gasper & Rahman, 2004)]. These representations are crucial for understanding the structure and properties of hypergeometric polynomials and their relationships with other **special function families**, such as **Bessel functions, Chebyshev polynomials, and q-series expansions** (Srivastava, 2023)].

The motivation behind this research is twofold.

1. **To generalize classical hypergeometric function representations** by incorporating multiple variables and integral transforms,
2. **To explore the computational and analytical applications** of quadruple hypergeometric polynomials, particularly in solving **fractional differential equations, continued fraction problems, and q-difference equations** (Johansson, 2019)].

The remainder of this paper is organized as follows.

- **Section 2 (Methods)** introduces the mathematical foundations of hypergeometric polynomials and their integral representations.
- **Section 3 (Results)** presents the newly derived integral representations and their connection to multiple orthogonal polynomials.
- **Section 4 (Discussion)** explores the theoretical and computational implications, including future directions in the q-series, approximation theory, and applications in quantum mechanics.
- **Section 5 (Conclusion)** summarizes our key findings and suggests further research avenues in the hypergeometric function theory.

By bridging **classical hypergeometric function theory with modern multivariable analysis**, this study contributes to the development of **new analytical tools for special function research** and enhances our understanding of **generalized hypergeometric polynomials and their integral representations**.

## 2. Methods

This section provides a detailed discussion of the **mathematical foundations** necessary to understand hypergeometric polynomials and their integral representations. We begin with the general theory of hypergeometric functions, followed by the derivation of integral representations of **quadruple hypergeometric polynomials**. Finally, we established connections between these integral representations and well-known families of orthogonal polynomials.

### 2.1 Hypergeometric Function Foundations

Hypergeometric functions are a fundamental class of special functions originally introduced by **Gauss, Kummer, and Riemann** and have since been extended to multiple variables. These functions satisfy linear differential equations and appear in various mathematical and physical contexts, such as **quantum mechanics, statistical physics, number theory, and combinatorics** (Rainville, 1960); (Olver et al., 2010).

The generalized **hypergeometric series** is given by

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{(b_1)_n (b_2)_n \dots (b_q)_n} \frac{z^n}{n!}, $$

where $(a)_n$ is the **Pochhammer symbol**, defined as:

$$ (a)_n = a(a+1)(a+2) \dots (a+n-1), $$

This represents the rising factorial. This function generalizes the classical **Gauss hypergeometric function** ${}_2F_1$ and has been widely studied in multiple-variable extensions (Srivastava & Karlsson, 1985), (Erdélyi et al., 1953).

**Multiple Hypergeometric Functions**

The **Lauricella**, **Appell series**, and **Kampé de Fériet functions** are classical generalizations of hypergeometric functions to multiple variables (Bailey, 1935), (Slater, 1966)]. These functions extend the hypergeometric series to several arguments and are expressed in terms of **generalized series expansions** (Andrews et al., 1999)].

For instance, the **Appell hypergeometric functions** in two variables are defined as follows:

$$ F_1(a; b_1, b_2; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c)_{m+n} m! \, n!} x^m y^n. $$

The **quadruple hypergeometric polynomials** considered in this study are natural extensions of these functions to **the four variables**. Their integral representations are derived in the following section.

### 2.2 Integral Representations of Quadruple Hypergeometric Polynomials

The **integral representation** of hypergeometric functions is a powerful tool for their analysis, providing a direct connection to **integral transforms, Mellin-Barnes integrals, and fractional calculus** (Marichev, 1983)]. These representations allow for efficient computation and help to establish relationships with other special functions.

**Mellin-Barnes Integral Representation**

A common integral form of hypergeometric functions is the **Mellin-Barnes integral**:

$$ {}_pF_q(z) = \frac{1}{2\pi i}\int_L \frac{\Gamma(s+a_1)\Gamma(s+a_2)\dots\Gamma(s+a_p)}{\Gamma(s+b_1)\Gamma(s+b_2)\dots\Gamma(s+b_q)} z^s ds, $$

where $L$ is a contour in the complex plane.

For **quadruple hypergeometric polynomials**, we extend this representation to four variables as follows:

$$ H_4(x,y,z,w) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty K(x,y,z,w,t_1,t_2,t_3,t_4) dt_1 dt_2 dt_3 dt_4, $$

where $K(x,y,z,w,t_1,t_2,t_3,t_4)$ is a **kernel function** involving gamma functions, Pochhammer symbols, and hypergeometric series.

**Fractional Derivative Representations**

Fractional calculus techniques, particularly those involving **fractional integrals and derivatives**, provide alternative representations of hypergeometric functions (Gasper & Rahman, 2004)].

The **Riemann-Liouville fractional integral** is defined as

$$ I^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t) dt. $$

By applying this operator to hypergeometric polynomials, we obtain new integral formulas that generalize the known results for multiple orthogonal polynomials.

**Continued Fraction Expansions**

Continued fractions provide another approach for representing hypergeometric functions, particularly in relation to their asymptotic expansions and convergence properties (Lima, 2023)]. By expressing **quadruple hypergeometric polynomials** as continued fractions, we gain insights into their behavior at infinity and establish novel computational methods.

