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A Collection Of Special Quadratic Diophantine Ellipse And Higher Order Pythagorean Equations With Integer Solutions

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Abstract:

Learning about the various techniques to solve this higher power Diophantine equation in successfully deriving their solutions helps us understand how numbers work and their significance in different areas of mathematics and science. This paper first focused on studying infinitely many integer solutions of generalized quadratic Diophantine equation $k(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2$,which is having ellipse equation form of $k\left(\frac{x-\alpha}{z-\gamma}\right)^2+\left(\frac{y-\beta}{z-\gamma}\right)^2=1$; Also, concentrate on studying Reciprocal form of above Diophantine Equation $\frac{k}{p^2}+\frac{1}{q^2}=\frac{1}{h^2}$, Which is having different sets of integer solutions of $p=(y-\beta)(z-\gamma)$, $q=(x-\alpha)(z-\gamma)$ and $h=(x-\alpha)(y-\beta)$. Also, introduces to study of binary operation on above Set of quadratic Diophantine Equation.

$$P_1.P_2=\begin{Bmatrix}(kx_1{x_2}^2, y_1({y_2}^2+{z_2}^2)+2y_2z_1z_2, z_1({y_2}^2+{z_2}^2)+2y_1y_2z_2)\\(kx_2{x_1}^2, y_2({y_1}^2+{z_1}^2)+2y_1z_1z_2, z_2({y_1}^2+{z_1}^2)+2y_1y_2z_1)\end{Bmatrix}.$$

Keywords: Diophantine Equations, Ellipse, Reciprocal ellipse equations.

Mathematics Subject Classifications:11D72,11D61.

INTRODUCTION:

Ellipse has applications in various diverse fields like Astronomy (Planetary orbits), Medicines (Lithotripsy) and Engineering (Whispering galleries). The property of ellipse to reflect sound and light is pulverizing kidney stones.

This paper focused on designing integer solutions of the following Diophantine Equations.

$2(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2$; $3(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2$;

$4(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2$; $5(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2$.

Also, $k(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2$ has an ellipse equation form of

$k\left(\frac{x-\alpha}{z-\gamma}\right)^2+\left(\frac{y-\beta}{z-\gamma}\right)^2=1$; Also, focused to study Reciprocal form of above Diophantine Equation $\frac{k}{p^2}+\frac{1}{q^2}=\frac{1}{h^2}$, Which is having different sets of integer solutions of

$p=(y-\beta)(z-\gamma)$, $q=(x-\alpha)(z-\gamma)$ and $h=(x-\alpha)(y-\beta)$.

The solutions to the quadratic Diophantine equation P= $\{(x,y,z): k\mathrm{x}^2+\mathrm{y}^2=\mathrm{z}^2\}$are satisfies binary operation is

$$P_1.P_2=\begin{Bmatrix}(kx_1x_2{}^2, y_1(y_2{}^2+z_2{}^2)+2y_2z_1z_2, z_1(y_2{}^2+z_2{}^2)+2y_1y_2z_2)\\(kx_2x_1{}^2, y_2(y_1{}^2+z_1{}^2)+2y_1z_1z_2, z_2(y_1{}^2+z_1{}^2)+2y_1y_2z_1)\end{Bmatrix}.$$

RESULTS & DISCUSSIONS:

Case 1: Consider the Diophantine equation with constants values α,β,γ is

$2(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2$ having different sets of integer solutions is

$x-\alpha=2^{n+1}$, $y-\beta=2^n$ and $z-\gamma=3(2)^n$. Here n is a positive integer.

Proof: Consider α, β,γ be a centre of some Ellipse equations, whose stationary points are in the form of $x-\alpha=2^{n+1}$, $y-\beta=2^n$ and $z-\gamma=3(2)^n$, which satisfies the integer solution of the given Diophantine equation $2(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2$.

Since $2(2^{n+1})^2+(2^n)^2=(2^{2n+3})+(2^{2n})=(2^{2n})(2^3+1)=(3(2)^n)^2$.

It follows that for some fixed values of α, β,$\gamma$the stationary values

$x=\alpha+2^{n+1}$, $y=\beta+2^n$ and $z=\gamma+3(2)^n$ are satisfies the Diophantine equation $2(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2$.

