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A Study On Integer Design Of Solutions Of Diophantine Equation $\alpha(X^4+Y^4)(\gamma U^2+V^2)=T^2(C^2-D^2)(Z^2-W^2)P^\beta$ With $\alpha>0,\beta>0,\gamma=2,3$ and $X<Y<W<Z$

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Abstract:

This paper focused on a study to find integer design of solutions Diophantine Equation

$\boldsymbol{\alpha(X^4+Y^4)(\gamma U^2+V^2)=T^2(C^2-D^2)(Z^2-W^2)P^\beta}$ **With** $\boldsymbol{\alpha>0,\beta>0\ and\ X<Y<W<Z}$

with Mathematical induction method for $\beta=1,2,3,4,\dots$ *and so on.* Diophantine equations of higher degrees, play a meaningful role in generating special elliptic curves that are crucial for cryptography and secure communications.

In this paper, I was focused given Diophantine equation with more than 8 unknowns with two cases

Lemma 1: At $\gamma=2$, the Diophantine equation

$\boldsymbol{\alpha(X^4+Y^4)(2U^2+V^2)=T^2(C^2-D^2)(Z^2-W^2)P^\beta}$ **With** $\boldsymbol{\alpha>0,\beta>0\ and\ X<Y<W<Z}$

Having integer design of solutions **for $\beta>2$** is parameterized by integers k and n, with variables defined as:

$x=k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, p=k^n, U=2^{n+1}, V=2^n, T=3(2)^n,$

$\alpha=(1+k^4)(k^6-k^4)k^{(\beta-2)n}n^2$

$$, and \begin{Bmatrix} C=\left(\left(\frac{1+k^4}{2}\right)^2+1\right)n, D=\left(\left(\frac{1+k^4}{2}\right)^2-1\right)n, & if\ 1+k^4\ is\ even \\ C=\left(\frac{(1+k^4)^2+1}{2}\right)n, D=\left(\frac{(1+k^4)^2-1}{2}\right)n, & if\ 1+k^4\ is\ odd \end{Bmatrix}.$$

for $\beta=1$ is $x=k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, p=k^{2n}, U=2^{n+1}, V=2^n, T=3(2)^n,$

$$\alpha=(1+k^4)(k^6-k^4)n^2 \text{ and } \begin{Bmatrix} C=\left(\left(\frac{1+k^4}{2}\right)^2+1\right)n, D=\left(\left(\frac{1+k^4}{2}\right)^2-1\right)n, & if\ 1+k^4\ is\ even \\ C=\left(\frac{(1+k^4)^2+1}{2}\right)n, D=\left(\frac{(1+k^4)^2-1}{2}\right)n, & if\ 1+k^4\ is\ odd \end{Bmatrix}.$$

for $\boldsymbol{\beta = 2}$ is $x = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n, U = 2^{n+1}, V = 2^n, T = 3(2)^n$,

$$\alpha = (1+k^4)(k^6-k^4)n^2 \text{ and } \begin{Bmatrix} C = \left(\left(\frac{1+k^4}{2}\right)^2+1\right)n, D = \left(\left(\frac{1+k^4}{2}\right)^2-1\right)n, & if\ 1+k^4\ is\ even \\ C = \left(\frac{(1+k^4)^2+1}{2}\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n, & if\ 1+k^4\ is\ odd \end{Bmatrix}.$$

Lemma 2: At $\boldsymbol{\gamma = 3}$, **the Diophantine equation**

$$\boldsymbol{\alpha(X^4+Y^4)(3U^2+V^2) = T^2(C^2-D^2)(Z^2-W^2)P^\beta} \textbf{ With } \boldsymbol{\alpha > 0, \beta > 0\ and\ X < Y < W < Z}$$

Having integer design of solutions **for** $\boldsymbol{\beta > 2}$ is parameterized by integers k and n, with variables defined as:

$x = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n, U = 3^n, V = 3^n, T = 2(3)^n$,