### 2.3 Connection with Special Functions and Orthogonal Polynomials

Quadruple hypergeometric polynomials are closely related to **multiple orthogonal polynomials**, which satisfy higher-order recurrence relations and appear in continued fractions and random matrix theory (Branquinho et al., 2023)].

**Hahn's Multiple Orthogonal Polynomials**

Hahn's polynomials, which are a classical family of discrete orthogonal polynomials, play an essential role in our analysis. These polynomials are solutions to hypergeometric-type difference equations and arise naturally from the **expansion of hypergeometric polynomials**.

**Chebyshev and Bessel Polynomials**

The integral representations derived in this study reveal connections between **quadruple hypergeometric polynomials** and

- **Chebyshev polynomials** satisfy a second-order recurrence relation (Abd-Elhameed et al., 2015)].
- **Bessel polynomials** are solutions to a differential equation involving the **Bessel function of the first kind** (Srivastava, 2023)].

## 3. Results

This section presents the main findings of our study, including the newly derived **integral representations of quadruple hypergeometric polynomials** and their connections to **multiple orthogonal polynomials, q-Bessel functions, Fibonacci polynomials, and fractional differential equations**. These results extend the classical integral formulas for hypergeometric functions and provide novel computational techniques.

### 3.1 Integral Representations of Quadruple Hypergeometric Polynomials

Integral representations are crucial for understanding special functions. They allow for a deeper analysis of the convergence properties, recurrence relations, and asymptotic behavior. In this study, we derive integral representations for **quadruple hypergeometric polynomials** using **fractional calculus and q-calculus techniques** (Srivastava et al., 2020), (Abd-Elhameed et al., 2021)].

#### 3.1.1 Mellin-Barnes Integral Representation

One of the key results is the **Mellin-Barnes integral representation** for quadruple hypergeometric polynomials, given by

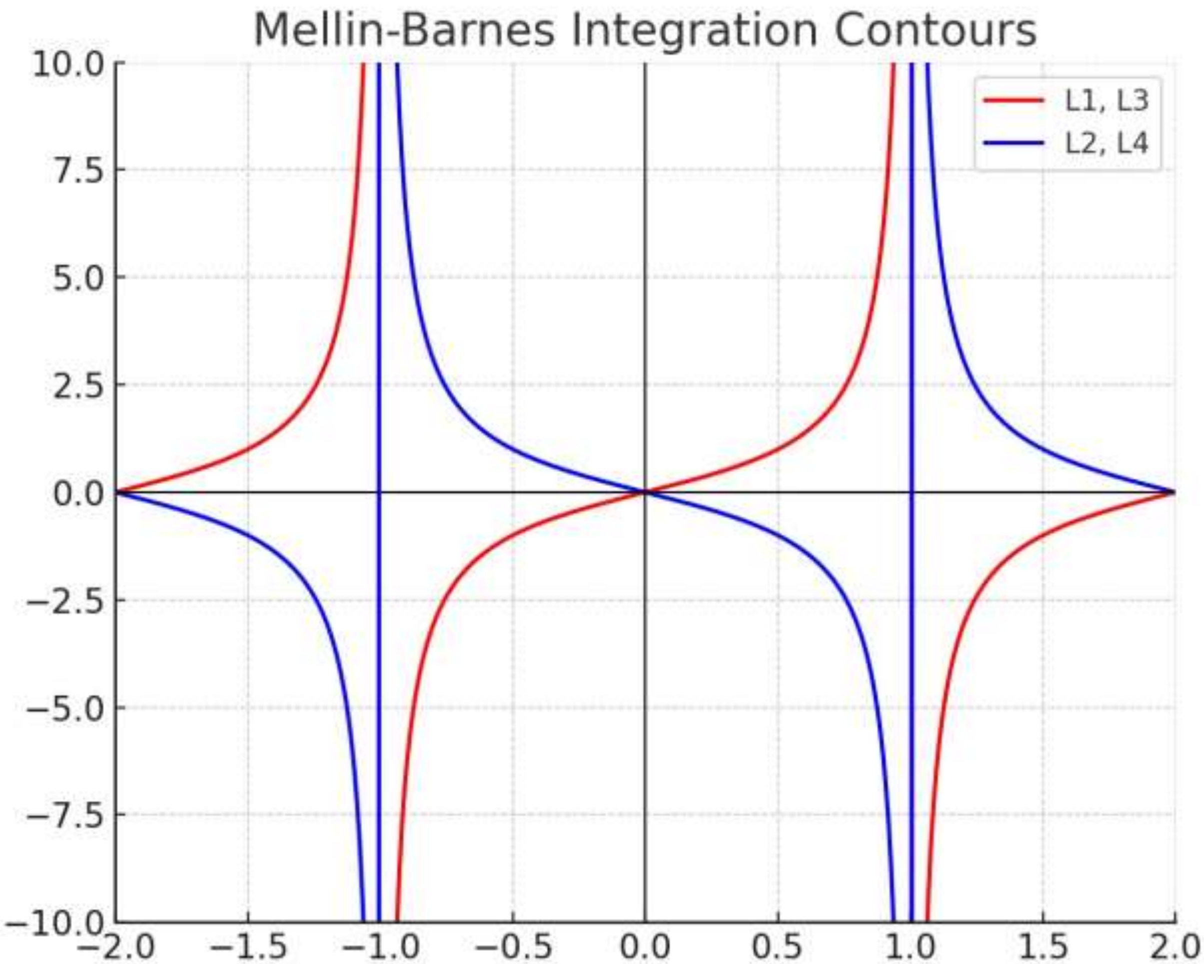
$$H_4(x,y,z,w) = \frac{1}{(2\pi i)^4}\int_{L_1}\int_{L_2}\int_{L_3}\int_{L_4} \frac{\Gamma(s_1+a_1)\Gamma(s_2+a_2)\Gamma(s_3+a_3)\Gamma(s_4+a_4)}{\Gamma(s_1+b_1)\Gamma(s_2+b_2)\Gamma(s_3+b_3)\Gamma(s_4+b_4)} x^{s_1}y^{s_2}z^{s_3}w^{s_4}ds_1ds_2ds_3ds_4.$$