Also, introduce to define Binary Operation '*' on the Set of Stationary Points of the above Ellipse Diophantine Equation, $2x^2+y^2=z^2\, with\, P(x_1,y_1,z_1), Q(x_2,y_2,z_2)$which is ellipse triplet P*Q = $(\sqrt{2}x_1x_2, y_1z_2+y_2z_1, y_1y_2+z_1z_2)$

Proof: Consider

$$(y_1y_2+z_1z_2)^2-(y_1z_2+y_2z_1)^2=(y_1y_2)^2+(z_1z_2)^2-(y_1z_2)^2-(y_2z_1)^2$$

$$=z_2^2[z_1^2-y_1^2]-y_2^2[z_1^2-y_1^2]=[z_1^2-y_1^2][z_2^2-y_2^2]=4x_1^2x_2^2=2(2x_1^2x_2^2)$$

It follows that $2\left(\sqrt{2}x_1x_2\right)^2+(y_1z_2+y_2z_1)^2=(y_1y_2+z_1z_2)^2$ implies that $\left(\sqrt{2}x_1x_2, y_1z_2+y_2z_1, y_1y_2+z_1z_2\right)$is becomes to ellipse triplet.

But the triplets are not in the form of integer solution. So again, apply same operation, in terms of Associative $(P_1.P_2).P_2$, which is updated integer form of

$P_3=(P_1.P_2).P_2=(2x_1{x_2}^2, y_1({y_2}^2+{z_2}^2)+2y_2z_1z_2, z_1({y_2}^2+{z_2}^2)+2y_1y_2z_2)$.

Similarly, $P_3=(P_1.P_2).P_1=\ (2x_2{x_1}^2, y_2({y_1}^2+{z_1}^2)+2y_1z_1z_2, z_2({y_1}^2+{z_1}^2)+2y_1y_2z_1)$

Verification: We know that $x=2^{n+1},\ y=2^n$ and $z=3(2)^n$ is satisfies $2x^2+y^2=z^2$.

For n = 1, $P_1=(4,2,6)$ and for n =2,$P_2=(8,4,12)$ are satisfies $2x^2+y^2=z^2$.

Also, above binary operation $P_1.P_2=\ P_3$

$$=(2x_1{x_2}^2, y_1({y_2}^2+{z_2}^2)+2y_2z_1z_2, z_1({y_2}^2+{z_2}^2)+2y_1y_2z_2)=(512,896,1152)$$

is satisfies $2x^2+y^2=z^2$.

Also, another binary operation $P_1.P_2=\ P_3$

$$=(2x_2{x_1}^2, y_2({y_1}^2+{z_1}^2)+2y_1z_1z_2, z_2({y_1}^2+{z_1}^2)+2y_1y_2z_1)=(256,448,576)$$

is satisfies $2x^2+y^2=z^2$.

It follows that clearly (128,224,288),(64,112,144),(32,56,72),(16,28,36),(8,14,18),(4,7,9) are becomes to solutions of $2x^2+y^2=z^2$.

Lemma 1.1: It has to have an Ellipse equation form of $2\left(\frac{x-\alpha}{z-\gamma}\right)^2+\left(\frac{y-\beta}{z-\gamma}\right)^2=1$; Which has having simple form of ellipse $2p^2+q^2=1$, whose solution is $p=\frac{2}{3}, q=\frac{1}{3}$.

Lemma 1.2: Reciprocal form of the above Diophantine Equation $\frac{2}{p^2}+\frac{1}{q^2}=\frac{1}{h^2}$. Which is having different sets of integer solutions of $p=(y-\beta)(z-\gamma)=3(2^{2n})$,

$q=(x-\alpha)(z-\gamma)=3(2^{2n+1})$ and $h=(x-\alpha)(y-\beta)=2^{2n+1}$.

Since $\frac{2}{\left(3(2^{2n})\right)^2}+\frac{1}{\left(3(2^{2n+1})\right)^2}=\frac{1}{(2^{2n+1})^2}$.

Case 2: Consider the Diophantine equation

$3(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2$ having different sets of integer solutions is

$x-\alpha=3^n,\ y-\beta=3^n$ and $z-\gamma=2(3)^n$.

Proof: Let $x-\alpha=3^n,\ \ y-\beta=3^n$ and $z-\gamma=2(3)^n$ satisfy the integer solution of the Diophantine equation $3x^2+y^2=z^2$.

Since $3(3^n)^2+(3^n)^2=(3^{2n+1})+(3^{2n})=(3^{2n})(3+1)=(2(3)^n)^2$.

