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Some Fixed Point Theorems InOrdered B-Metric Spaces With Auxiliary Functions

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Abstract

The main aim of these paper is to present some fixed point theorems for self mappings satisfying certain contraction which is involving an auxiliary function. Also ,the results is obtained for the existence of a common fixed point and coincidence point for generalize and ordered complete b- metric space ,Our results generalizes and extends some well-known results existing in the literature.

Keywords : Ordered b- metric space, Rational type generalized (ϕ, ψ) contraction mapping ,fixed point ,complete metric space, weakly compatible mapping

Introduction Metric fixed point theory has been the centre of expansive research for several researchers fixed point theory has become an important tool for solving many non-linear problems related to science and engineering because of its applications. The Banach contraction Principle is one of the most versatile result in fixed point theory and approximation theory. It plays an important role in solving many existing problems in pure and applied mathematics. There is a vast literature dealing with technical extended generalization of Banach contraction Principle, see, for ex [2].In recent times, fixed point of mappings in ordered metric spaces are of great use in many branches of mathematical analysis for solving non linear equations .The first result in direction was initiated by Walk [11] and later monjardet [12] in partially ordered sets. Ram and Reurings [13] Studied the existence of fixed points for certain mappings in partially ordered metric spaces and applied their results to matrix equations. The result of[13] were extended by Nieto et al. [2,3] for non-decreasing mappings and obtained the solutions of certain Partial differential equations with periodic boundary conditions. At the same time , the results regarded to generalized contractions in ordered space were studied by O' Regan et al.[5]. There have been a lot of generalizations and improvements of the results for single valued and multivalued operators in various ordered spaces with topological properties , Some of which are in [18-32] in[32], Dass and Gupta [32] proved the following fixed point result for rational contraction in a complete metric space.

Theorem 1.1: [32] Suppose (X, d) is a complete metric space, Let $S: X \to X$ be a mapping such that there exist $\alpha, \beta \epsilon [0,1]$ with $\alpha + \beta < 1$ satisfying

$$d(Sx, Sy) \leq \alpha \frac{d(y, Sy)[1 + d(x, Sx)]}{1 + d(x, y)} + \beta d(x, y)$$

For any distinct $x, y \in X$. Then S has a unique fixed point in X.
The generalization of above result in partially ordered metric space was given by Cabrera et al. [33] in 2013. Later, Chandok et al. [34] generalized the result of [33] by use of control functions in the space, again ,Theorem 1.1 was generalized by Jaggi [35] in 1977 and proved the following Theorem 1.2 [35]. Suppose (X, d) is a complete metric space. A Self maps on X such that

$$d(Sx, Sy) \leq \alpha \frac{d(y, Sy)d(x, Sx)}{d(x, y)} + \beta d(x, y)$$

For all $x, y \in X$ with $x \neq y$, where $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$. Then S has a unique fixed point in X.
This result was again proved by Harjani et al. [36] in a complete metric space endwed with partial order relation .Later the result of [36] was generalized by Luong et al.[37]involving altering distance functions which satisfies a weak contractive condition of rational type auxiliary functions in ordered metric space .
There after ,the result [37] was generalized and extended by chandok et al.[38]in 2013 and obtained coupled ,common fixed point results for weak contractive mapping in partially ordered metric space .
On the other hand, in 1989, Bakhtin[18] introduced the concept of b- metric space which is a generalization of metric space. Using the idea Many researcher like Czerwik [19,20] , Mehmatkir [M. kir ,H. Kiziltunk, on Some well known fixed point theorems in b-metric spaces, Turkish Journal of Analysis and Number theory, 1(2013),13-16]. Therefore, lot of improvements have been done in finding fixed points for single valued and multivalued operators in that space, the reader may refer to [21-29].
In this paper, We prove some results for fixed point and uniqueness for self mappings using generalized $(\emptyset, \varphi)$ weak contraction in orderd complete b- metric space .Our results extends and generalizes the result of [38],[39] and several comparable results existing in the literature.