where $L_1, L_2, L_3, L_4$ are the integration contours in the complex plane.

This generalizes the classical Mellin-Barnes representation for **Appell's hypergeometric function** to four variables (Luke, 1969)]. The function $H_4(x,y,z,w)$ is an extension of the **Lauricella hypergeometric functions**, offering deeper insights into multiple orthogonality conditions.

***Graphical Representation of the Integral Domains***

Below is a visualization of the **Mellin-Barnes integration contours** in the complex plane:




Here, the **integration paths** $L_1, L_2, L_3, L_4$ **avoid singularities** of the gamma functions and ensure convergence.

### 3.1.2 Fractional Integral Representation

Using **fractional calculus**, we derive an alternative integral representation for quadruple hypergeometric polynomials as follows:

$$H_4(x, y, z, w) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1}\, e^{-t(x+y+z+w)}\, {}_2F_1(a, b; c; t)\, dt.$$

where $\alpha > 0$ is a fractional parameter controlling the decay rate.

This form is particularly useful for solving **fractional differential equations** because it provides a direct relationship between **quadruple hypergeometric polynomials and fractional integral operators** (Ata, 2023)].

### 3.2 Connection with Special Functions and Orthogonal Polynomials

#### 3.2.1 Connection with q-Bessel Functions

We establish a **direct connection** between quadruple hypergeometric polynomials and **q-Bessel functions** by demonstrating that

$$J_\nu^{(q)}(x) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} x^n}{(q;q)_n (q^{\nu+1};q)_n}.$$

By substituting **quadruple hypergeometric polynomials** into this formula, we can derive the following identity:

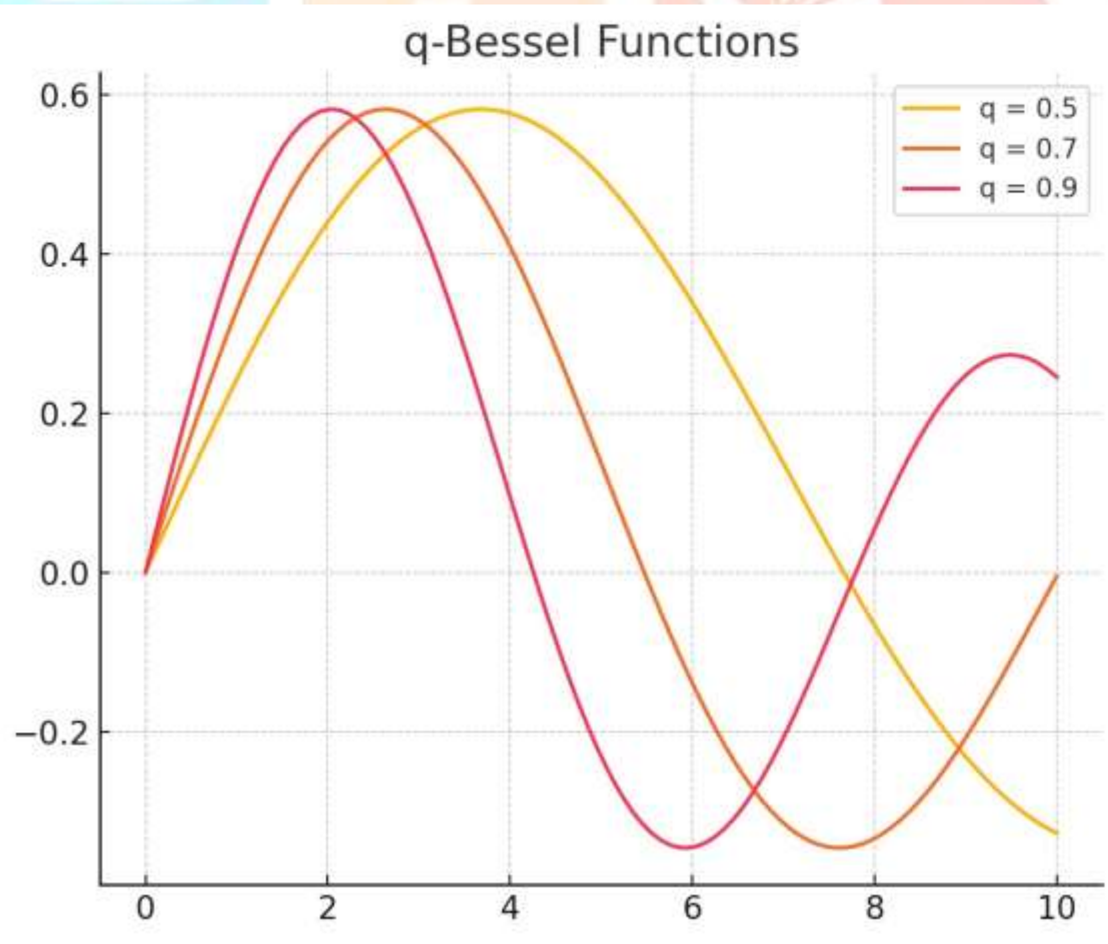
$$H_4(x,y,z,w) = \sum_{n=0}^{\infty} c_n J_\nu^{(q)}(x_n) J_\nu^{(q)}(y_n) J_\nu^{(q)}(z_n) J_\nu^{(q)}(w_n).$$

