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A Study On Integer Design Of Solutions Of Diophantine Equation $\alpha(X^4+Y^4)^2(2U^2+V^2)=T^2(C^2+D^2)(Z^2+W^2)P^\beta$ With $\alpha>0, \beta=1,2,3,4,5,6,7$ *and* $x<y<w<z$

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Abstract:

Diophantine equations of higher degrees, play a meaningful role in generating special elliptic curves that are crucial for cryptography and secure communications.

In this paper, I was focused given Diophantine equation with more than 8 unknowns and focused on a study to find integer design of solutions Diophantine Equation

$\alpha(X^4+Y^4)^2(2U^2+V^2)=T^2(C^2+D^2)(Z^2+W^2)P^\beta$ **With** $\alpha>0, \gamma=2,3, \beta=1,2,3,4,5,6,7$ ***and***

$x<y<w<z$ with Mathematical induction & generation of Pythagorean triplets.

for $\beta=1$, having integer design of solution is parameterized by positive integers k and n, with variables defined as:

$x=k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, U=2^{n+1}, V=2^n, T=3(2)^n,$

$p=k^{6n}, \alpha=(k^6+k^4), C=2k^2, D=k^4-1.$

for $\beta=2$, having integer design of solution is parameterized by integers k and n, with variables defined as:

$x=k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, U=2^{n+1}, V=2^n, T=3(2)^n,$

$p=k^{3n}, \alpha=(k^6+k^4), C=2k^2, D=k^4-1.$

for $\beta=3$, having integer design of solution is parameterized by integers k and n, with variables defined as:

$x=k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, U=2^{n+1}, V=2^n, T=3(2)^n,$

$p=k^{2n}, \alpha=(k^6+k^4), C=2k^2, D=k^4-1.$

for $\beta = 4$, having integer design of solution is parameterized by integers k and n, with variables defined as:

$x = k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, U = 2^{n+1},\ V = 2^n, T = 3(2)^n,$

$p = k^n, \alpha = (k^6 + k^4), C = 2k^2, D = k^4 - 1.$

for $\beta = 5$, having integer design of solution is parameterized by integers k and n, with variables defined as:

$x = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, U = 2^{n+1},\ V = 2^n, T = 3(2)^n,$

$p = k^{2n}, \alpha = k^n(k^6 + k^4), C = 2k^2, D = k^4 - 1.$

for $\beta = 6$, having integer design of solution is parameterized by integers k and n, with variables defined as:

$x = k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, U = 2^{n+1},\ V = 2^n, T = 3(2)^n,$

$p = k^n, \alpha = k^{2n}(k^6 + k^4), C = 2k^2, D = k^4 - 1.$

for $\beta = 7$, having integer design of solution is parameterized by integers k and n, with variables defined as:

$x = k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, U = 2^{n+1},\ V = 2^n, T = 3(2)^n,$

$p = k^n, \alpha = k^{3n}(k^6 + k^4), C = 2k^2, D = k^4 - 1.$

Keywords: Diophantine Equation, exponential, Pythagorean triplet, Integer design.

Introduction:

Diophantine equations—polynomial equations with integer solutions—are a central topic in number theory. Among their many variants, **exponential Diophantine equations** involve terms where variables appear as exponents. Finding integer solutions to such equations is notably complex and has implications in mathematics, cryptography, and several scientific fields. Historical Context and Theoretical Background

Classical Diophantine Equations: Traditionally, research started with linear and polynomial forms, such as the well-known cases of Pythagorean triples .

Exponential Generalization: The study of exponential forms expanded from these roots, posing questions that often lack general solution methods and in some cases are proven to be undecidable. In this paper, focused to find the general exponential integer solution of

The general exponential integer solution of $\boldsymbol{\alpha(X^4 + Y^4)^2(2U^2 + V^2) = T^2(C^2 + D^2)(Z^2 + W^2)P^{\beta}}$

With $\alpha > 0$, is derived from fixed value of $\beta = 1, 2, 3, 4, 5, 6, 7$ *and* $x < y < w < z$.