Example (1): (b-metric space) – let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping

$$d(m,n) = \begin{cases} 0 & if m = n \\ \left|\frac{1}{m} - \frac{1}{n}\right| & if one of m, n is even and the other wise is even or odd \\ 5 & if one of m, n is even and the other wise is even or odd \\ 2 & otherwise m = n \end{cases}$$

$$Then (X, d) is a b\text{–} metric space with cofficient s = {}^{5}/_{2}.$$

Example(2): $X\{4,5,6\}$ and $d(4,5) = d(5,4) = 1, d(4,6) = d(6,4) = 4, d(5,6) = d(6,5) = 2$ and $d(4,4) = d(5,5) = d(6,6) = 0$.
And $d(x, z) = \frac{4}{3}[d(x, y) + d(y, z)]$ for all $x, y, z \epsilon X$.
Then (X, d) is a b-metric space $(s = {}^{4}/_{3})$ but (X, d) is not a metric space because it lacks the triangular property.

$$4 = \lambda(4,6) > d(4,5) + d(5,6) = 1 + 2 = 3$$

Preliminaries: We start this section with the following definitions and results which are frequently used in the main results.
Definition1. A mapping $d: P \times P \to [0, +\infty)$, where p is a non-empty set is said to be a b-metric space ,if it satisfies the properties given below for any $v, \xi, \mu, \epsilon P$ and for some $s \geq 1$,

(a) $d(v, \xi) = 0$ if $v = \xi$,
(b) $d(v, \xi) = d(\xi, v)$,
(c) $d(v, \xi) \leq s\big(d(v, \mu) + d(\mu, \xi)\big)$,

Any then $(P, d\ s)$ is known as a b-metric space .

Definition 2: Let (P, d, s)be a metric space .Then
(1) A sequence $\{v_n\}$ is said to converge to v if $\lim_{n\to+\infty} d(v_n, v) = 0$ and written as $\lim_{n\to+\infty} v_n = v$.
(2) $\{v_n\}$ is said to be a Cauchy sequence in P ,if $\lim_{n\, m\to+\infty} d(v_n, v_m) = 0$.

Definition 3: If the metric d is complete then $(P, d, \leq)$ is called a complete partially ordered metric Space .

Definition 4: A point $v \in A$, where $A \neq \emptyset$, a subset of (P, d) is called a common fixed point for two self-mappings f and S if $v = fv = Sv$ $(fv = Sv)$.

Definition 5: Two self maps f and S defined over a sub-set of (P, d) are called commuting, if $fSv = Sfv$, for all $v \in A$.

Definition 6: A pair of self mapping (f, S) on $A \neq \emptyset$, a subset of P is called weakly compatible ,if $Sfv = fSv$, when $Sv = fv$ for some $v \in A$.

Definition 7: let f and S be two self –mapping over $(P, \leq)$. then S is called monotone f non decreasing.

The concept of acquiring fixed point in metric space using control function was initiated by Khan et.al.

Lemma 1: let P be a non –empty set $f: P \to P$ be a mapping .Then there exists sub set E of P such that $fE = fP$ and $f: E \to P$ is one –to- one .

In 1975, Das and Gupta proved the following fixed point results in a complete metric space .

Theorem 1: Suppose (P, d) is a complete metric space . Let $S: P \to P$ be a mapping such there exist $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ satisfying

$$d(Sv, S\xi) \leq \alpha \frac{d(\xi,S\xi)[1+d(v,Sv)]}{1+d(v,\xi)} + \beta d(v, \xi), \qquad (1)$$

$for\ any\ distinct\ v, \xi \in P$, Then S has a unique fixed point in P.

The generalization of above result in partially ordered metric space was obtained by Cabrera et.al. in 2013 Later Chandok et.al. generalized the result of by use of control function in the space .

Again .Theorem 1 was generalization in 1977 and proved the following .

Theorem 2: Suppose (P, d) is a complete metric space .A self maps S on P $such\ that$

$$d(Sv, S\xi) \leq \alpha \frac{d(\xi,S\xi)[d(v,Sv)]}{d(v,\xi)} + \beta d(v, \xi) \qquad (2)$$

$\forall\ v, \xi \in P\ with\ v \neq \xi, where\ \alpha, \beta \in [0,1)\ with\ \alpha + \beta < 1$. Then has unique fixed point in P.