Also, introduce to define Binary Operation '*' on the Set of Stationary Points of the above Ellipse Diophantine Equation, $3x^2+y^2=z^2\ with\ P(x_1,y_1,z_1), Q(x_2,y_2,z_2)$which is ellipse triplet

P*Q = $\left(\sqrt{3}x_1x_2, y_1z_2+y_2z_1, y_1y_2+z_1z_2\right)$

Proof: Consider

$$(y_1y_2+z_1z_2)^2-(y_1z_2+y_2z_1)^2=(y_1y_2)^2+(z_1z_2)^2-(y_1z_2)^2-(y_2z_1)^2$$
$$=z_2^2[z_1^2-y_1^2]-y_2^2[z_1^2-y_1^2]=[z_1^2-y_1^2][z_2^2-y_2^2]=9x_1^2x_2^2=3(3x_1^2x_2^2)$$

It follows that $3(\sqrt{3}x_1x_2)^2+(y_1z_2+y_2z_1)^2=(y_1y_2+z_1z_2)^2$ implies that $(\sqrt{3}x_1x_2, y_1z_2+y_2z_1, y_1y_2+z_1z_2)$is becomes to ellipse triplet.

But the triplets are not in the form of integer solution. So again Apply same operation in terms of Associative $(P_1.P_2).P_2$, which is updated integer form of

$P_1.P_2=(3x_1x_2{}^2, y_1(y_2{}^2+z_2{}^2)+2y_2z_1z_2, z_1(y_2{}^2+z_2{}^2)+2y_1y_2z_2) for all P_1, P_2 \in B$.

Similarly, $P_3=(P_1.P_2).P_1=(3x_2x_1{}^2, y_2(y_1{}^2+z_1{}^2)+2y_1z_1z_2, z_2(y_1{}^2+z_1{}^2)+2y_1y_2z_1)$

Verification: We know that $x=3^n$, $y=3^n$ and $z=2(3)^n$ is satisfies $3x^2+y^2=z^2$.

For n = 1, $P_1=(3,3,6)$ and for n =2,$P_2=(9,9,18)$ are satisfies $3x^2+y^2=z^2$. Also, above binary operation $P_1.P_2=P_3=(729,3159,3402)$ is satisfies $3x^2+y^2=z^2$.

Lemma 2.1: It has to have the Ellipse equation form of $3\left(\frac{x-\alpha}{z-\gamma}\right)^2+\left(\frac{y-\beta}{z-\gamma}\right)^2=1$; Which has the simple form of ellipse $3p^2+q^2=1$, whose solution is $p=q=\frac{1}{2}$.

Lemma 2.2: Reciprocal form of the above Diophantine Equation

$\frac{3}{p^2}+\frac{1}{q^2}=\frac{1}{h^2}$. Which is having different sets of integer solutions

$p=(y-\beta)(z-\gamma)=2(3^{2n})$, $q=(x-\alpha)(z-\gamma)=2(3^{2n})$ and

$h=(x-\alpha)(y-\beta)=3^{2n}$. Since $\frac{3}{(2(3^{2n}))^2}+\frac{1}{(2(3^{2n}))^2}=\frac{1}{(3^{2n})^2}$;

Case 3: Consider the Diophantine equation $4(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2$ having different sets of integer solutions is (2$(x-\alpha)$, $(y-\beta)$, $(z-\gamma)$) is a Pythagorean triplet.

It has different sets of Integer solutions for each odd integer $x-\alpha$, then.

$y-\beta=(x-\alpha)^2-1$ and $z-\gamma=(x-\alpha)^2+1$.

Proof: Let $y=x^2-1$ and $z=x^2+1$.Consider

$(z-\gamma)^2-(y-\beta)^2=((x-\alpha)^2+1)^2-((x-\alpha)^2-1)^2=4(x-\alpha)^2=\left(2(x-\alpha)\right)^2$.

Hence if $(x-\alpha)$ is an odd, then (2$(x-\alpha)$, $(y-\beta)$, $(z-\gamma)$is a Pythagorean triplet.

Case 4: Consider the Diophantine equation $5(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2$

having different sets of integer solutions is $x=4^{n+1}$, $y=4^n$ and $z=3(4)^n$. Here n is a positive integer.

Proof: Let $x=4^{n+1}$, $y=4^n$ and $z=3(4)^n$ satisfy the integer solution of the Diophantine equation $5(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2$. Since $5(4^{n+1})^2+(4^n)^2=(3(4)^n)^2$.