Results & Discussions:

Proportion 1: A Study on integer design of solution of above Diophantine Equation at

$\boldsymbol{\beta = 1}$ **is** $\boldsymbol{\alpha(X^4 + Y^4)^2(2U^2 + V^2) = T^2(C^2 + D^2)(Z^2 + W^2)}\ P$

Explanation: Let $x = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{6n}, U = 2^{n+1},\ V = 2^n, T = 3(2)^n.$

Consider $\alpha(X^4 + Y^4)(\boldsymbol{2U^2 + V^2}) = \alpha k^{8n}(1 + k^4)^2(3(2)^n)^2$

Again consider $(Z^2 + W^2)P = k^{8n}(k^6 + k^4).$

It follows that $\boldsymbol{\alpha}(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = \boldsymbol{T^2}(\boldsymbol{C^2}+\boldsymbol{D^2})(\boldsymbol{Z^2}+\boldsymbol{W^2})\,P$ implies that

$\alpha k^{8n}(1+k^4)^2(3(2)^n)^2 = (C^2+D^2)k^{8n}(k^6+k^4)(3(2)^n)^2$ implies $\alpha(1+k^4)^2 = (C^2+D^2)(k^6+k^4)$.

Solve for α, whenever $(C, D, 1+k^4)$ is a Pythagorean Triplet.

From the References [1],[2],[3],[4],[5],[6],[7],[8], we know that

$(C, D, 1+k^4)$ becomes a Pythagorean Triplet with $C = (2k^2)$, $D = (k^4-1)$,

$\boldsymbol{C^2}+\boldsymbol{D^2} = (\boldsymbol{1}+\boldsymbol{k^4})^{\boldsymbol{2}}$. Hence $\alpha = (k^6+k^4)$.

Hence $\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = \boldsymbol{T^2}(C^2+D^2)(Z^2+W^2)P$ having integer design of solution is parameterized by integers k and n, with variables defined as:

x $= k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{6n}, \alpha = (k^6+k^4), C = 2k^2, D = k^4-1,$

$U = 2^{n+1},\ V = 2^n, T = 3(2)^n.$

Verification: Consider **LHS**

$\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = (k^6+k^4)(\boldsymbol{k^{4n}}+\boldsymbol{k^{4n+4}})^2 = \boldsymbol{k^{8n}}(k^6+k^4)(\boldsymbol{1}+\boldsymbol{k^4})^2(3(2)^n)^2.$

Consider RHS

$\boldsymbol{T^2}(C^2+D^2)(Z^2+W^2)P = (3(2)^n)^2(\boldsymbol{1}+\boldsymbol{k^4})^2(k^{2n+6}+k^{2n+4})k^{6n} = \boldsymbol{k^{8n}}(k^6+k^4)(\boldsymbol{1}+\boldsymbol{k^4})^2(3(2)^n)^2.$

Hence LHS $=$ ***RHS*.**

Proportion 2: A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 2}$ **is** $\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = \boldsymbol{T^2}(C^2+D^2)(Z^2+W^2)P^2$

Explanation: Let x $= k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{3n}, U = 2^{n+1},\ V = 2^n, T = 3(2)^n.$

Consider $\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = \alpha k^{8n}(1+k^4)^2(3(2)^n)^2.$

Again consider $(Z^2+W^2)\boldsymbol{P^2} = k^{8n}(k^6+k^4).$

It follows that $\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = \boldsymbol{T^2}(C^2+D^2)(Z^2+W^2)P^2$ implies that

$\alpha k^{8n}(1+k^4)^2(3(2)^n)^2 = (C^2+D^2)k^{8n}(k^6+k^4)(3(2)^n)^2$ implies $\alpha(1+k^4)^2 = (C^2+D^2)(k^6+k^4)$. Solve for α, whenever $(C, D, 1+k^4)$ is a Pythagorean Triplet.

From the References [1],[2],[3],[4],[5],[6],[7],[8], we know that

$(C, D, 1+k^4)$ becomes a Pythagorean Triplet with $C = (2k^2)$, $D = (k^4-1)$,

$\boldsymbol{C^2}+\boldsymbol{D^2} = (\boldsymbol{1}+\boldsymbol{k^4})^{\boldsymbol{2}}$Hence $\alpha = (k^6+k^4)$.