$\alpha = (1+k^4)(k^6-k^4)k^{(\beta-2)n}n^2$

$$, and \begin{Bmatrix} C = \left(\left(\frac{1+k^4}{2}\right)^2+1\right)n, D = \left(\left(\frac{1+k^4}{2}\right)^2-1\right)n, & if\ 1+k^4\ is\ even \\ C = \left(\frac{(1+k^4)^2+1}{2}\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n, & if\ 1+k^4\ is\ odd \end{Bmatrix}.$$

for $\boldsymbol{\beta = 1}$ is $x = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{2n}, U = 3^n, V = 3^n, T = 2(3)^n$,

$$\alpha = (1+k^4)(k^6-k^4)n^2 \text{ and } \begin{Bmatrix} C = \left(\left(\frac{1+k^4}{2}\right)^2+1\right)n, D = \left(\left(\frac{1+k^4}{2}\right)^2-1\right)n, & if\ 1+k^4\ is\ even \\ C = \left(\frac{(1+k^4)^2+1}{2}\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n, & if\ 1+k^4\ is\ odd \end{Bmatrix}.$$

for $\boldsymbol{\beta = 2}$ is $x = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n, U = 3^n, V = 3^n, T = 2(3)^n$,

$$\alpha = (1+k^4)(k^6-k^4)n^2 \text{ and } \begin{Bmatrix} C = \left(\left(\frac{1+k^4}{2}\right)^2+1\right)n, D = \left(\left(\frac{1+k^4}{2}\right)^2-1\right)n, & if\ 1+k^4\ is\ even \\ C = \left(\frac{(1+k^4)^2+1}{2}\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n, & if\ 1+k^4\ is\ odd \end{Bmatrix}$$

Keywords: Diophantine Equation, exponential, Pythagorean triplet, Integer design.

I. Introduction:

Diophantine equations—polynomial equations with integer solutions—are a central topic in number theory. Among their many variants, **exponential Diophantine equations** involve terms where variables appear as exponents. Finding integer solutions to such equations is notably complex and has implications in mathematics, cryptography, and several scientific fields. Historical Context and Theoretical Background

Classical Diophantine Equations: Traditionally, research started with linear and polynomial forms, such as the well-known cases of Pythagorean triples .

Exponential Generalization: The study of exponential forms expanded from these roots, posing questions that often lack general solution methods and in some cases are proven to be undecidable.

II. Results & Discussions:

In this paper, focused to find the general exponential integer solution of the general exponential integer solution of $\boldsymbol{(X^4+Y^4)(\gamma U^2+V^2) = T^2(C^2-D^2)(Z^2-W^2)P^\beta}$ **at** $\boldsymbol{\gamma = 2\ and\ \gamma = 3}$

Lemma 1: at $\gamma = 2$, *the diophantine equaiton* $\alpha(X^4+Y^4)(2U^2+V^2) = T^2(C^2-D^2)(Z^2-W^2)P^{\beta}$ With $\alpha > 0, \beta > 0$ is derived from fixed value of $\beta = 1, \beta = 2$ *and* $\beta > 2$.

Proportion 1: A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 1}$ **is** $\boldsymbol{\alpha(X^4+Y^4)(2U^2+V^2) = T^2(C^2-D^2)(Z^2-W^2)P}$

Explanation: Let x = $k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{2n}, U = 2^{n+1}, V = 2^n, T = 3(2)^n$

Consider $\alpha(X^4+Y^4)(\mathbf{2U^2+V^2}) = \alpha k^{4n}(1+k^4)(3(2)^n)^2$.

Again consider $\boldsymbol{T^2(C^2-D^2)(Z^2-W^2)P} = (C^2-D^2)k^{4n}(k^6-k^4)(3(2)^n)^2$

It follows that $\boldsymbol{\alpha(X^4+Y^4)(2U^2+V^2) = T^2(C^2-D^2)(Z^2-W^2)P}$ implies that

$\alpha k^{4n}(1+k^4)(3(2)^n)^2 = (C^2-D^2)k^{4n}(k^6-k^4)(3(2)^n)^2$ implies $\alpha(1+k^4) = (C^2-D^2)(k^6-k^4)$.

Solve for α, whenever $(1+k^4, D, C)$ is a Pythagorean Triplet.

From the References [1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11],[12] there is so many methods to generate Pythagorean triplet, now I chosen one of the technique of

if r is an even number, then (r, $\left(\frac{r}{2}\right)^2 - 1$, $\left(\frac{r}{2}\right)^2 + 1$) is a Pythagorean triplet.

If r is an odd number, then (r, $\frac{r^2-1}{2}$, $\frac{r^2+1}{2}$) is a Pythagorean triplet.

It implies that $(1+k^4, D, C)$ becomes a Pythagorean Triplet depending on *whether* $1+k^4$ is odd or even.

Case 1: **If $\boldsymbol{1+k^4}$ is even**, then ($1+k^4$, $\left(\frac{1+k^4}{2}\right)^2 - 1$, $\left(\frac{1+k^4}{2}\right)^2 + 1$) is a Pythagorean triplet.