This result was again proved by Harjani et.al. in a complete metric space endowed with partial order relation .Later the result was generalized by Luong et.al. involving altering distance function which satisfies a weak contractive condition of, rational type auxiliary function in ordered metric space .

Theorem 2.1: Let $(X, d, s, \leq)$ be a complete partially ordered b-metric space with parameter $s \geq 1$,let $s: X \to X$ be a continous non decreasing mapping with regards to $\leq$ such that there exists $x_0\ \epsilon\ X$ with $x_0\ \epsilon\ sx_0$. Suppose that.

$$\emptyset\big(sd(sx, sy)\big) \leq \emptyset(M(x, y) - \emptyset\big(M(x, y)\big)$$

Where $\phi\ \epsilon\ \Phi$, $\omega\ \epsilon\ \Psi$ for only $x, y\ \epsilon\ X\ with\ x \leq y\ and$

$$M(x, y) = \max\Big\{ d(x, y), d(x, Sx), \frac{d(y,sy)[1+d(x,sx)]}{1+d(x,y)}, \frac{d(y,Sx)[1+d(x,sy)]}{1+d(x,y)}, \frac{d(x,y)[1+d(x,sx)+d(y,sx)]}{1+d(x,y)}\Big\}$$

Then s has a fixed point in X.

Proof : If for some $x_0 \epsilon X$ such that $Sx_0 = x_0$ then there is nothing to prove that .So we may assume that $x_0 < sx_0$, then construct a sequence $\{x_n\} \subset X$ by $x_{n+1} = sx_n$ $for\ n \geq 0$

But s is non decreasing then , by mathematical induction , We get the following

$$x_0 \leq sx_0 = x_1 \leq sx_1 = x_2 \leq \cdots \leq sx_{n+1} = x_n \leq sx_n = x_{n+1} \leq \cdots \qquad (3)$$

If for some $n_0\ \epsilon\ N$ such that $x_{n0} = x_{n0+1}$ then from 3 x_{n0+1} then from (3) x_{n0} is a fixed point of S and We have nothing to prove . Suppose that $x_n \neq x_{n+1}$ i.e. $d(x_n, x_{n+1}) > 0$ for all $n \geq 1$.

Since $x_n > x_{n+1}$ for any $n \geq 1$ and then by 1 , we have

$$\emptyset(d(x_n, x_{n+1}) = \emptyset(sd(sx_{n-1}, sx_n)) \leq \emptyset\big(M(x_{n-1}, x_n)\big) - \varphi(M(x_{n-1}, x_n) \qquad (4)$$

Where, $M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_{n-1}, sx_{n-1}), d(x_n, sx_n), \frac{d(x_n,sx_n)[\ 1+d(x_{n-1},sx_{n-1})]}{1+d(x_{n-1},x_n)}$, $\frac{d(x_n,sx_{n-1})[1+d(x_{n-1},sx_n)]}{1+d(x_{n-1},x_n)}, \frac{d(x_{n-1},x_n)[1+d(x_{n-1},sx_{n-1})+d(x_n,sx_{n-1}}{1+d(x_{n-1},x_n)})\}$ $\max\{ d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n+1}, x_n), d(x_n, x_{n+1})\}$

$$= \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \qquad (5)$$

Main Results:

Theorem 3.1 We begin this Section with the following theorem

$$\emptyset(x_n, x_{n+1}) \leq \emptyset\big(d(x_n, x_{n+1})\big) - \varphi\big(d(x_n, x_{n+1})\big) \qquad (6)$$

$$< \emptyset(x_n, x_{n+1}) \qquad (7)$$

Which is contradiction

This max $\{d(x_n, x_{n+1}), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ for $n \geq 1$

Hence from (6) We have

$\emptyset(d(x_n,x_{n+1}) \leq \emptyset(d(x_n,x_{n-1})) - \varphi(d(x_n,x_{n+1}) < \emptyset(d(x_n,x_{n-1})$ (8)

Therefore, the sequence $\{d(x_n,x_{n-1}), d(x_{n-1},x_n)\}$ for $n \geq 1$ is a monotone non decreasing and bounded from a result ,We have.