Verification: Consider **LHS**

$\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = (k^6+k^4)(\boldsymbol{k^{4n}}+\boldsymbol{k^{4n+4}})^2(3(2)^n)^2 = (3(2)^n)^2\boldsymbol{k^{8n}}(k^6+k^4)(\boldsymbol{1}+\boldsymbol{k^4})^2.$

Consider RHS

$\boldsymbol{T^2}(C^2+D^2)(Z^2+W^2)P^2 = (3(2)^n)^2(\boldsymbol{1}+\boldsymbol{k^4})^2(k^{2n+6}+k^{2n+4})k^{6n} = \boldsymbol{k^{8n}}(k^6+k^4)(\boldsymbol{1}+\boldsymbol{k^4})^2.$

Hence LHS $=$ ***RHS*.**

Proportion 3: A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 3}$ **is** $\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = \boldsymbol{T^2}(\boldsymbol{C^2}+\boldsymbol{D^2})(\boldsymbol{Z^2}+\boldsymbol{W^2})\boldsymbol{P^3}$

Explanation: Let $x = k^n, y = k^{n+1}, z = k^{n+3}, w = k^{n+2}, p = k^{2n}, U = 2^{n+1},\ V = 2^n, T = 3(2)^n$.

Consider $\alpha(\boldsymbol{X^4} + \boldsymbol{Y^4})^2(\boldsymbol{2U^2} + \boldsymbol{V^2}) = \alpha k^{8n}(1 + k^4)^2(3(2)^n)^2$.

Again consider $(Z^2 + W^2)\boldsymbol{P^3} = k^{8n}(k^6 + k^4)$.

It follows that $\alpha(\boldsymbol{X^4} + \boldsymbol{Y^4})^2(\boldsymbol{2U^2} + \boldsymbol{V^2}) = \boldsymbol{T^2}(\boldsymbol{C^2} + \boldsymbol{D^2})(\boldsymbol{Z^2} + \boldsymbol{W^2})\boldsymbol{P^3}$ implies that

$\alpha k^{8n}(1 + k^4)^2(3(2)^n)^2 = (3(2)^n)^2(C^2 + D^2)k^{8n}(k^6 + k^4)$ implies $\alpha(1 + k^4)^2 = (C^2 + D^2)(k^6 + k^4)$. Solve for α, whenever $(C, D, 1 + k^4)$ is a Pythagorean Triplet.

From the References [1],[2],[3],[4],[5],[6],[7],[8], we know that

$(C, D, 1 + k^4)$ becomes a Pythagorean Triplet with $C = (2k^2),\ D = (k^4 - 1)$,

$\boldsymbol{C^2} + \boldsymbol{D^2} = (\boldsymbol{1} + \boldsymbol{k^4})^2$Hence $\alpha = (k^6 + k^4)$.

Verification: Consider **LHS**

$\alpha(\boldsymbol{X^4} + \boldsymbol{Y^4})^2(\boldsymbol{2U^2} + \boldsymbol{V^2}) = (k^6 + k^4)(\boldsymbol{k^{4n}} + \boldsymbol{k^{4n+4}})^2(3(2)^n)^2 = \boldsymbol{k^{8n}}(k^6 + k^4)(\boldsymbol{1} + \boldsymbol{k^4})^2\ (3(2)^n)^2$.

Consider RHS

$\boldsymbol{T^2}(C^2 + D^2)(Z^2 + W^2)P^3 = (3(2)^n)^2\ (\boldsymbol{1} + \boldsymbol{k^4})^2(k^{2n+6} + k^{2n+4})k^{6n} = (3(2)^n)^2\boldsymbol{k^{8n}}(k^6 + k^4)(\boldsymbol{1} + \boldsymbol{k^4})^2$.

Hence LHS $=$ ***RHS*.**

Proportion 4 A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 4}$ **is** $\boldsymbol{\alpha(X^4 + Y^4)^2(2U^2 + V^2) = T^2(C^2 + D^2)(Z^2 + W^2)P^4}$.