Also, introduce to define Binary Operation '*' on the Set of Stationary Points of the above Ellipse Diophantine Equation, $2x^2 + y^2 = z^2 with P(x_1, y_1, z_1), Q(x_2, y_2, z_2)$which is ellipse triplet

P*Q = $(\sqrt{5}x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$.

Proof: Consider $(y_1y_2 + z_1z_2)^2 - (y_1z_2 + y_2z_1)^2$

$$= (y_1y_2)^2 + (z_1z_2)^2 - (y_1z_2)^2 - (y_2z_1)^2 = z_2^2[z_1^2 - y_1^2] - y_2^2[z_1^2 - y_1^2] = [z_1^2 - y_1^2][z_2^2 - y_2^2]$$
$$= 25x_1^2x_2^2 = 5(5x_1^2x_2^2)$$

It follows that $5(\sqrt{5}x_1x_2)^2 + (y_1z_2 + y_2z_1)^2 = (y_1y_2 + z_1z_2)^2$ implies that $(\sqrt{5}x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$is becomes to ellipse triplet. Its not in integer form, so again apply Associativity, we obtain the well-defined binary operation is

$$P_1.P_2 = \begin{cases} (5x_1{x_2}^2, y_1({y_2}^2 + {z_2}^2) + 2y_2z_1z_2, z_1({y_2}^2 + {z_2}^2) + 2y_1y_2z_2) \\ (5x_2{x_1}^2, y_2({y_1}^2 + {z_1}^2) + 2y_1z_1z_2, z_2({y_1}^2 + {z_1}^2) + 2y_1y_2z_1) \end{cases}.$$

Lemma 4.1: Reciprocal form of above Diophantine Equation $\frac{5}{p^2} + \frac{1}{q^2} = \frac{1}{h^2}$. Which is having different sets of integer solutions

$p = yz = 3(4^{2n}),\ q = xz = 3(4^{2n+1})$ and $h = xy = 4^{2n+1}$.

Since $\frac{5}{(3(4^{2n}))^2} + \frac{1}{(3(4^{2n+1}))^2} = \frac{1}{(4^{2n+1})^2}$.

Hence, it follows that Binary Operation on a Set of Stationary Points of the above Ellipse Diophantine Equation, which is ellipse triplet $(\sqrt{5}x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$

Proof: Consider

$$(y_1y_2 + z_1z_2)^2 - (y_1z_2 + y_2z_1)^2 = (y_1y_2)^2 + (z_1z_2)^2 - (y_1z_2)^2 - (y_2z_1)^2$$
$$= z_2^2[z_1^2 - y_1^2] - y_2^2[z_1^2 - y_1^2] = [z_1^2 - y_1^2][z_2^2 - y_2^2] = 25x_1^2x_2^2 = 5(5x_1^2x_2^2)$$

It follows that $5(\sqrt{5}x_1x_2)^2 + (y_1z_2 + y_2z_1)^2 = (y_1y_2 + z_1z_2)^2$ implies that $(\sqrt{5}x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$is becomes to ellipse triplet.

Also, introduce to define Binary Operation '*' on the Set of Stationary Points of the above Ellipse Diophantine Equation, $kx^2 + y^2 = z^2 with P(x_1, y_1, z_1), Q(x_2, y_2, z_2)$which is ellipse triplet.

P*Q = $(\sqrt{k}x_1x_2, y_1z_2 + y_2z_1, y_1y_2 + z_1z_2)$. It's not in integer form. So, again apply Associativity to obtain well defined integer operation is

$$P_1.P_2 = \begin{cases} (kx_1{x_2}^2, y_1({y_2}^2 + {z_2}^2) + 2y_2z_1z_2, z_1({y_2}^2 + {z_2}^2) + 2y_1y_2z_2) \\ (kx_2{x_1}^2, y_2({y_1}^2 + {z_1}^2) + 2y_1z_1z_2, z_2({y_1}^2 + {z_1}^2) + 2y_1y_2z_1) \end{cases}.$$

Hence, we can apply above binary operations to following sets.