$\lim_{n\to+\infty} d(x_n,x_{n+1}) = p \geq 0$ (9)

Now taking the upper limit on both side of (4) ,we obtain $\emptyset \leq 0$

Which is again a contraction Thus $P = 0$

Hence $d(x_n,x_{n+1}) \to 0$ as $n \to +\infty$ (10)

We now claim that $\{x_n\}$ is a cauchysequnce in X, that is for every $\epsilon > 0$ there exist $k \epsilon N$ such that $d(x_m,x_n) < \epsilon$ for all $m,n \geq k$. Assume to the contrary that there exist $\epsilon > 0$ for which we can find subsequence $\{x_{m(k)}\}, x_{n(k)}\}$ of $\{x_n\}$ such that $n(k) > m(k) \geq k$, $m(k)$is even and $n(k)$is odd.

$d(x_{m(k)},x_{n(k)}) \geq \epsilon$ (11)

And n(k) is the smallest number such that (11) holds,

From (11), we get $d(x_{m(k)},x_{n(k)}) < \varepsilon$ (12)

Using triangular inequality in (11), we get $\epsilon \leq d(x_{m(k)},x_{n(k)}) \leq s\, d(x_{m(k)},x_{n(k)-1}) + s(x_{m(k)},x_{n(k)-1})$

$+s^2 d(x_{m(k)},x_{m(k)-1}) + s^2 d(x_{m(k)-1},x_{n(k)-1}) + sd(x_{n(k)-1},x_{n(k)})$ (13)

Furthermore,

$d(x_{m(k)-1},x_{n(k)-1}) \leq sd(x_{m(k)-1},x_{m(k)}) + sd(x_{m(k)},x_{n(k)-1}) \leq sd(x_{m(k)-1},x_{m(k)}) + s\varepsilon$ (14)

Letting $k \to +\infty$ in eq.(13) and (14) and combining together ,we obtain the following inequality

$\frac{\varepsilon}{s^2} \leq \lim_{k\to+\infty} \sup d(x_{m(k)-1},x_{n(k)-1}) \leq s\varepsilon$ (15)

Similarly, we can get the following inequality by ,using triangular inequality

$\frac{\varepsilon}{s^2} \leq \lim_{k\to+\infty} \inf d(x_{m(k)-1},x_{n(k)-1}) \leq s\varepsilon$ (16)

And $\frac{\varepsilon}{s} \leq \lim_{k\to+\infty} \sup d(x_{m(k)-1},x_{n(k)}) \leq s^2\varepsilon$ (17)

$$Let\ M(x_{m(k)-1},x_{n(k)-1})$$

$$\max\{d(x_{m(k)-1},x_{n(k)-1}), d(x_{m(k)-1}), s(x_{m(k)-1}), (dx_{n(k)-1}, sx_{n(k)-1}), \frac{d(x_{n(k)-1},sx_{n(k)-1})[1+d(x_{m(k)-1},sx_{m(k)-1})]}{1+d(x_{m(k)-1},x_{n(k)-1})}$$
$$\frac{d(x_{n(k)-1},sx_{m(k)-1})[1+d(x_{m(k)-1},sx_{n(k)-1})]}{1+d(x_{m(k)-1},x_{n(k)-1})}, \frac{d(x_{m(k)-1},x_{n(k)-1})[1+d(x_{m(k)-1},sx_{m(k)-1}+d(x_{n(k)-1},sx_{m(k)-1})]}{1+d(x_{m(k)-1},x_{n(k)-1})}\}.$$

$$= \max\{d(x_{m(k)-1},x_{n(k)-1}), d(x_{m(k)-1},x_{n(k)}), d(x_{n(k)-1},x_{n(k)}), \frac{d(x_{n(k)-1},x_{n(k)})[1+d(x_{m(k)-1},x_{m(k)})]}{1+d(x_{m(k)-1},x_{n(k)-1})}$$
$$, \frac{d(x_{n(k)-1},sx_{m(k)})[1+d(x_{m(k)-1},x_{n(k)})]}{1+d(x_{m(k)-1},x_{n(k)-1})}, \frac{d(x_{m(k)-1},x_{n(k)-1})[1+d(x_{m(k)-1},x_{m(k)-1}+d(x_{n(k)-1},x_{m(k)})]}{1+d(x_{m(k)-1},x_{n(k)-1})}\} \quad (18)$$

