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On The Interdependence Of Stable And Pseudo-Stable Properties In Near-Ring Substructures

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Abstract: This paper presents a detailed study on the behaviour of idempotent elements in the structural framework of stable and pseudo-stable near-rings. Idempotent elements play a central role in the decomposition and organization of algebraic systems, and their influence on the formation of ideals provides valuable insight into the internal properties of near-rings. The study of stability conditions within algebraic systems has been a central theme in understanding their internal symmetries and operational consistency. In this paper, we investigate stable and pseudo-stable properties in the context of near ring substructures. A near ring, being a generalization of a ring without the requirement of full distributivity or commutativity, provides a rich framework to analyze weaker forms of algebraic regularity. We define and explore the conditions under which substructures such as ideals, subnear-rings, and quasi-ideals exhibit stability under homomorphic images, endomorphisms, and composition mappings. The concept of pseudo-stability is introduced as a relaxed form of stability, wherein closure properties hold under specific mappings or restricted operations rather than universally. Comparative results between stable and pseudo-stable substructures are established, highlighting their interrelationships and dependencies on the structural constraints of the parent near ring. These findings contribute to the deeper understanding of near ring theory and open pathways for further exploration in algebraic systems exhibiting partial distributivity.

Index Terms **-** Idempotents, central, decomposition, ideals, distributivity, Commutativity, sub near rings, homomorphism.

Introduction

Near-rings are algebraic structures that extend the concept of rings by relaxing one or more of their defining conditions. A near-ring $(N,+,\cdot)$ consists of a non-empty set N endowed with an additive group structure $(N,+)$ and a semigroup structure $(N,\cdot)$, where multiplication is usually required to be only one-sided distributive over addition. This relaxation makes near-rings more flexible than rings and allows them to capture a wide range of algebraic systems arising in diverse mathematical contexts. Near-rings appear naturally in the study of endomorphism sets of groups, combinatorial designs, switching functions, and certain geometric configurations, thereby offering a broad platform for theoretical exploration. Within near-ring theory, the study of substructures such as ideals, sub near-rings, and idempotent elements has become a vital area of research. These components often reveal deep connections between the additive and multiplicative operations of a near-ring. In particular, the role of idempotent elements that remain unchanged under their own multiplication has gained importance because they influence the formation and decomposition of ideals and substructures.

Idempotents can act as building blocks that define invariant or symmetric substructures within a near-ring, making their behaviour crucial to understanding internal regularity and balance.

I. PRELIMINARIES

Definition 2.1

A *right near ring* is a non-empty set N together with the two binary operations "+" and "·" such that:

(i) $(N, +)$ is a group (not necessarily abelian)

(ii) $(N, \cdot)$ is a semigroup

(iii) $(n_1 + n_2)n_3 = n_1n_3 + n_2n_3$ for all $n_1, n_2, n_3 \in N$

Definition 2.2

A near ring N is said to be *zero symmetric* if $N = N_0$ where $N_0 = \{n \in N / n0 = 0\}$.

Definition 2.3

A subgroup M of a near-ring N with $M.M \subseteq M$ is called a *sub near-ring* of N and is denoted by $M \leq N$.

Definition 2.4

We define N to be *stable (pseudo stable)* if for all x in N satisfies $xN = xNx = Nx$ $(aN = bN \Longrightarrow Na = Nb$ for a, b in $N)$.

Definition 2.5

An ideal I of N is said to be a *prime ideal (completely prime)* if for all ideal A, B of N, $AB \subseteq I \Longrightarrow A \subseteq I$ *or* $B \subseteq I$ $(xy \in I \Longrightarrow x \in I$ *or* $y \in I.)$.

Definition 2.6

A subgroup M of the additive group $(N, +)$ is called a N-*subgroup* of N if $NM \subseteq M$.

Notation 2.7

- E denotes the set of all idempotent of N (e in N is called an *idempotent* if $e^2 = e$).
- L denotes the set of all nilpotent of N (a in N is *nilpotent* if $a^k = 0$ for some positive integer k).
- $C(N) = \{n \in N / nx = xn \ \forall\ x \ in \ N\}$- centre of N.