Explanation: Let $x = k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, p = k^n, U = 2^{n+1},\ V = 2^n, T = 3(2)^n$

Consider $\alpha(\boldsymbol{X^4} + \boldsymbol{Y^4})^2(\boldsymbol{2U^2} + \boldsymbol{V^2}) = \alpha k^{8n}(1 + k^4)^2\ (3(2)^n)^2$.

Again consider $(Z^2 + W^2)\boldsymbol{P^4} = k^{8n}(k^6 + k^4)$.

It follows that $\boldsymbol{\alpha}(\boldsymbol{X^4} + \boldsymbol{Y^4})^2(\boldsymbol{2U^2} + \boldsymbol{V^2}) = \boldsymbol{T^2}(\boldsymbol{C^2} + \boldsymbol{D^2})(\boldsymbol{Z^2} + \boldsymbol{W^2})\boldsymbol{P^4}$ implies that

$\alpha k^{8n}(1 + k^4)^2(3(2)^n)^2 = (3(2)^n)^2(C^2 + D^2)k^{8n}(k^6 + k^4)$ implies $\alpha(1 + k^4)^2 = (C^2 + D^2)(k^6 + k^4)$. Solve for α, whenever $(C, D, 1 + k^4)$ is a Pythagorean Triplet.

From the References [1],[2],[3],[4],[5],[6],[7],[8], we know that

$(C, D, 1 + k^4)$ becomes a Pythagorean Triplet with $C = (2k^2),\ D = (k^4 - 1)$,

$\boldsymbol{C^2} + \boldsymbol{D^2} = (\boldsymbol{1} + \boldsymbol{k^4})^2$Hence $\alpha = (k^6 - k^4)$.

Verification: Consider **LHS**

is $\boldsymbol{\alpha}(\boldsymbol{X^4} + \boldsymbol{Y^4})^2(\boldsymbol{2U^2} + \boldsymbol{V^2}) = (k^6 + k^4)(\boldsymbol{k^{4n}} + \boldsymbol{k^{4n+4}})^2(3(2)^n)^2 = \boldsymbol{k^{8n}}(k^6 + k^4)(\boldsymbol{1} + \boldsymbol{k^4})^2\ (3(2)^n)^2$

Consider RHS

$$\boldsymbol{T^2}(\boldsymbol{C^2} + \boldsymbol{D^2})(\boldsymbol{Z^2} + \boldsymbol{W^2})\boldsymbol{P^4} = (3(2)^n)^2\ (\boldsymbol{1} + \boldsymbol{k^4})^2(k^{2n+6} + k^{2n+4})k^{6n} = \boldsymbol{k^{8n}}(k^6 + k^4)(\boldsymbol{1} + \boldsymbol{k^4})^2(3(2)^n)^2$$

Hence LHS $=$ ***RHS*.**

Proportion 5: A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 5}$ **is** $\boldsymbol{\alpha(X^4 + Y^4)^2(2U^2 + V^2) = T^2(C^2 + D^2)(Z^2 + W^2)P^5}$.

Explanation: Let $x = k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, p = k^n, U = 2^{n+1},\ V = 2^n, T = 3(2)^n$

Consider $\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = \alpha k^{8n}(1+k^4)^2\,(3(2)^n)^2$.

Again consider $(Z^2+W^2)\boldsymbol{P^5} = k^{9n}(k^6+k^4)$.

It follows that $\boldsymbol{\alpha(X^4+Y^4)^2(2U^2+V^2) = T^2(C^2+D^2)(Z^2+W^2)P^5}$ implies that

$\alpha k^{8n}(1+k^4)^2(3(2)^n)^2 = (3(2)^n)^2(C^2+D^2)k^{9n}(k^6+k^4)$ implies $\alpha(1+k^4)^2 = (C^2+D^2)k^n(k^6+k^4)$. Solve for α, whenever $(C, D, 1+k^4)$ is a Pythagorean Triplet.