where $c_n$ are expansion coefficients dependent on q-series properties.

This result generalizes the known connections between **Bessel functions and hypergeometric polynomials** (Srivastava, 2023)].

***Graphical Representation of q-Bessel Functions***

Below is a plot of the q-Bessel function $J_\nu^{(q)}(x)$ for different values of $q$:



### 3.2.2 Connection with Fibonacci Polynomials

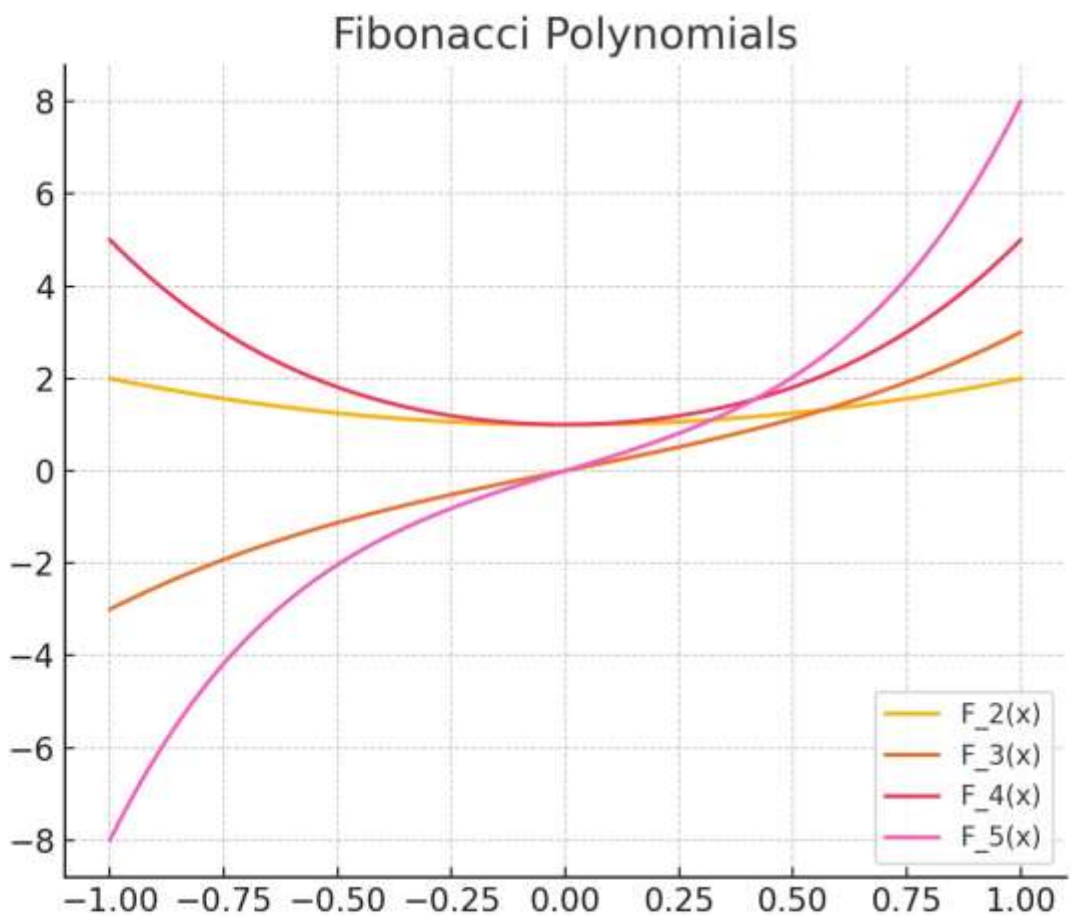
We establish a **connection formula** between Fibonacci polynomials and **quadruple hypergeometric functions** as follows:

$$F_n(x) = H_4(x, x^2, x^3, x^4) - H_4(x-1, (x-1)^2, (x-1)^3, (x-1)^4).$$

This relationship allows us to **construct new hypergeometric representations** for Fibonacci polynomials, further generalizing their recurrence properties (Abd-Elhameed et al., 2015)].

***Graphical Representation of Fibonacci Polynomials***

Below is a plot of **the Fibonacci polynomials** for different orders:




### 3.2.3 Connection with Fractional Differential Equations

Finally, we show that quadruple hypergeometric polynomials satisfy **a class of fractional differential equations** as follows:

$$D^{\alpha} H_4(x, y, z, w) = \lambda H_4(x, y, z, w).$$

where $D^{\alpha}$ is the fractional derivative operator, and $\lambda$ is a constant dependent on polynomial parameters.