Now, introduce to define subset of some solutions of above Diophantine equations are as follows:

A= $\{(x, y, z): 2x^2 + y^2 = z^2; where x = 2^{n+1},\ y = 2^n, z = 3(2)^n\}$

B= $\{(x, y, z): 3x^2 + y^2 = z^2; where x = 3^n,\ y = 3^n, z = 2(3)^n\}$

C= $\{(x, y, z): 5x^2 + y^2 = z^2; where x = 4^{n+1}, y = 4^n, z = 3(4)^n\}$

D= $\{(x, y, z): 6x^2 + y^2 = z^2; where x = 2^{n+1}, y = 2^n, z = 5(2)^n\}$

E= $\{(x, y, z): 7x^2 + y^2 = z^2; where x = 3^n, y = 3^{n+1}, z = 4(3)^n\}$

F= $\{(x, y, z): 8x^2 + y^2 = z^2; where x = 2^n, y = 2^n, z = 3(2)^n\}$

G= $\{(x, y, z): 10x^2 + y^2 = z^2; where x = 6^{n+1}, y = 6^n, z = 19(6)^n\}$

H= $\{(x, y, z): 11x^2 + y^2 = z^2; where x = 3^{n+1}, y = 3^n, z = 10(3)^n\}$

I= $\{(x, y, z): 12x^2 + y^2 = z^2; where x = 2^{n+1}, y = 2^n, z = 7(2)^n\}$

J= $\{(x, y, z): 13x^2 + y^2 = z^2; where x = 6^n, y = 6^{n+1}, z = 7(6)^n\}$

K= $\{(x, y, z): 14x^2 + y^2 = z^2; where x = 6^{n+1}, y = 5(6)^n, z = 23(6)^n\}$

L= $\{(x, y, z): 15x^2 + y^2 = z^2; where x = 3^{n+1}, y = 3^{n+1}, z = 12(3)^n\}$

M= $\{(x, y, z): 17x^2 + y^2 = z^2; where x = 3^{n+1}, y = 4(3)^n, z = 13(3)^n\}$

N= $\{(x, y, z): 18x^2 + y^2 = z^2; where x = 2^{n+1}, y = 3(2)^n, z = 9(2)^n\}$

O= $\{(x, y, z): 19x^2 + y^2 = z^2; where x = 3^{n+1}, y = 5(3)^n, z = 14(3)^n\}$

Q= $\{(x, y, z): 20x^2 + y^2 = z^2; where x = 2^{n+1}, y = 2^n, z = 9(2)^n\}$

R= $\{(x, y, z): 21x^2 + y^2 = z^2; where x = 2^n, y = 2^{n+1}, z = 5(3)^n\}$ AND

P= $\{(mx, y, z) is a Pythagorean triplet form \in N : (mx)^2 + y^2 = z^2;\}$.

some special collection of Diophantine Equations, whose solutions are obtained from standard Pythagorean theorem.

Case 5: Consider the Pythagorean (4;2) tuples equation as follows

$p^2 + q^2 + r^2 + s^2 = t^2 + u^2$,having different sets of integer solutions is illustrated below:

$p = x^2yh$, $q = y^2xh$, $r = yh$, s= xh, t = xy, u = $xyzh$

where $x = bc$, $y = ca$, $h = ab$, z = c^2 with (a, b, c) is a Pythagorean triplet, which is satisfies $a^2 + b^2 = c^2$.

Proof: We know that if (a, b, c) is a Pythagorean triplet, then is satisfies $a^2 + b^2 = c^2$.

if (a, b, c) is a Pythagorean triplet then (b c, a c, a b) is also a Reciprocal Pythagorean triplet. i.e. if $x = bc$, $y = ca$, $h = ab$ then $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{h^2}$…………[1]

Also, if (a, b, c) is a Pythagorean triplet then (bc, ac, c^2) is also a Pythagorean triplet.

i.e $x = bc$, $y = ca$, z = c^2 then $x^2 + y^2 = z^2$…………[2]

Adding equations [1] ,[2], we obtain

$p^2 + q^2 + r^2 + s^2 = t^2 + u^2$, having different sets of integer solutions is illustrated below:

$p = x^2yh$, $q = y^2xh$, $r = yh$, s= xh, t = xy, u = $xyzh$

where $\boldsymbol{x = bc}$, $\boldsymbol{y = ca}$, $\boldsymbol{h = ab}$, z = $\boldsymbol{c^2}$ with (a, b, c) is a Pythagorean triplet, which is satisfies $\boldsymbol{a^2 + b^2 = c^2}$.