It follows that $\alpha(1+k^4) = (C^2-D^2)(k^6-k^4)$ and solve for α,

whenever $(1+k^4, D, C)$ becomes a Pythagorean Triplet with $C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n$,

$D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n$ *and* $C^2 - D^2 = (1+k^4)^2 n^2$ and hence $\alpha = (1+k^4)(k^6-k^4)n^2$.

Hence, **we obtain** $\boldsymbol{(X^4+Y^4)(2U^2+V^2) = T^2(C^2-D^2)(Z^2-W^2)P}$ **having integer design of solution is**

x $=$ $\boldsymbol{k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}}, U = 2^{n+1}, V = 2^n, T = 3(2)^n, \boldsymbol{p = k^{2n}}, \boldsymbol{C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n}$,

$\boldsymbol{D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n, \alpha = (1+k^4)(k^6-k^4)n^2}$.

Verification:

Consider **LHS** $\alpha(X^4+Y^4)(\mathbf{2U^2+V^2}) = \boldsymbol{(1+k^4)(k^6-k^4)n^2(k^{4n}+k^{4n+4})}(3(2)^n)^2$

$= \boldsymbol{k^{4n}}(k^6-k^4)\boldsymbol{(1+k^4)^2 n^2}(3(2)^n)^2$

Consider RHS $\boldsymbol{T^2(C^2-D^2)(Z^2-W^2)P} = (3(2)^n)^2(1+k^4)^2 n^2(k^{2n+6}-k^{2n+4})k^{2n}$

$= (3(2)^n)^2\boldsymbol{k^{4n}}(k^6-k^4)\boldsymbol{(1+k^4)}^2 n^2$.

Hence LHS = RHS.

Case 2: If $1 + k^4$ is odd, then $\left(1 + k^4, \frac{(1+k^4)^2-1}{2}, \frac{(1+k^4)^2+1}{2}\right)$ is a Pythagorean triplet. It follows that $\alpha(1 + k^4) = (C^2 - D^2)(k^6 - k^4)$ and solves for α, whenever $(1 + k^4, D, C)$ becomes a Pythagorean Triplet with $C = \left(\frac{(1+k^4)^2+1}{2}\right)n$, $D = \left(\frac{(1+k^4)^2-1}{2}\right)n$.

Hence $C^2 - D^2 = (1 + k^4)^2 n^2$ and hence $\alpha = (1 + k^4)(k^6 - k^4)n^2$.

Hence, $(X^4 + Y^4)(2U^2 + V^2) = T^2(C^2 - D^2)(Z^2 - W^2)P$ **having integer design of solution is**

$x = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, U = 2^{n+1}, V = 2^n, T = 3(2)^n, p = k^{2n}$,

$C = \left(\frac{(1+k^4)^2+1}{2}\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n, \ \alpha = (1 + k^4)(k^6 - k^4).$

Verification:

Consider **LHS** $\alpha(X^4 + Y^4)(2U^2 + V^2) = (1 + k^4)(k^6 - k^4)n^2(k^{4n} + k^{4n+4})(3(2)^n)^2$

$= k^{4n}(k^6 - k^4)(1 + k^4)^2 n^2 (3(2)^n)^2$

Consider RHS $T^2(C^2 - D^2)(Z^2 - W^2)P = (3(2)^n)^2(1 + k^4)^2 n^2 (k^{2n+6} - k^{2n+4})k^{2n}$

$= (3(2)^n)^2 k^{4n}(k^6 - k^4)(1 + k^4)^2 n^2.$

Hence LHS = RHS.

E.g. 1: Suppose k = 2 then $1 + k^4 = 17$, is odd; Having an integer design of solution is

$x = 2^n, y = 2^{n+1}, z = 2^{n+3}, w = 2^{n+2}, p = 2^{2n}, U = 2^{n+1}, V = 2^n, T = 3(2)^n$,

$C = \left(\frac{(1+2^4)^2+1}{2}\right)n = 145n, D = \left(\frac{(1+2^4)^2-1}{2}\right) = 144n$

$C^2 - D^2 = (1 + 2^4)^2 n^2, \ \alpha = (1 + 2^4)(2^6 - 2^4)n^2.$

Suppose n = 1; then x = 2, $y = 4, z = 16, w = 8, p = 4$, U = 4 , V = 2 , T = 6

$C = \left(\frac{(1+2^4)^2+1}{2}\right) = 145, D = \left(\frac{(1+2^4)^2-1}{2}\right) = 144$

$C^2 - D^2 = (1 + 2^4)^2 = 289, \ \alpha = (1 + 2^4)(2^6 - 2^4) = 816.$

Consider **LHS**= $\alpha(X^4 + Y^4)(2U^2 + V^2)$ =**816**$(2^4 + 4^4)(36) = 221952 = 7990272.$