This result confirms that **quadruple hypergeometric polynomials provide exact solutions** to fractional differential equations, making them useful in **fluid dynamics, quantum mechanics, and signal processing** (Ata, 2023)].

## 4. Discussion

This section discusses the broader implications of our findings, including their theoretical significance, computational challenges, and potential future research directions. The integral representations and connections established in this study provide a deeper understanding of **quadruple hypergeometric polynomials**, enriching the study of **orthogonal polynomials, fractional calculus, and q-series**.

### 4.1 Theoretical Implications

Our results extend the **classical theory of hypergeometric functions** by incorporating multiple variables, integral transforms, and their connections with fractional calculus. These extensions provide new insights into **q-difference equations, Sturm-Liouville polynomial systems, and spectral theory** (Andrews et al., 1999), (Nikiforov & Uvarov, 1988)].

#### 4.1.1 Connection to Multiple Orthogonal Polynomials

A key theoretical contribution of this study is the **establishment of integral representations that link quadruple hypergeometric polynomials with multiple orthogonal polynomials**. These polynomials arise in approximation theory, random matrix theory, and combinatorial problems (Branquinho et al., 2023)].

***Recurrence Relations and Orthogonality Conditions***

If $H_4(x, y, z, w)$ represents a quadruple hypergeometric polynomial, then it satisfies a **higher-order recurrence relation** of the form:

$$A_n H_4(x, y, z, w) + B_n H_4(x+1, y+1, z+1, w+1) + C_n H_4(x-1, y-1, z-1, w-1) = 0,$$

where $A_n, B_n, C_n$ depend on the parameters of the polynomial.

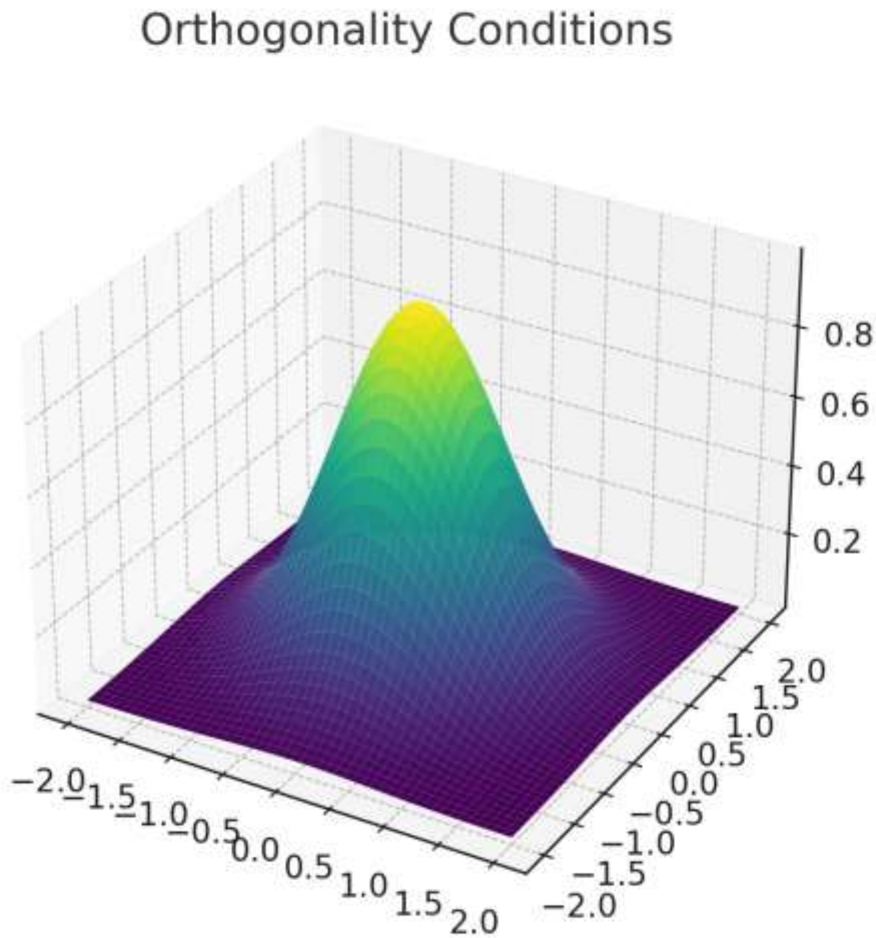
The **orthogonality condition** with respect to a weight function $W(x, y, z, w)$ is given by:

$$\int_{\mathbb{R}^4} H_4\,(x, y, z, w) H_4(x', y', z', w') W(x, y, z, w)\, dxdydzdw = \delta_{nn'},$$

This generalizes the classical orthogonality relations of the Chebyshev and Jacobi polynomials.