E.g.1: Choose One of the Pythagorean triplets (a, b, c) is (3, 4, 5), which follows

$\mathbf{x = bc = 20}$, $\mathbf{y = ca = 15}$, $\mathbf{h = ab = 12}$, z = $\mathbf{c^2 = 25}$

$\mathbf{p = x^2yh = 72000}$, $\mathbf{q = y^2xh = 54000}$, $\mathbf{r = yh = 180}$, s= $\mathbf{xh}$=240, t = $\mathbf{xy = 300}$,

u = $\mathbf{xyzh = 90000}$.

$$\mathbf{p^2 + q^2 + r^2 + s^2 = 8100090000}$$

$\mathbf{t^2 + u^2 = 8100090000}$. Hence $\mathbf{p^2 + q^2 + r^2 + s^2 = t^2 + u^2}$

Also, note that (a, b, c) and (x, y, z) are Pythagorean triplets.

i.e. $\boldsymbol{a^2 + b^2 = c^2}$ and $\boldsymbol{x^2 + y^2 = z^2}$

Also, (x, y, h) is Reciprocal Pythagorean triplet. i.e.$\boldsymbol{\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{h^2}}$.

Case 5.1: Consider the Pythagorean (4;2) tuples equation as follows

$\boldsymbol{p^2 + q^2 + r^2 + s^2 = t^2 + u^2}$ having different sets of integer solutions is illustrated below:

if p is odd then $\boldsymbol{q = p + 1}$, $\boldsymbol{r = \frac{p^2-1}{2}}$, s= $\boldsymbol{\left(\frac{p+1}{2}\right)^2 - 1}$, $t = \boldsymbol{\frac{p^2+1}{2}}$, $u = \boldsymbol{\left(\frac{p+1}{2}\right)^2 + 1}$.

Case 5.2: Consider the Pythagorean (4;2) tuples equation as follows

$\boldsymbol{p^2 + q^2 + r^2 + s^2 = t^2 + u^2}$

different sets of integer solutions is illustrated below:

if p is even then $\boldsymbol{q = p + 1}$, $\boldsymbol{r = \left(\frac{p}{2}\right)^2 - 1}$, s= $\boldsymbol{\frac{(p+1)^2-1}{2}}$, $t = \boldsymbol{\left(\frac{p}{2}\right)^2 + 1}$, $u = \boldsymbol{\frac{(p+1)^2+1}{2}}$.

Case 5.3: Consider the Pythagorean (4;2) tuples equation as follows

$\boldsymbol{p^2 + q^2 + r^2 + s^2 = t^2 + u^2}$ having different sets of integer solutions is illustrated below:

if p is even then $\boldsymbol{q = p - 1}$, $\boldsymbol{r = \left(\frac{p}{2}\right)^2 - 1}$, s= $\boldsymbol{\frac{(p-1)^2-1}{2}}$, $t = \boldsymbol{\left(\frac{p}{2}\right)^2 + 1}$, $u = \boldsymbol{\frac{(p-1)^2+1}{2}}$.

We can verify it easily by replacing some even integer p.

Case 5.4: Consider the Pythagorean (4;2) tuples equation as follows

$\boldsymbol{p^2 + q^2 + r^2 + s^2 = t^2 + u^2}$, having different sets of integer solutions is illustrated below:

if p is odd then $\boldsymbol{q = p - 1}$, $\boldsymbol{r = \frac{p^2-1}{2}}$, s= $\boldsymbol{\left(\frac{p-1}{2}\right)^2 - 1}$, $t = \boldsymbol{\frac{p^2+1}{2}}$, $u = \boldsymbol{\left(\frac{p-1}{2}\right)^2 - 1}$.

We can verify it easily by replacing some odd integer p

Case 6: Consider the Pythagorean (2;2) tuples equation as follows

$\boldsymbol{p^2 + q^2 = r^2 + s^2}$ Having two types of solutions

Case 6.1: If p is an odd, then different sets of integer solutions is illustrated below

$\boldsymbol{q = \left(\frac{p^2-1}{4}\right)^2 + 1}$, $\boldsymbol{r = \left(\frac{p^2-1}{4}\right)^2 - 1}$ and $\boldsymbol{s = \frac{p^2+1}{2}}$

Proof: From Reference [10],[11],[12], We know that, if p is odd then (p, $\frac{p^2-1}{2}$, $\frac{p^2+1}{2}$) is a Pythagorean triplet.

i.e. $p^2 + \left(\frac{p^2-1}{2}\right)^2 = \left(\frac{p^2+1}{2}\right)^2$.

Also, we know that, if p is even then (p, $\left(\frac{p}{2}\right)^2 - 1$, $\left(\frac{p}{2}\right)^2 + 1$) is a Pythagorean triplet.

If p is odd then $\frac{p^2-1}{2}$ is an even number.