From the References [1],[2],[3],[4],[5],[6],[7],[8], we know that

$(C, D, 1+k^4)$ becomes a Pythagorean Triplet with $C = (2k^2)$, $D = (k^4-1)$,

$\boldsymbol{C^2+D^2 = (1+k^4)^2}$Hence $\alpha = k^n(k^6+k^4)$.

Verification: Consider **LHS**

$\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = k^n\,(k^6+k^4)(\boldsymbol{k^{4n}}+\boldsymbol{k^{4n+4}})^2(3(2)^n)^2 = (3(2)^n)^2\boldsymbol{k^{9n}}(k^6+k^4)(\boldsymbol{1+k^4})^2$.

Consider RHS

$\boldsymbol{T^2}(C^2+D^2)(Z^2+W^2)P^5 = (3(2)^n)^2(\boldsymbol{1+k^4})^2(k^{4n+6}+k^{4n+4})k^{5n} = (3(2)^n)^2\boldsymbol{k^{9n}}(k^6+k^4)(\boldsymbol{1+k^4})^2$.

Hence LHS = *RHS*.

Proportion 6: A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 6}$ **is** $\boldsymbol{\alpha(X^4+Y^4)^2(2U^2+V^2) = T^2(C^2+D^2)(Z^2+W^2)P^6}$.

Explanation: Let x = $k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, p = k^n, U = 2^{n+1},\ V = 2^n, T = 3(2)^n$

Consider $\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = \alpha k^{8n}(1+k^4)^2(3(2)^n)^2$.

Again consider $(Z^2+W^2)\boldsymbol{P^6} = k^{10n}(k^6+k^4)$.

It follows that $\boldsymbol{\alpha(X^4+Y^4)^2(2U^2+V^2) = T^2(C^2+D^2)(Z^2+W^2)P^6}$ implies that

$\alpha k^{8n}(1+k^4)^2(3(2)^n)^2 = (3(2)^n)^2(C^2+D^2)k^{10n}(k^6+k^4)$ implies $\alpha(1+k^4)^2 = (C^2+D^2)k^{2n}(k^6+k^4)$. Solve for α, whenever $(C, D, 1+k^4)$ is a Pythagorean Triplet.

From the References [1],[2],[3],[4],[5],[6],[7],[8], we know that

$(C, D, 1+k^4)$ becomes a Pythagorean Triplet with $C = (2k^2)$, $D = (k^4-1)$,

$\boldsymbol{C^2+D^2 = (1+k^4)^2}$Hence $\alpha = k^{2n}(k^6+k^4)$.

Verification: Consider **LHS**

$\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = k^{2n}\,(k^6+k^4)(\boldsymbol{k^{4n}}+\boldsymbol{k^{4n+4}})^2(3(2)^n)^2 = (3(2)^n)^2\boldsymbol{k^{10n}}(k^6+k^4)(\boldsymbol{1+k^4})^2$.

Consider RHS

$\boldsymbol{T^2(C^2+D^2)(Z^2+W^2)P^6} = (3(2)^n)^2(\boldsymbol{1+k^4})^2(k^{4n+6}+k^{4n+4})k^{6n} = (3(2)^n)^2\boldsymbol{k^{10n}}(k^6+k^4)(\boldsymbol{1+k^4})^2$.

Hence LHS = *RHS*.

Proportion 7: A Study on exponential integer solution of above Diophantine Equation at

$\boldsymbol{\beta = 7}$ **is** $\boldsymbol{\alpha(X^4+Y^4)^2(2U^2+V^2) = T^2(C^2+D^2)(Z^2+W^2)P^7}$.

Explanation: Let x = $k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, p = k^n, U = 2^{n+1},\ V = 2^n, T = 3(2)^n$

Consider $\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = \alpha k^{8n}(1+k^4)^2(3(2)^n)^2$.

Again consider $(Z^2+W^2)\boldsymbol{P^6} = k^{10n}(k^6+k^4)$.

It follows that $\alpha\boldsymbol{\alpha}(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = \boldsymbol{T^2}(\boldsymbol{C^2}+\boldsymbol{D^2})(\boldsymbol{Z^2}+\boldsymbol{W^2})\boldsymbol{P^7}$**.** implies that

$\alpha k^{8n}(1+k^4)^2(3(2)^n)^2 = (3(2)^n)^2(C^2+D^2)k^{11n}(k^6+k^4)$ implies $\alpha(1+k^4)^2 = (C^2+D^2)k^{3n}(k^6+k^4)$. Solve for α, whenever $(C, D, 1+k^4)$ is a Pythagorean Triplet.