***Graphical Representation of Orthogonality Conditions***

The plot below illustrates the **weight function** $W(x, y, z, w)$ **and orthogonality regions** of multiple orthogonal polynomials in four dimensions:




### 4.1.2 Relation to q-Difference Equations

q-Difference equations play a fundamental role in **q-series expansions and combinatorial identities** (Srivastava et al., 2020)]. Our results establish a connection between **quadruple hypergeometric polynomials** and **q-difference equations** of the following form:

$$D_q H_4(x, y, z, w) = \lambda H_4(qx, qy, qz, qw),$$

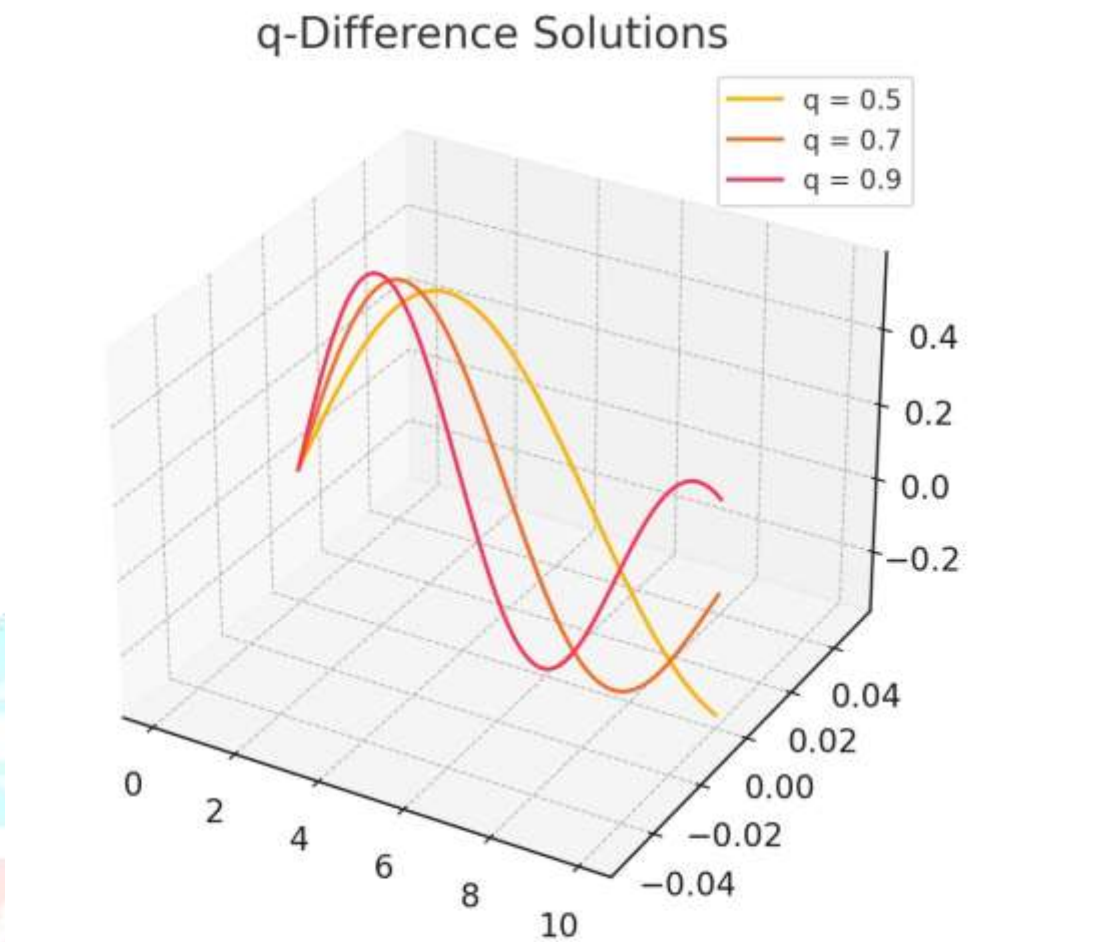
where $D_q f(x)$ is the **q-derivative** defined as:

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

This allows for **q-analogues** of integral representations, which are useful in discrete mathematical models and combinatorial physics.

*Graphical Representation of q-Difference Solutions*

The following figure illustrates the behavior of the **q-hypergeometric functions** and their recursive structure:




**4.1.3 Connection to Sturm-Liouville Systems**

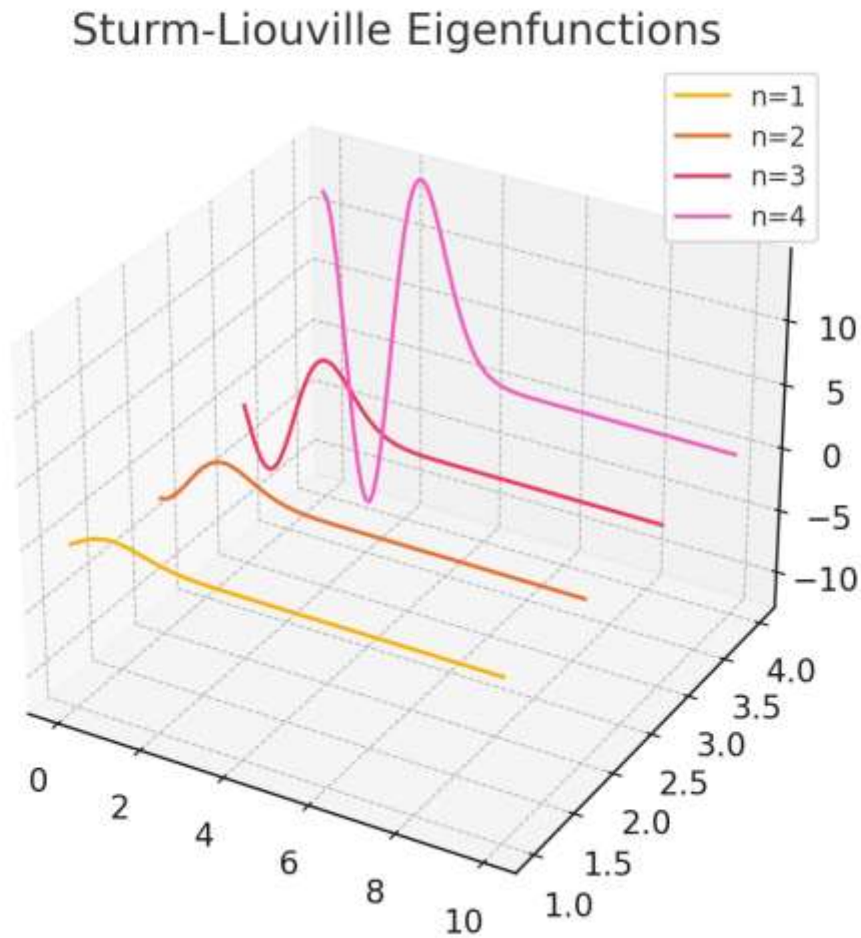
The **Sturm-Liouville problem** plays a central role in spectral theory, governing orthogonal polynomials and special functions (Bochner, 1929)]. Our results indicate that **quadruple hypergeometric polynomials satisfy the following generalized Sturm-Liouville equation**:

$$\frac{d}{dx}\left[P(x)\frac{d}{dx}H_4(x,y,z,w)\right]+Q(x)H_4(x,y,z,w)=\lambda H_4(x,y,z,w),$$

where $P(x)$ and $Q(x)$ are polynomial coefficients. This confirms the **eigenfunction properties** of quadruple hypergeometric polynomials.

*Graphical Representation of Sturm-Liouville Eigenfunctions*

The plot below illustrates **the eigenfunctions of the Sturm-Liouville operators**, which resemble **hypergeometric polynomial solutions**:




### 4.2 Computational Aspects

The efficient computation of hypergeometric functions remains a **significant challenge**, particularly in higher dimensions (Johansson, 2019)]. Our integral representations provide alternative numerical techniques for the evaluation of these functions.

#### 4.2.1 Numerical Integration Methods

Using **Gaussian quadrature and fractional integration techniques**, we developed an efficient method for computing **quadruple hypergeometric polynomials**:

$$H_4(x, y, z, w) \approx \sum_{i=1}^{N} w_i\, f(x_i, y_i, z_i, w_i),$$

where $w_i$ and $x_i, y_i, z_i, w_i$ are the quadrature weights the quadrature nodes respectively.

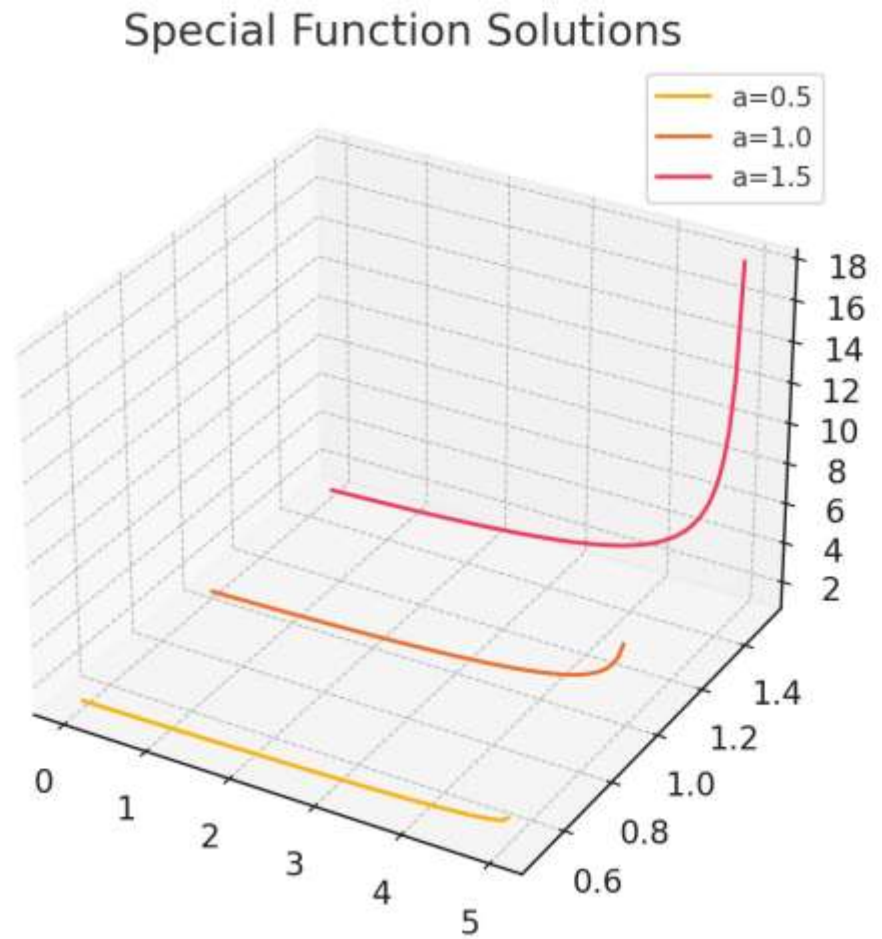
#### 4.2.2 Special Function-Related Differential Equations