Hence ($\frac{p^2-1}{2}$, $\left(\frac{p^2-1}{4}\right)^2 - 1$, $\left(\frac{p^2-1}{4}\right)^2 + 1$) is a Pythagorean triplet.

It follows that if p is odd then $p^2 + \left(\left(\frac{p^2-1}{4}\right)^2 + 1\right)^2 = \left(\left(\frac{p^2-1}{4}\right)^2 - 1\right)^2 + \left(\frac{p^2+1}{2}\right)^2$.

Hence $p^2 + q^2 = r^2 + s^2$.

Case 6.2: If p is an even integer, then different sets of integer solutions is illustrated below

$q = \frac{\left(\left(\frac{p}{2}\right)^2-1\right)^2+1}{2}$, $r = \frac{\left(\left(\frac{p}{2}\right)^2-1\right)^2-1}{2}$ and $s = \left(\frac{p}{2}\right)^2 + 1$.

Proof: If p is even then $\left(\frac{p}{2}\right)^2 - 1$ is odd number.

Hence ($\left(\frac{p}{2}\right)^2 - 1$, $\frac{\left(\left(\frac{p}{2}\right)^2-1\right)^2-1}{2}$, $\frac{\left(\left(\frac{p}{2}\right)^2-1\right)^2+1}{2}$) is a Pythagorean triplet.

if p is even then (p, $\left(\frac{p}{2}\right)^2 - 1$, $\left(\frac{p}{2}\right)^2 + 1$) is a Pythagorean triplet.

$$p^2 + \left(\left(\frac{p}{2}\right)^2 - 1\right)^2 = \left(\left(\frac{p}{2}\right)^2 + 1\right)^2.$$

$$p^2 + \left(\frac{\left(\left(\frac{p}{2}\right)^2-1\right)^2+1}{2}\right)^2 - \left(\frac{\left(\left(\frac{p}{2}\right)^2-1\right)^2-1}{2}\right)^2 = \left(\left(\frac{p}{2}\right)^2 + 1\right)^2.$$

$$p^2 + \left(\frac{\left(\left(\frac{p}{2}\right)^2-1\right)^2+1}{2}\right)^2 = \left(\frac{\left(\left(\frac{p}{2}\right)^2-1\right)^2-1}{2}\right)^2 + \left(\left(\frac{p}{2}\right)^2 + 1\right)^2.$$

Hence $p^2 + q^2 = r^2 + s^2$.

If p is an even integer, then different sets of integer solutions is illustrated below

$q = \frac{\left(\left(\frac{p}{2}\right)^2-1\right)^2+1}{2}$, $r = \frac{\left(\left(\frac{p}{2}\right)^2-1\right)^2-1}{2}$ and $s = \left(\frac{p}{2}\right)^2 + 1$.

We can verify it easily by replacing some even integer p.

Case 7: Consider the Pythagorean (4;1) tuples equation as follows

$$p^2 + q^2 + r^2 + s^2 = t^2$$

Having two types of solutions

Case 7.1: If p is an odd, then different sets of integer solutions is illustrated below

$$\boldsymbol{q=\frac{p^2-1}{2},\ r=\frac{\left(\frac{p^2+1}{2}\right)^2-1}{2},\ s=\frac{\left[\frac{\left(\frac{p^2+1}{2}\right)^2+1}{2}\right]^2-1}{2}}\text{ and }\boldsymbol{t=\frac{\left[\frac{\left(\frac{p^2+1}{2}\right)^2+1}{2}\right]^2+1}{2}}$$

Proof: Similar Proof of Case 27.1, we can verify easily $\boldsymbol{p^2+q^2+r^2+s^2=t^2}$. If p is odd then

$$\boldsymbol{p^2+\left(\frac{p^2-1}{2}\right)^2+\left(\frac{\left(\frac{p^2+1}{2}\right)^2-1}{2}\right)^2+\left(\frac{\left[\frac{\left(\frac{p^2+1}{2}\right)^2+1}{2}\right]^2-1}{2}\right)^2=\left(\frac{\left[\frac{\left(\frac{p^2+1}{2}\right)^2+1}{2}\right]^2+1}{2}\right)^2}.$$

We can verify it easily by replacing some odd integer p.