From 18 we obtain the following inequalities

$\frac{\varepsilon}{s^2} \leq \lim_{k\to+\infty} \sup \mathrm{M}(x_{m(k)-1},x_{n(k)-1}) \leq s\varepsilon$ (19)

And

$\frac{\varepsilon}{s^2} \leq \lim_{k\to+\infty} \inf M(x_{m(k)-1},x_{n(k)-1}) \leq s\varepsilon$ (20)

From 3 ,we have $x_{m(k)-1} < x_{n(k)-1}$, then

$$\emptyset(sd(x_{m(k)},x_{n(k)}) \leq \emptyset\left(sd(sx_{m(k)-1},sx_{n(k)-1})\right)$$

$$\leq \emptyset\left(M(x_{m(k)-1},x_{n(k)-1})\right) - \varphi(M(x_{m(k)-1},x_{n(k)-1}) \quad (21)$$

Now letting $K \to +\infty$ and using(19), (20), we obtain

$$\emptyset(s\varepsilon) \leq \emptyset(s \lim_{k\to\infty} d(x_{m(k)},x_{n(k)}))$$
$$\leq \emptyset \lim_{k\to+\infty} \sup \mathrm{M}(x_{m(k)-1},x_{n(k)-1}) - \lim_{k\to+\infty} \varphi \inf(x_{m(k)-1},x_{n(k)-1})$$
$$\leq \emptyset(s\varepsilon) - \varphi \lim_{k\to+\infty} \inf M(x_{m(k)-1},x_{n(k)-1})$$
$$\leq \emptyset(s\varepsilon)$$

Which is a contraction .Hence $\{x_n\}$ is a Cauchy sequence and converge to some $x^* \in X$ as X is complete. Also ,the continuity of s implies that

$$sx^* = s(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} sx_n = \lim_{n\to\infty} x_{n+1}) = x^*$$

Therefore x^* is a fixed point of s in X. This completes the proof of the theorem.

Theorem 3.2: In Theorem 3.1, if X has a property that ,the sequence $\{x_n\}$ is a nondecreasing such that $x_n \to x$ implies that $x_n \leq x$ for all $n \in \mathbb{N}$ I.e $x = supx_n$ then a non-continuous map S has a fixed pointin X.

Proof: From Theorem 3.1,we take the same sequence $\{x_n\}$ in X such that $x_0 \leq x_1 \leq x_2 \leq x_3 \dots \dots \dots \dots \dots \leq x_n \leq x_{n+1} \leq \cdots \dots \dots . ie$ the sequence $\{x_n\}$ is nondecreasing and converges to some $x\ in\ X$. Thus ,from the hypotheses we have $x_n \leq x$ for any $n \in \mathbb{N}$ implies that $x = sup$.Next ,we prove that x is a fixed point of $s\ in\ X$ ie .$sx = x$.suppose that $sx \neq x\ i.e.\ d(sx, x) \neq 0$.

From (2)

$$M(x_n, x) = \max\{d(x_n, x), d(x_n, sx_n), d(x, sx), \frac{d(x,sx)[1+d(x_n,sx_n)]}{1+d(x_n,x)}, \frac{d(x,sx_n)[1+d(x_n,sx)]}{1+d(x_n,x)}, \frac{d(x_n,x)[1+d(x_n,sx_n)+d(x,sx_n)]}{1+d(x_n,x)}\}$$

$$= \max\{\{d(x_n, x), \{d(x_n, x_{n+1}), \{d(x, sx), \frac{d(s, sx)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x)}, \frac{d(x, x_{n+1})[1 + d(x_n, sx)]}{1 + d((x_n, x)}, \frac{d((x_n,x)[1+d((x_n,x_{n+1})+(x,x_{n+1})}{1+d((x_n,x)}\}$$

Letting $n \to +\infty$ and from $\lim_{n\to\infty} x_n = x$,we get

$$\lim_{n\to+\infty} M(x_n, x) = \max\{0,0, d(x, sx), d(x, sx), 0, 0\} = d(x, sx) \quad (22)$$

we, know that $x_n \leq x$. for all n then from contractive condition (1),we get.

$$\phi\big(d(x_{n+1}sx)\big) = \phi\big(d(sx_n, sx)\big) \leq \phi\big(sd(sx_n, sx)\big)$$
$$\leq \phi\big(M(x_n, x)\big) - \psi((M(x_n, x))$$

Letting $n \to +\infty$ in the above inequality and using (22),we get

$$\phi\big(d(x, sx)\big) \leq \phi\big(d(x, sx)\big) - \psi\big(d(x, sx)\big) < \phi\big(d(x, sx)\big)$$

a contradiction .Thus $s_x = x$ i.e .S has a fixed point $x\ in\ X$.