Definition 2.8

A normal subgroup of I of $(N, +)$ is called *ideal* of N if

a) $IN \subseteq I$

b) b) $\forall\ n, n' \in N$ and $\forall\ i \in N$ such that $n(n' + i) - nn' \in I$

Normal subgroup R of $(N, +)$ with a) is called right ideals of N and normal subgroup R of $(N, +)$ is said to be left ideal.

II. Main Results

Theorem 3.1

Let N be stable near-ring then every completely prime ideal of N is prime.

Proof

Let Na and Nb be two N-subgroups of N

And let P be ideal of N.

Consider $NaNb = NNab = Nab$ [since stable near rings]

Since P is completely prime, $Nab \subseteq P$

Let $NaNb \subseteq P$.

$Na \subseteq Nab \subseteq P$ and $Nb \subseteq Nab \subseteq P$

$\Longrightarrow P$ is prime ideal.

Theorem 3.2

Let N be stable near-ring with I as semiprime ideal of N and e be idempotent of N. Then the principal ideal eN is also semiprime.

Proof

Let J be an ideal generated by e

By semiprimeness, $J^2 \subseteq I \Longrightarrow J \subseteq I$

Now consider, $(eN)^2 \subseteq I$

Since $e \in J \subseteq I$

$\Longrightarrow e \in I$

$\Longrightarrow eN \subseteq I$

Theorem 3.3

Let N be stable near-ring with idempotent e then Ne, eN, eNe are also idempotent.

Proof

Since N is stable,

For $e \in E$ we have $eN = Ne = eNe$

Consider

$$\begin{aligned}(Ne)^2 &= (Ne)(Ne)\\ &= Ne(eN)\\ &= N(eN)\\ &= NNe\\ &= Ne\end{aligned}$$

$$\begin{aligned}(eN)^2 &= (eN)(eN)\\ &= eN(Ne)\\ &= e(Ne)\\ &= e(eN)\end{aligned}$$

$= eeN = eN$

$(eNe)^2 = (eNe)(eNe)$

$= eNeeNe$

$= eN(eNe)$

$= eN(Ne)$

$= eNe$

Theorem 3.4

Let N be pseudo-stable and reverse pseudo stable then for all $a, b \in N, aN = bN \Leftrightarrow Na = Nb$

Proof

By definition of pseudo-stable near-ring, for all $a, b \in N, aN = bN \Longrightarrow Na = Nb$

And by definition of pseudo-stable near-ring,

for all $a, b \in N, Na = Nb \Longrightarrow aN = bN$

Therefore, the proof is direct.

Theorem 3.5

Let N be pseudo-stable and e be its idempotent then $eN = Ne = eNe$.

Proof

Since $e^2 = e$

$eN = e^2N \subseteq eNe \subseteq eN$

$\Longrightarrow eN = eNe$

Also, since N is pseudo-stable,

$eN = e^2N \Longrightarrow Ne = Ne^2 \subseteq eNe \subseteq Ne$

$\Longrightarrow Ne = eNe$

Therefore, we have,

$eN = Ne = eNe$

Theorem 3.6

If N is pseudo-stable then $E \subseteq C(N)$.

Proof

By previous theorem we have,

$eN = Ne = eNe$

Then for every $n \in N$, there exist u and v in N such that

$en = eue$ and $ne = eve$

It follows that $en = ene = ne$

Therefore, $E \subseteq C(N)$.

Theorem 3.7

Stability is preserved by surjective homomorphisms

Proof

Let $\varphi: N \longrightarrow M$ be a surjective near-ring homomorphism. If N is stable, then M is stable.

Now consider $y \in M$

Since φ is surjective there exists $x \in N$ with $\varphi(x) = y$

Consider the right principal ideal yM of M.

Every element of yM is of the form ym for some $m \in M$.

Choose $n \in N$ with $\varphi(n) = m$.

Then $ym = \varphi(x)\varphi(n)$

Therefore $yM = \varphi(xN)$

Similarly, we get $yMy = \varphi(xNx)$ and $My = \varphi(Nx)$

Since N is stable we have, $xN = xNx = Nx$

Homomorphisms preserve products, $\varphi(xN) = \varphi(xNx) = \varphi(Nx)$

$\Longrightarrow yM = yMy = My$

Therefore M is stable.

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