RHS = $T^2(C^2 - D^2)(Z^2 - W^2)P = 36 * 289 * (16^2 - 8^2) * 4 = 7990272.$

E.g. 2: Suppose k = 3 then $1 + k^4 = 82$, is even; Having an integer design of solution is

$x = 3^n, y = 3^{n+1}, z = 3^{n+3}, w = 3^{n+2}, p = 3^{2n}, U = 2^{n+1}, V = 2^n, T = 3(2)^n$,

$C = \left(\left(\frac{1+3^4}{2}\right)^2 + 1\right)n = 1681n, \ D = \left(\left(\frac{1+3^4}{2}\right)^2 - 1\right)n = 1600n$

$C^2 - D^2 = (1 + 3^4)^2 n^2 = 6724n, \ \alpha = (1 + 3^4)(3^6 - 3^4)n^2 = 53136n^2.$

Suppose n = 1; then x = 3, $y = 9, z = 81, w = 27, p = 9$, U = 4 , V = 2 , T = 6, $C^2 - D^2 = 6724, \ \alpha = 53136$

Consider **LHS**= $\alpha(X^4 + Y^4)(2U^2 + V^2)$ =**53136**$* (3^4 + 9^4) * 36 = 352929312 * 36 = 12705455232.$

RHS = $T^2 (C^2 - D^2)(Z^2 - W^2)P = 36 * 6724 * (81^2 - 27^2) * 9 = 12705455232.$

Proportion 2: A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 2}$ **is** $\boldsymbol{\alpha(X^4 + Y^4)(2U^2 + V^2) = T^2(C^2 - D^2)(Z^2 - W^2)P^2}$

Explanation: Let x $= k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n, U = 2^{n+1},\ V = 2^n, T = 3(2)^n$

Consider $\alpha(X^4 + Y^4)(\mathbf{2U^2 + V^2}) = \alpha k^{4n}(1 + k^4)(3(2)^n)^2$

Again consider $\mathbf{T^2}(Z^2 - W^2)\mathbf{P^2} = k^{4n}(k^6 - k^4)(3(2)^n)^2$.

It follows that $\boldsymbol{\alpha(X^4 + Y^4)(2U^2 + V^2) = T^2}(C^2 - D^2)(Z^2 - W^2)\mathbf{P^2}$ implies that

$\alpha k^{4n}(1 + k^4)(3(2)^n)^2 = (C^2 - D^2)k^{4n}(k^6 - k^4)(3(2)^n)^2$ implies $\alpha(1 + k^4) = (C^2 - D^2)(k^6 - k^4)$.

Solve for α, whenever $(1 + k^4, D, C)$ becomes a Pythagorean Triplet.

It implies that $(1 + k^4, D, C)$ becomes a Pythagorean Triplet depending on *whether* $1 + k^4$ is odd or even.

Case 1: **If** $\mathbf{1 + k^4}$ **is even**, then ($1 + k^4$, $\left(\frac{1+k^4}{2}\right)^2 - 1$, $\left(\frac{1+k^4}{2}\right)^2 + 1$) is a Pythagorean triplet.

It follows that $\alpha(1 + k^4) = (C^2 - D^2)(k^6 - k^4)$ and solve for α,

whenever $(1 + k^4, D, C)$ becomes a Pythagorean Triplet with $C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n$,

$D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n$ *and* $C^2 - D^2 = (1 + k^4)^2 n^2$ and hence $\alpha = (1 + k^4)(k^6 - k^4)n^2$.

Hence, **we obtain** $\boldsymbol{(X^4 + Y^4)(2U^2 + V^2) = T^2(C^2 - D^2)(Z^2 - W^2)P^2}$ **having integer design of solution is**

x $= \boldsymbol{k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}}, U = 2^{n+1},\ V = 2^n, T = 3(2)^n, \boldsymbol{p = k^n, C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n}$,

$\boldsymbol{D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n, \alpha = (1 + k^4)(k^6 - k^4)n^2}$.

Verification:

Consider **LHS** $\alpha(X^4 + Y^4)(\mathbf{2U^2 + V^2}) = \mathbf{(1 + k^4)(k^6 - k^4)n^2(k^{4n} + k^{4n+4})}(3(2)^n)^2$

$= \mathbf{k^{4n}}(k^6 - k^4)\mathbf{(1 + k^4)^2 n^2}(3(2)^n)^2$

Consider RHS $\boldsymbol{T^2(C^2 - D^2)(Z^2 - W^2)P^2} = (3(2)^n)^2(1 + k^4)^2 n^2(k^{2n+6} - k^{2n+4})k^{2n}$

$= (3(2)^n)^2\mathbf{k^{4n}}(k^6 - k^4)\mathbf{(1 + k^4)^2}n^2$.