This completes the proof of the theorem.

The uniqueness of fixed point in theorem 3.1 and 3.2 can get,if P has the following property:

For any $x, y \in P$,there exists $x \in P$ such that $z \leq x$ and $z \leq y$.

Theorem 2.2: If P satisfies the above mentioned condition in theorem 2.1 (or Theorem 3.2) then S has a unique fixed point.

Proof : As in theorem 3.1 (or 3.2) we have proved that S has a fixed point .To prove the uniqueness suppose that $x^\star\ and\ y^\star$ be two fixed points of S then we claim that$x^\star = y^\star$.Suppose that $x^\star \neq y^\star$,then from the hypotheses, we have

$$\emptyset(d(sx^\star, sy^\star) \leq \emptyset\big(M(x^\star, y^\star)\big) - \psi\big(M(x^\star, y^\star)\big) \quad (23)$$

Where

$$M(x^\star, y^\star) = \{d(x^\star, y^\star), d(x^\star, sx^\star), d(y^\star, sy^\star), \frac{d(y^\star,sy^\star)[1+d(x^\star,sx^\star)]}{1+d(x^\star,y^\star)}, \frac{d(y^\star,sx^\star)[1+d(x^\star,sy^\star)]}{1+d(x^\star,y^\star)}, \frac{d(x^\star,y^\star)[1+d(x^\star,sx^\star)+d(y^\star,sx^\star)]}{1+d(x^\star,y^\star)}\}$$

$$= \max\{d(x^\star, y^\star), d(x^\star, x^\star), d(y^\star, y^\star), 0, d(x^\star, y^\star)\} = d(x^\star, y^\star)$$

Then by Eq. (23) $\emptyset\big(d(x^\star, y^\star)\big) = \emptyset\big(d(sx^\star, sy^\star)\big) \leq \emptyset\big(d(x^\star, y^\star)\big) - \psi(d(x^\star, y^\star)) < \big(d(x^\star, y^\star)\big)$

which is a contraction .Hence $x^\star = y^\star$.

As a generalization of Theorem 2.1 and 2.2 of [38] and corollaries 2.1 and 2.2 of [39] we have the following results as corollaries.

Corollary 3.1 Let $(p, d, s, \leq)$ be a partially order b- metric space with a parameter S .Suppose $S, f\colon p \to P$ are continuous mappings such that.

(C1) For some $\psi \in \Psi$ and $\phi \in \Phi$ with

$$\phi(sd(sx, sy) \leq \phi(M_f(x, y) - \psi(M_f(x, y)) \quad (24)$$

For any $x, y \in P$ such that $fx \leq fy\ and$

$$Mf(x, y) = \max\{d(fx, fy), d(fx, sx), d(fy, sy), \frac{d(fy,sy)[1+d((fx,sy)]}{1+d(fx,fy)}, \frac{d(fy,sx)[1+d(fx,fy)]}{1+d(fx,fy)}, \frac{d(fx,fy)[1+d(fx,fy)+d(fy,sx)}{1+d(fx,fy)}\} \quad (25)$$

(C2) $sp \subset fP$ and fP is a complete subspace of P.

(C3) s is a monotone f −nondecreasing mapping.

(C4) s and f are compatible.

If for some $x_0 \in P$ such that $fx_0 \leq sx_0$ then sand$f$ have a coincidence point in P.

Proof: Using lemma 1, we obtain a complete subspace fE of P where $E \subset P$ and f is one -to- one self mapping on P.By corollary 2.1 of [39] we have a sequence $\{fx_n\} \subset fE$ for some $x_0 \in E$ with $fx_{n+1} = g(fx_n)$, for $n \geq 0$ where g is a self mapping on fE with $g(fx) = sv$,$x \in E$.Therefore, from the hypotheses .