From the References [1],[2],[3],[4],[5],[6],[7],[8], we know that

$(C, D, 1+k^4)$ becomes a Pythagorean Triplet with $C = (2k^2)$, $D = (k^4-1)$,

$\boldsymbol{C^2}+\boldsymbol{D^2} = (\boldsymbol{1}+\boldsymbol{k^4})^2$. Hence $\alpha = k^{3n}(k^6+k^4)$.

Verification: Consider **LHS**

$\boldsymbol{\alpha}(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = k^{3n}(k^6+k^4)(\boldsymbol{k^{4n}}+\boldsymbol{k^{4n+4}})^2(3(2)^n)^2 = (3(2)^n)^2\,\boldsymbol{k^{11n}}(k^6+k^4)(\boldsymbol{1}+\boldsymbol{k^4})^2$**.**

Consider RHS

$\boldsymbol{T^2}(\boldsymbol{C^2}+\boldsymbol{D^2})(\boldsymbol{Z^2}+\boldsymbol{W^2})\boldsymbol{P^7} = (3(2)^n)^2(\boldsymbol{1}+\boldsymbol{k^4})^2(k^{4n+6}+k^{4n+4})k^{7n} = (3(2)^n)^2\,\boldsymbol{k^{11n}}(k^6+k^4)(\boldsymbol{1}+\boldsymbol{k^4})^2$**.**

Hence LHS = $\boldsymbol{RHS}$**.**

Main Result:

A Study on exponential integer solution of above Diophantine Equation at

$$\boldsymbol{\alpha}(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = \boldsymbol{T^2}(\boldsymbol{C^2}+\boldsymbol{D^2})(\boldsymbol{Z^2}+\boldsymbol{W^2})\boldsymbol{P^\beta}.$$

Explanation: Let x = $k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, p = k^n, U = 2^{n+1}, V = 2^n, T = 3(2)^n$

Consider $\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = \alpha k^{8n}(1+k^4)^2(3(2)^n)^2$.

Again consider $(Z^2+W^2)\boldsymbol{P^\beta} = k^{4n+n\boldsymbol{\beta}}(k^6+k^4)$.

It follows that $\boldsymbol{\alpha}(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = \boldsymbol{T^2}(\boldsymbol{C^2}+\boldsymbol{D^2})(\boldsymbol{Z^2}+\boldsymbol{W^2})\boldsymbol{P^\beta}$**.**

implies that

$\alpha k^{8n}(1+k^4)^2(3(2)^n)^2 = (3(2)^n)^2(C^2+D^2)k^{4n+n\boldsymbol{\beta}}(k^6+k^4)$ implies

$\alpha(1+k^4)^2 = (C^2+D^2)k^{-4n+n\boldsymbol{\beta}}(k^6+k^4)$. Solve for α, whenever $(C, D, 1+k^4)$ is a Pythagorean Triplet.

From the References [1],[2],[3],[4],[5],[6],[7],[8], we know that

$(C, D, 1+k^4)$ becomes a Pythagorean Triplet with $C = (2k^2)$, $D = (k^4-1)$,

$\boldsymbol{C^2}+\boldsymbol{D^2} = (\boldsymbol{1}+\boldsymbol{k^4})^2$. Hence $\alpha = k^{-4n+n\boldsymbol{\beta}}(k^6+k^4) = k^{(\beta-4)n}(k^6+k^4)$.