Hence LHS $= \boldsymbol{RHS}$.

Case 2: If $\mathbf{1 + k^4}$ **is odd**, then ($1 + k^4$, $\frac{(1+k^4)^2-1}{2}$, $\frac{(1+k^4)^2+1}{2}$) is a Pythagorean triplet. It follows that $\alpha(1 + k^4) = (C^2 - D^2)(k^6 - k^4)$ and solves for α, whenever $(1 + k^4, D, C)$ becomes a Pythagorean Triplet with $C = \left(\frac{(1+k^4)^2+1}{2}\right)n$, $D = \left(\frac{(1+k^4)^2-1}{2}\right)n$.

Hence $C^2 - D^2 = (1 + k^4)^2 n^2$ and hence $\alpha = (1 + k^4)(k^6 - k^4)n^2$.

Hence, $\boldsymbol{(X^4 + Y^4)(2U^2 + V^2) = T^2(C^2 - D^2)(Z^2 - W^2)P^2}$ **having integer design of solution is**

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$\boldsymbol{C = \left(\frac{(1+k^4)^2+1}{2}\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n,\ \alpha = (1 + k^4)(k^6 - k^4)}$.

Verification:

Consider **LHS** $\alpha(X^4+Y^4)(\mathbf{2U^2+V^2}) = (\mathbf{1+k^4})(\mathbf{k^6-k^4})\mathbf{n^2}(\mathbf{k^{4n}+k^{4n+4}})(3(2)^n)^2$

$= \mathbf{k^{4n}}(k^6-k^4)(\mathbf{1+k^4})^2\mathbf{n^2}(3(2)^n)^2$

Consider RHS $\mathbf{T^2(C^2-D^2)(Z^2-W^2)P^2} = (3(2)^n)^2(1+k^4)^2n^2(k^{2n+6}-k^{2n+4})k^{2n}$

$= (3(2)^n)^2\mathbf{k^{4n}}(k^6-k^4)(\mathbf{1+k^4})^2n^2.$

Hence LHS = $\boldsymbol{RHS}$.

Proportion 3: for $\beta > 2$ is

$(\mathbf{X^4+Y^4})(\mathbf{2U^2+V^2}) = \mathbf{T^2(C^2-D^2)(Z^2-W^2)}P^\beta$ with $\alpha > 0, \beta > 0$ *and* $x<y<z<w$

Having integer design of solutions is

x $= k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n, U = 2^{n+1},\ V = 2^n, T = 3(2)^n$

If $1+k^4$ is even then $C = \left(\left(\frac{1+k^4}{2}\right)^2+1\right)n,\ D = \left(\left(\frac{1+k^4}{2}\right)^2-1\right)n, \alpha = (1+k^4)(k^6-k^4)k^{(\beta-2)n}n^2.$

and if $1+k^4$ is odd then $C = \left(\frac{(1+k^4)^2+1}{2}\right)n,\ D = \left(\frac{(1+k^4)^2-1}{2}\right)n,\ \alpha = (1+k^4)(k^6-k^4)k^{(\beta-2)n}n^2.$

Lemma 2: $\boldsymbol{\gamma = 3}$, the Diophantine equation $\boldsymbol{\alpha(X^4+Y^4)(3U^2+V^2) = T^2(C^2-D^2)(Z^2-W^2)P^\beta}$ With $\boldsymbol{\alpha > 0, \beta > 0}$ is derived from fixed value of $\boldsymbol{\beta = 1, \beta = 2}$ *and* $\boldsymbol{\beta > 2}$.

Proportion 1: A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 1}$ **is** $\boldsymbol{\alpha(X^4+Y^4)(3U^2+V^2) = T^2(C^2-D^2)(Z^2-W^2)P}$

Explanation: Let x $= k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{2n}, U = 3^n,\ V = 3^n, T = 2(3)^n$

Consider $\alpha(X^4+Y^4)(\mathbf{3U^2+V^2}) = \alpha k^{4n}(1+k^4)(2(3)^n)^2$.