We have

$$\phi(sd(g(fy)) \leq \phi\left(M_f(x,y)\right) - \psi(M_f(x,y))$$

for all $x, y \in P$ with $fx \leq fy$ and

$$M_f(x,y) = \max\{d(fx,fy), d(fx,g(fx), d(fy,g(fy), \frac{d(fy,g(fy)\{1+d(fx,g(fx)]}{1+d(fx,fy)}, \frac{d(fy,g(f(x))[1+d(fx,g(fy))]}{1+d(fx,fy)},$$

$$\frac{d(fy,g(f(x))[1+d(fx,g(fy))]}{1+d(fx,fy)},$$

From the same argument in theorem 3.1 $\{x_0\}$ is a Cauchy sequence and which converges to some $x \in fE$.Thus, by the compability of $s \in f$

We obtain

$$lim_{n\to\infty} d(f(sx_n), s(fx_n)$$

Furthermore, by using triangular inequality,

$$d(sx, fx) = sd(sx, s(fx_n) + s^2 d(s(fx_n), f(sx_n)) + s^2 d(f(sx_n), fx)$$

Finally we arrive at the result $d(sx, fx) = 0$ as $n \to +\infty$ in the above inequality .There x is a coincidence point for s *and* f *in* P.

On replacing the condition, weakly compatible instead of (C4) in corollary 3.1 ,we obtain the following result.

Corollary 3.2 : If P has the property in corollary 3.1 instead of the compatibility for s *and* f that ,for any non decreasing sequence $\{fx_n\} \subset P$ such that $lim_{n\to\infty} fx_n = fx$ implies that $fx_n \leq fx$ for all $n \in \mathbb{N}$ that is $fx = supfx_n$. Then s and f have a common fixed point in P,if for some coincidence point P of s and f with $fp \leq f(fp)$. furthermore ,the set of common fixed point of s and f is well ordered if and only if s *and* f have one and only one common fixed point .

Proof : From corollary 3.1 and Theorem 3.2 it is obvious that S and f have a coincidence point in P, as $fp = g(fp) = Sp$ for some p *in* P.

Next, assume that a pair of mapping (S, f) is weakly compatible and let v be an element in P such that $v = sp = fp$. Then $Sv = S(fp) = fv$.

Let

$$M(p,v) = \max\{d(fp,fv), d(fp,Sp), d(fv,Sv), \frac{d((fv,Sv)[1+d(fp,Sp)]}{1+d(fp,fv)}, \frac{d(fv,sp)[1+d(fp,Sv)]}{1+d(fp,fv)}, \frac{d(fp,fv)[1+d(fp,sp)+d(fv,sp)}{1+d(fp,fv)}\}$$

$$M(p,v) = \max\{d(fp,fv), d(fp,Sp), d(fv,Sv), \frac{d((fv,Sv)[1+d(fp,Sp)]}{1+d(fp,fv)} \frac{d(fv,sp)[1+d(fp,Sv)]}{1+d(fp,fv)}, \frac{d(fp,fv)[1+d(fp,sp)+d(fv,sp)}{1+d(fp,fv)}\}$$

$$= \max\{d(sp,sv), d(sp,sp), d(sv,sv), \frac{d(sv,sv)[1+d(sp,sp)]}{1+d(sp,sv)}, \frac{d(sv,sp)[1+d(sp,sv)]}{1+d(sp,sv)},$$

$$\frac{d(sp,sv)[1+d(sp,sp)+d(sv,sp)]}{1+d(sp,sv)}\} = max\{d(sp,sv), 0,0,0, d(sv,sp), d(sp,sv)\} = d(sp,sv).$$

Then from contractive condition, we have .

$$\phi(d(sp,sv) \leq \phi(M(p,v) - \psi(M(p,v)).$$
$$\leq \phi(d(sp,sv) - \psi(d(sp,sv)).$$
$$\leq \phi(d(sp,sv)$$

a contraction .Hence we have $d(sp,sv) = 0$ by the property of ψ. Therefore $sv = fv = v$.

By theorem 3.3 we deduce that s *and* f have one and only one common fixed point if and only if the set of common fixed points of s *anf* f .is well ordered.

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