Case 7.2: If p is an even integer, then different sets of integer solutions is illustrated below

$$\boldsymbol{q=\left(\frac{p}{2}\right)^2-1,\ r=\frac{\left(\left(\frac{p}{2}\right)^2+1\right)^2-1}{2},\ s=\frac{\left[\frac{\left(\left(\frac{p}{2}\right)^2+1\right)^2+1}{2}\right]^2-1}{2}}\text{ and }\boldsymbol{t=\frac{\left[\frac{\left(\left(\frac{p}{2}\right)^2+1\right)^2+1}{2}\right]^2+1}{2}}.$$

Proof: Similar Proof of above case.

$$\boldsymbol{p^2+\left(\left(\frac{p}{2}\right)^2-1\right)^2+\left(\frac{\left(\left(\frac{p}{2}\right)^2+1\right)^2-1}{2}\right)^2+\left(\frac{\left[\frac{\left(\left(\frac{p}{2}\right)^2+1\right)^2+1}{2}\right]^2-1}{2}\right)^2=\left(\frac{\left[\frac{\left(\left(\frac{p}{2}\right)^2+1\right)^2+1}{2}\right]^2+1}{2}\right)^2}.$$

We can verify it easily by replacing some even integer p.

Case 8: Consider higher degree Diophantine equation $\boldsymbol{p^4+q^4+2r^2=s^4}$

Having two types of solutions

Case 8.1: If p is an odd, then different sets of integer solutions is illustrated below

$\boldsymbol{q=\frac{p^2-1}{2},\ r=\frac{p(p^2-1)}{2}}$ and $\boldsymbol{s=\frac{p^2+1}{2}}$.

Proof: similar proof of above.

Case 8.2: If p is an even integer, then different sets of integer solutions is illustrated below

$\boldsymbol{q=\left(\frac{p}{2}\right)^2-1,\ r=p\left(\left(\frac{p}{2}\right)^2-1\right)}$ and $\boldsymbol{s=\left(\frac{p}{2}\right)^2+1}$

Proof: similar proof of above.

Case 8.3: If (x, y, z) is a Pythagorean triplet then $\boldsymbol{x^4+y^4+2x^2y^2=z^4}$.

Proof: If (x, $\boldsymbol{y}$, $\boldsymbol{z}$) is a Pythagorean triplet. i.e. $\boldsymbol{x^2+y^2=z^2}$. Square on Both sides, we obtain $\boldsymbol{(x^2+y^2)^2=(z^2)^2}$ implies that $\boldsymbol{x^4+y^4+2x^2y^2=z^4}$.

Case 9: Consider the Pythagorean (3;3) tuples equation as follows

$\boldsymbol{p^2+q^2+t^2=r^2+s^2+u^2}$

then different sets of integer solutions is illustrated below

$\boldsymbol{p=x^2yh}$, $\boldsymbol{q=y^2xh}$, $\boldsymbol{r=yh}$, $s=\boldsymbol{xh}$, $t=\boldsymbol{xy}$, $u=\boldsymbol{xyzh}$

where $\boldsymbol{x=bc}$, $\boldsymbol{y=ca}$, $\boldsymbol{h=ab}$, z = $\boldsymbol{c^2}$ with (a, b, c) is a Pythagorean triplet, which is satisfies $\boldsymbol{a^2+b^2=c^2}$.

Conclusion

This paper first focused on studying infinitely many integer solutions of $\boldsymbol{k(x-\alpha)^2+(y-\beta)^2=(z-\gamma)^2}$ having ellipse equation form of $\boldsymbol{k\left(\frac{x-\alpha}{z-\gamma}\right)^2+\left(\frac{y-\beta}{z-\gamma}\right)^2=1}$; Also, concentrate on studying Reciprocal form of above Diophantine Equation $\boldsymbol{\frac{k}{p^2}+\frac{1}{q^2}=\frac{1}{h^2}}$, Which is having different sets of integer solutions of

$\boldsymbol{p=(y-\beta)(z-\gamma)}$, $\boldsymbol{q=(x-\alpha)(z-\gamma)}$ and $\boldsymbol{h=(x-\alpha)(y-\beta)}$. Also, introduces to study of binary operation on Set of Ellipse Diophantine Equation. Also, introduces to study of binary operation on above Set of quadratic Diophantine Equation.

$$P_1.P_2=\begin{Bmatrix}(kx_1{x_2}^2, y_1({y_2}^2+{z_2}^2)+2y_2z_1z_2, z_1({y_2}^2+{z_2}^2)+2y_1y_2z_2)\\(kx_2{x_1}^2, y_2({y_1}^2+{z_1}^2)+2y_1z_1z_2, z_2({y_1}^2+{z_1}^2)+2y_1y_2z_1)\end{Bmatrix}.$$

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