Verification: Consider **LHS**

$\alpha(\boldsymbol{X^4}+\boldsymbol{Y^4})^2(\boldsymbol{2U^2}+\boldsymbol{V^2}) = k^{(\beta-4)n}(k^6+k^4)\,(\boldsymbol{k^{4n}}+\boldsymbol{k^{4n+4}})^2(3(2)^n)^2$

$= (3(2)^n)^2\boldsymbol{k}^{(\beta+4)n}(k^6+k^4)(\boldsymbol{1}+\boldsymbol{k^4})^2$**.**

Consider RHS

$\boldsymbol{T^2}(\boldsymbol{C^2}+\boldsymbol{D^2})(\boldsymbol{Z^2}+\boldsymbol{W^2})\boldsymbol{P^\beta} = (3(2)^n)^2(\boldsymbol{1}+\boldsymbol{k^4})^2(k^{4n+6}+k^{4n+4})k^{\beta n}$

$= (3(2)^n)^2\boldsymbol{k}^{(\beta+4)n}(k^6+k^4)(\boldsymbol{1}+\boldsymbol{k^4})^2$**.**

Hence LHS = RHS.

Conclusion:

This equation generalizes classical Diophantine problems, blending sums of fourth powers with multiplicative factorizations. While challenging, targeted parametrization and modular analysis can yield solutions. Future work may classify solutions for specific α, β or link to broader number-theoretic frameworks. The parametric framework provides infinite families of solutions by exploiting algebraic identities and modular arithmetic. Future work could explore non-parametric solutions or generalizations to higher exponents.

This paper focused on a study to find integer design of solutions Diophantine Equation

$\boldsymbol{\alpha(X^4+Y^4)^2(2U^2+V^2)=T^2(C^2+D^2)(Z^2+W^2)P^{\beta}}$ **With** $\boldsymbol{\alpha>0, \gamma=2,3, \beta=1,2,3,4,5,6,7}$ ***and***

$\boldsymbol{x<y<w<z}$ with Mathematical induction & generation of Pythagorean triplets.

for $\boldsymbol{\beta=1}$, having integer design of solution is parameterized by positive integers k and n, with variables defined as:

$x=k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, U=2^{n+1},\ V=2^n, T=3(2)^n,$

$p=k^{6n}, \alpha=(k^6+k^4), C=2k^2, D=k^4-1.$

for $\boldsymbol{\beta=2}$, having integer design of solution is parameterized by integers k and n, with variables defined as:

$x=k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, U=2^{n+1},\ V=2^n, T=3(2)^n,$

$p=k^{3n}, \alpha=(k^6+k^4), C=2k^2, D=k^4-1.$

for $\boldsymbol{\beta=3}$, having integer design of solution is parameterized by integers k and n, with variables defined as:

$x=k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, U=2^{n+1},\ V=2^n, T=3(2)^n,$

$p=k^{2n}, \alpha=(k^6+k^4), C=2k^2, D=k^4-1.$

for $\boldsymbol{\beta=4}$, having integer design of solution is parameterized by integers k and n, with variables defined as:

$x=k^n, y=k^{n+1}, z=k^{2n+3}, w=k^{2n+2}, U=2^{n+1},\ V=2^n, T=3(2)^n,$

$p=k^n, \alpha=(k^6+k^4), C=2k^2, D=k^4-1.$

for $\boldsymbol{\beta=5}$, having integer design of solution is parameterized by integers k and n, with variables defined as:

$x=k^n, y=k^{n+1}, z=k^{n+3}, w=k^{n+2}, U=2^{n+1},\ V=2^n, T=3(2)^n,$

$p=k^{2n}, \alpha=k^n(k^6+k^4), C=2k^2, D=k^4-1.$

for $\boldsymbol{\beta=6}$, having integer design of solution is parameterized by integers k and n, with variables defined as:

$x=k^n, y=k^{n+1}, z=k^{2n+3}, w=k^{2n+2}, U=2^{n+1},\ V=2^n, T=3(2)^n,$

$p=k^n, \alpha=k^{2n}(k^6+k^4), C=2k^2, D=k^4-1.$

for $\beta = 7$, having integer design of solution is parameterized by integers k and n, with variables defined as:

$x = k^n, y = k^{n+1}, z = k^{2n+3}, w = k^{2n+2}, U = 2^{n+1},\ V = 2^n\ , T = 3(2)^n,$

$p = k^n, \alpha = k^{3n}(k^6 + k^4), C = 2k^2, D = k^4 - 1.$

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