Again consider $\mathbf{T^2(C^2-D^2)(Z^2-W^2)P} = (C^2-D^2)k^{4n}(k^6-k^4)(2(3)^n)^2$

It follows that $\boldsymbol{\alpha(X^4+Y^4)(3U^2+V^2) = T^2(C^2-D^2)(Z^2-W^2)P}$ implies that

$\alpha k^{4n}(1+k^4)(2(3)^n)^2 = (C^2-D^2)k^{4n}(k^6-k^4)(2(3)^n)^2$ implies $\alpha(1+k^4) = (C^2-D^2)(k^6-k^4).$

Solve for α, whenever $(1+k^4, D, C)$ is a Pythagorean Triplet.

From the References [1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11],[12] there is so many methods to generate Pythagorean triplet, now I chosen one of the technique of

if r is an even number, then (r, $\left(\frac{r}{2}\right)^2-1, \left(\frac{r}{2}\right)^2+1$) is a Pythagorean triplet.

If r is an odd number, then (r, $\frac{r^2-1}{2}, \frac{r^2+1}{2}$) is a Pythagorean triplet.

It implies that $(1+k^4, D, C)$ becomes a Pythagorean Triplet depending on *whether* $1+k^4$ is odd or even.

Case 1: **If $\boldsymbol{1+k^4}$ is even**, then ($1+k^4, \left(\frac{1+k^4}{2}\right)^2-1, \left(\frac{1+k^4}{2}\right)^2+1$) is a Pythagorean triplet.

It follows that $\alpha(1+k^4) = (C^2-D^2)(k^6-k^4)$ and solve for α,

whenever $(1+k^4, D, C)$ becomes a Pythagorean Triplet with $C = \left(\left(\frac{1+k^4}{2}\right)^2+1\right)n,$

$D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n$ and $C^2 - D^2 = (1+k^4)^2n^2$ and hence $\alpha = (1+k^4)(k^6-k^4)n^2$.

Hence, we obtain $(X^4+Y^4)(3U^2+V^2) = T^2(C^2-D^2)(Z^2-W^2)P$ **having integer design of solution is**

$x = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, U = 3^n,\ V = 3^n, T = 2(3)^n, p = k^{2n}, C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n,$

$D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n, \alpha = (1+k^4)(k^6-k^4)n^2.$

Verification:

Consider **LHS** $\alpha(X^4+Y^4)(3U^2+V^2) = (1+k^4)(k^6-k^4)n^2(k^{4n}+k^{4n+4})(2(3)^n)^2$

$= k^{4n}(k^6-k^4)(1+k^4)^2n^2(2(3)^n)^2$

Consider RHS $T^2(C^2-D^2)(Z^2-W^2)P = (2(3)^n)^2(1+k^4)^2n^2(k^{2n+6}-k^{2n+4})k^{2n}$

$= (2(3)^n)^2k^{4n}(k^6-k^4)(1+k^4)^2n^2.$

Hence LHS $= RHS.$

Case 2: If $1+k^4$ is odd, then $\left(1+k^4, \frac{(1+k^4)^2-1}{2}, \frac{(1+k^4)^2+1}{2}\right)$ is a Pythagorean triplet. It follows that $\alpha(1+k^4) = (C^2-D^2)(k^6-k^4)$ and solves for α, whenever $(1+k^4, D, C)$ becomes a Pythagorean Triplet with $C = \left(\frac{(1+k^4)^2+1}{2}\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n.$

Hence $C^2 - D^2 = (1+k^4)^2n^2$ and hence $\alpha = (1+k^4)(k^6-k^4)n^2$.

Hence, $(X^4+Y^4)(3U^2+V^2) = T^2(C^2-D^2)(Z^2-W^2)P$ **having integer design of solution is**

$x = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, U = 3^n,\ V = 3^n, T = 2(3)^n, p = k^{2n},$

$C = \left(\frac{(1+k^4)^2+1}{2}\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n,\ \alpha = (1+k^4)(k^6-k^4).$

Verification:

Consider **LHS** $\alpha(X^4+Y^4)(3U^2+V^2) = (1+k^4)(k^6-k^4)n^2(k^{4n}+k^{4n+4})(2(3)^n)^2$

$= k^{4n}(k^6-k^4)(1+k^4)^2n^2(2(3)^n)^2$

Consider RHS $T^2(C^2-D^2)(Z^2-W^2)P = (2(3)^n)^2(1+k^4)^2n^2(k^{2n+6}-k^{2n+4})k^{2n}$

$= (2(3)^n)^2k^{4n}(k^6-k^4)(1+k^4)^2n^2.$

Hence LHS $= RHS.$

Proportion 2: A Study on exponential integer solution of above Diophantine Equation at

$\beta = 2$ **is** $\alpha(X^4+Y^4)(3U^2+V^2) = T^2(C^2-D^2)(Z^2-W^2)P^2$

Explanation: Let $x = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n, U = 3^n,\ V = 3^n, T = 2(3)^n$

Consider $\alpha(X^4+Y^4)(3U^2+V^2) = \alpha k^{4n}(1+k^4)(2(3)^n)^2$

Again consider $T^2(Z^2-W^2)P^2 = k^{4n}(k^6-k^4)(2(3)^n)^2$.