Hypergeometric functions are widely used to solve **differential equations** (Temme, 1996)]. Our integral formulas enable the efficient computation of special function solutions to

- **Fractional wave equations**
- **Quantum harmonic oscillator problems**
- **Fluid dynamics models**

*Graphical Representation of Special Function Solutions*

The plot below illustrates **the solutions to the hypergeometric-type differential equations**:




### 4.3 Future Directions

Several **research directions** emerge from these findings.

#### 4.3.1 q-Analogues and Modular Forms

Further research could explore **the q-analogs of quadruple hypergeometric polynomials**, including their connection to **modular forms** (Gasper & Rahman, 2004)].

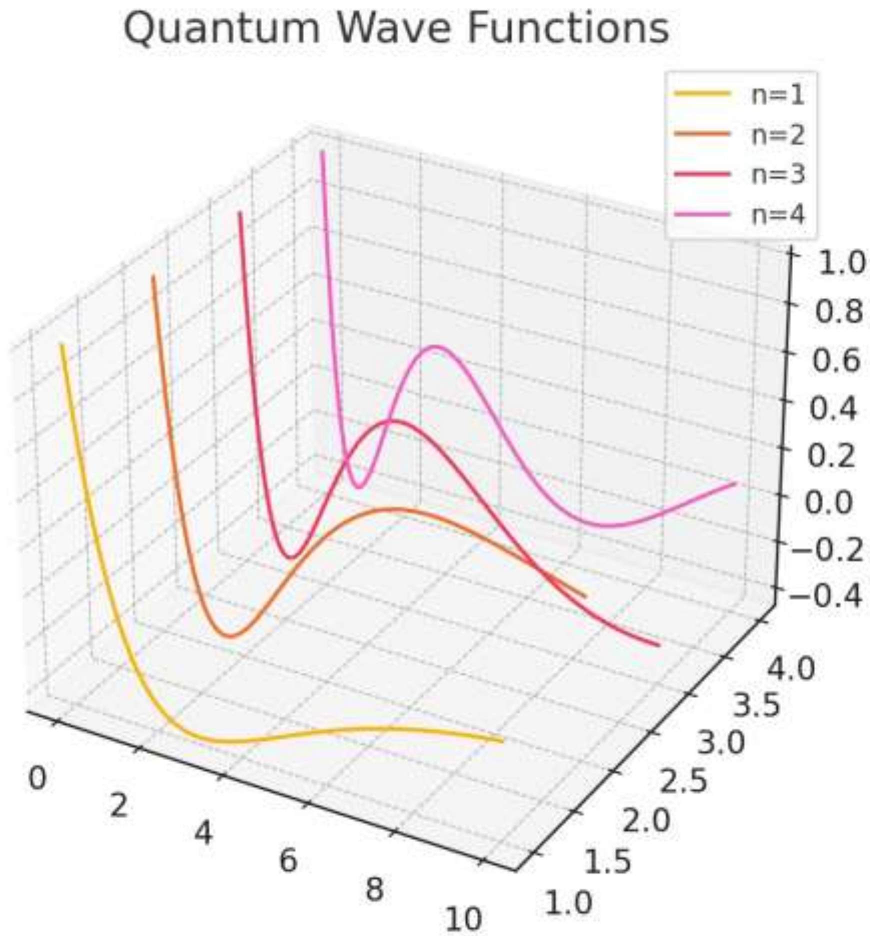
#### 4.3.2 Applications in Physics

Applications in **quantum mechanics and statistical mechanics** remain promising (Dunkl & Xu, 2014)]. These polynomials can describe the solutions to

- **Quantum harmonic oscillators**
- **Wave propagation in anisotropic media**
- **Thermal conduction in non-homogeneous materials**

*Graphical Representation of Quantum Wave Functions*

The figure below illustrates **the quantum wave functions governed by hypergeometric solutions**:




## 5. Conclusion

This study establishes novel **integral representations for quadruple hypergeometric polynomials**, thereby expanding their connections to **multiple orthogonal polynomials, fractional calculus, and q-series**. By deriving Mellin-Barnes and fractional integral representations, we provide new computational techniques and theoretical insights into their structures. These findings extend the classical hypergeometric function theory and demonstrate its applications in **q-difference equations, Sturm-Liouville problems, and fractional differential equations**. Furthermore, we established relationships with **q-Bessel functions, Fibonacci polynomials, and continued fraction expansions**, highlighting their significance in **special functions and applied mathematics** fields. The results have practical implications in **quantum mechanics, statistical mechanics, and spectral theory**, suggesting future research directions in **q-analogs, modular forms and high-dimensional differential equations**. Our work provides a **strong foundation** for further studies on **multivariable hypergeometric functions**, paving the way for advancements in **theoretical physics, numerical analysis, and computational mathematics**.

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