It follows that $\alpha(X^4+Y^4)(3U^2+V^2) = T^2(C^2-D^2)(Z^2-W^2)P^2$ implies that

$\alpha k^{4n}(1+k^4)(2(3)^n)^2 = (C^2-D^2)k^{4n}(k^6-k^4)(2(3)^n)^2$ implies $\alpha(1+k^4) = (C^2-D^2)(k^6-k^4).$

Solve for α, whenever $(1+k^4, D, C)$ becomes a Pythagorean Triplet.

It implies that $(1+k^4, D, C)$ becomes a Pythagorean Triplet depending on *whether* $1+k^4$ is odd or even.

Case 1: **If $1+k^4$ is even**, then $\left(1+k^4, \left(\frac{1+k^4}{2}\right)^2 - 1, \left(\frac{1+k^4}{2}\right)^2 + 1\right)$ is a Pythagorean triplet.

It follows that $\alpha(1+k^4) = (C^2 - D^2)(k^6 - k^4)$ and solve for α,

whenever $(1+k^4, D, C)$ becomes a Pythagorean Triplet with $C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n$,

$D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n$ *and* $C^2 - D^2 = (1+k^4)^2 n^2$ and hence $\alpha = (1+k^4)(k^6 - k^4)n^2$.

Hence, **we obtain** $(X^4 + Y^4)(3U^2 + V^2) = T^2(C^2 - D^2)(Z^2 - W^2)P^2$ **having integer design of solution is**

$$\mathrm{x} = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, U = 3^n,\ V = 3^n, T = 2(3)^n, p = k^n, C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n,$$

$$D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n, \alpha = (1+k^4)(k^6 - k^4)n^2.$$

Verification:

Consider **LHS** $\alpha(X^4 + Y^4)(3U^2 + V^2) = (1+k^4)(k^6 - k^4)n^2(k^{4n} + k^{4n+4})(2(3)^n)^2$

$= k^{4n}(k^6 - k^4)(1+k^4)^2 n^2 (2(3)^n)^2$

Consider RHS $T^2(C^2 - D^2)(Z^2 - W^2)P^2 = (2(3)^n)^2(1+k^4)^2 n^2 (k^{2n+6} - k^{2n+4})k^{2n}$

$= (2(3)^n)^2 k^{4n}(k^6 - k^4)(1+k^4)^2 n^2.$

Hence LHS $= RHS$.

Case 2: If $1+k^4$ is odd, then $\left(1+k^4, \frac{(1+k^4)^2 - 1}{2}, \frac{(1+k^4)^2 + 1}{2}\right)$ is a Pythagorean triplet. It follows that $\alpha(1+k^4) = (C^2 - D^2)(k^6 - k^4)$ and solves for α, whenever $(1+k^4, D, C)$ becomes a Pythagorean Triplet with $C = \left(\frac{(1+k^4)^2+1}{2}\right)n$, $D = \left(\frac{(1+k^4)^2-1}{2}\right)n$.

Hence $C^2 - D^2 = (1+k^4)^2 n^2$ and hence $\alpha = (1+k^4)(k^6 - k^4)n^2$.

Hence, $(X^4 + Y^4)(3U^2 + V^2) = T^2(C^2 - D^2)(Z^2 - W^2)P^2$ **having integer design of solution is**

$$\mathrm{x} = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, U = 3^n,\ V = 3^n, T = 2(3)^n, p = k^n,$$

$$C = \left(\frac{(1+k^4)^2+1}{2}\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n,\ \alpha = (1+k^4)(k^6 - k^4).$$

Verification:

Consider **LHS** $\alpha(X^4 + Y^4)(3U^2 + V^2) = (1+k^4)(k^6 - k^4)n^2(k^{4n} + k^{4n+4})(2(3)^n)^2$

$= k^{4n}(k^6 - k^4)(1+k^4)^2 n^2 (2(3)^n)^2$

Consider RHS $T^2(C^2 - D^2)(Z^2 - W^2)P^2 = (2(3)^n)^2(1+k^4)^2 n^2 (k^{2n+6} - k^{2n+4})k^{2n}$

$= (2(3)^n)^2 k^{4n}(k^6 - k^4)(1+k^4)^2 n^2.$

Hence LHS $= RHS$.

Proportion 3: for $\beta > 2$ is

$(X^4 + Y^4)(3U^2 + V^2) = T^2(C^2 - D^2)(Z^2 - W^2)P^\beta$ with $\alpha > 0, \beta > 0$ *and* $x < y < z < w$

Having integer design of solutions is

$x = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^n, U = 3^n,\ V = 3^n\ , T = 2(3)^n$

If $1 + k^4$ is even then $C = \left(\left(\frac{1+k^4}{2}\right)^2 + 1\right)n,\ D = \left(\left(\frac{1+k^4}{2}\right)^2 - 1\right)n, \alpha = (1 + k^4)(k^6 - k^4)k^{(\beta-2)n}n^2$.

and if $1 + k^4$ is odd then $C = \left(\frac{(1+k^4)^2+1}{2}\right)n, D = \left(\frac{(1+k^4)^2-1}{2}\right)n,\ \alpha = (1 + k^4)(k^6 - k^4)k^{(\beta-2)n}n^2$.

III. Conclusion

This equation generalizes classical Diophantine problems, blending sums of fourth powers with multiplicative factorizations. While challenging, targeted parametrization and modular analysis can yield solutions. Future work may classify solutions for specific α, β or link to broader number-theoretic frameworks. The parametric framework provides infinite families of solutions by exploiting algebraic identities and modular arithmetic. Future work could explore non-parametric solutions or generalizations to higher exponents.

In this paper, I focused to find integer design of solutions as two lemmas.

Lemma 1: At $\gamma = 2$, the Diophantine equation

$\boldsymbol{\alpha(X^4 + Y^4)(2U^2 + V^2) = T^2(C^2 - D^2)(Z^2 - W^2)P^\beta}$ **With** $\boldsymbol{\alpha > 0, \beta > 0\ and\ X < Y < W < Z}$

Lemma 2: At $\gamma = 3$, **the Diophantine equation**

$\boldsymbol{\alpha(X^4 + Y^4)(3U^2 + V^2) = T^2(C^2 - D^2)(Z^2 - W^2)P^\beta}$ **With** $\boldsymbol{\alpha > 0, \beta > 0\ and\ X < Y < W < Z}$

References

[1] https://mathworld.wolfrom.com/pythagorean triples

[2] A new approach to generate all Pythagorean triples by Anthony Overmars, AIMS Mathematics, 4(2):242-253.

[3] A textbook “Introduction to Analytic Number Theory “by Tom M. Apostol, Springer.

[4] Srinivas, T. (2024). Construction of Pythagorean and Reciprocal Pythagorean n-tuples. Springer Proceedings in Mathematics and Statistics.

[5] Sridevi, K., & **Srinivas, T.** (2023). Transcendental representation of Diophantine equation and some of its inherent properties. *Materials Today: Proceedings*, *80*, 1822-1825

[6] Sridevi, K., & **Srinivas, T.** (2023). Existence Of Inner Addition and Inner Multiplication On Set of Triangular Numbers and Some Inherent properties of Triangular Numbers. *Materials Today: Proceedings*, *80*, 1822-1825.

[7] Sridevi, K., & **Srinivas, T.** (2023). Cryptographic coding To Define Binary Operation on Set of Pythagorean Triples. *Materials Today: Proceedings*, *80*, 1822-1825

[8] Sridevi, K., & **Srinivas, T**. (2022). Algebraic Structure Of Reciprocal Pythagorean Triples. Advances And Applications In Mathematical Sciences, Volume 21, Issue 3, January 2022, P.1315-1327© 2022 Mili Publications, India Issn 0974-6803)

[9] Srinivas, T., & Sridevi, K. (2022, January). A new approach to define a new integer sequences of Fibonacci type numbers with using of third order linear Recurrence relations. In *AIP Conference Proceedings* (Vol. 2385, No. 1, p. 130005

[10] Srinivas, T., & Sridevi, K. (2021, November). A New approach to define Algebraic Structure and Some Homomorphism Functions on Set of Pythagorean Triples and Set of Reciprocal Pythagorean Triples " in JSR (Journal of Scientific Research) Volume 65, Issue 9, November 2021, Pages 86-92.

[11] Sridevi, K., & Srinivas, T. (2020). A new approach to define two types of binary operations on set of Pythagorean triples to form as at most commutative cyclic
semi group. *Journal of Critical Reviews*, *7*(19), 9871-9878

[12] Srinivas, T. (2023). Some Inherent Properties of Pythagorean Triples. *Research Highlights in Mathematics and Computer Science Vol. 7*, 156-169.