IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

Fixed Point Theorems Using Generalized Distance Function Ordered Metric Spaces

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Abstract : In this paper we prove certain fixed point theorems for multi valued and single valued mappings in such spaces, by using Generalized distance function. Our results extend some existing results.

Keywords: Ordered set, Multi valued mapping, single valued mapping, fixed point.

2000 Mathematics subject classification: 47H10, 54H25.

Introduction and Preliminaries

Let (X, d) be a metric space. We denote the class of non empty and bounded subsets of X by $B(X)$. For $A, B \in B(X)$, function $D(A, B)$ and $\delta(A, B)$ are defined as follows:

$$D(A,B) = inf\ \{ d(a,b) : a \in A, b \in B \}$$

$$\delta(A,B) = sup\ \{ d(a,b) : a \in A, b \in B \}$$

If $A = \{ a \}$ then we write $D(A,B) = D(a,B)$ and $\delta(A,B) = \delta(a,B)$. Also in addition, if $B = \{ b \}$, then $D(A,B) = d(a,b)$ and $\delta(A,B) = d(a,b)$. Obviously, $D(A,B) \le \delta(A,B)$. For all $A, B, C \in B(X)$, the definition of $\delta(A,B)$ yields the following:

$$\delta(A,B) = \delta(B,A)$$

$$\delta(A,B) \le \delta(A,C) + \delta(C,B)$$

$$\delta(A,B) = 0 \text{ iff } A = B = \{ a \}$$

$\delta(A,B) = diam\ A$ (Fisher 1981, and Iseki, 1983).

Definition 1: (Beg and Butt, [2]) : Let A and B be two non empty subsets of a ordered set $(X, \preccurlyeq)$. The relation between A and B is denoted and defined as follows:

$A \preccurlyeq B$, if for every $a \in A$ there *exists* $b \in B$ such that $a \preccurlyeq b$.

We will utilize the following control function which is also referred to a Generalized distance function.

Definition 2 : (Khan et al. [9]): A function $\psi : [0, \infty) \to [0, \infty)$ is called a Generalized distance function if the following properties are satisfied:

i. ψ is monotone increasing and continuous,

ii. $\psi(t) = 0$ if and only if $t = 0$.

In this paper we prove certain fixed point theorems for multi valued and single valued mappings in such spaces, by using Generalized distance function. Our results extend some existing results.

Main Results

Theorem 2.1: Let $(X, \preccurlyeq)$ be a ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to B(X)$ be a multivalued mapping such that the following conditions are satisfied;

i. there exists $x_0 \in X$ such that $\{x_0\} \preccurlyeq Tx\_0$,

ii. for $x, y \in X, x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$,

iii. if $x_n \to x$ is a non decreasing sequence in X, then $x_n \preccurlyeq x$ for all n,

iv. $\psi\big(\delta(Tx, Ty)\big) \leq \alpha\, \psi(max\,\{D(x, Tx), D(y, Ty)\}) + \beta\, \psi\,(max\,\{D(x, Ty), D(y, Tx)\}) + \gamma\, \psi\,(d(x, y))$

For all comparable $x, y \in X$ where $\alpha, \beta, \gamma \in (0,1)$ such that $0 < \alpha + 2\beta + \gamma < 1$ and ψ is an Generalized distance function. Then T has a fixed point.

Proof: By the assumption (i) there exists $x_1 \in Tx_0$ such that $x_0 \preccurlyeq x_1$. By the assumption (ii), $Tx_0 \preccurlyeq Tx_1$. Then there *exists* $x_2 \in Tx_1$ such that $x_1 \preccurlyeq x_2$. Continuing the process we construct a monotone increasing sequence $\{x_n\}$ in X such *that* $x_{n+1} \in Tx_n$ for all $n \geq 0$. Thus we have $x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq x_3 \preccurlyeq \ldots\ldots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \ldots..$

If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T. Hence we shall assume that $x_n \neq x_{n+1}$ for all $n \geq 0$.

Using the monotone property *of* ψ and the condition (iv), we have for all $n \geq 0$,

$$\psi\,\big(d(x_{n+1}, x_{n+2})\big) \leq \psi\big(\delta\,(Tx_n, Tx_{n+1})\big)$$

$$\psi\,\big(\delta\,(Tx_n, Tx_{n+1})\big) \leq \alpha\, \psi(max\,\{D(x_n, Tx_n), D(x_{n+1}, Tx_{n+1})\})$$
$$+\beta\, \psi(max\,\{D(x_n, Tx_{n+1}), D(x_{n+1}, Tx_n)\}) + \gamma \psi\,\big(d(x_n, x_{n+1})\big)$$

$$\psi\left(d(x_{n+1},x_{n+2})\right) \leq \alpha\, \psi(max\,\{\,d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\})$$
$$+\beta\,\psi\,(max\,\{\,d(x_n,x_{n+2}),d(x_{n+1},x_{n+1})\}) \;\; +\gamma\,\psi\left(d(x_n,x_{n+1})\right)$$

There arise two cases.

Case - 1, if we take $max\,\{\,d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\} \;=\; d(x_n,x_{n+1})$ then,

$$\psi\left(d(x_{n+1},x_{n+2})\right) \leq \frac{\alpha+\beta+\gamma}{1-\beta}\;\psi\left(d(x_n,x_{n+1})\right)$$

Case - 2, if we take $max\,\{\,d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\} \;=\; d(x_{n+1},x_{n+2})$ then,

$$\psi\left(d(x_{n+1},x_{n+2})\right) \leq \frac{\beta+\gamma}{1-\alpha-\beta}\;\psi\left(d(x_n,x_{n+1})\right)$$

Since $0 < \alpha + 2\beta + \gamma < 1$ in both cases, which implies

$$\psi\left(d(x_{n+1},x_{n+2})\right) \leq k\;\psi\left(d(x_n,x_{n+1})\right) \qquad (2.1)$$

where $k = max\left\{\frac{\beta+\gamma}{1-\alpha-\beta},\frac{\alpha+\beta+\gamma}{1-\beta}\right\}$.

Therefore, $d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1})$ for all $n \geq 0$ and $\{d(x_n,x_{n+1})\}$ is monotone decreasing sequence of non negative real numbers. Hence there exists an $r \geq 0$ such that,

$$d(x_n,x_{n+1}) \to r \;\; as \;\; n \to \infty\,. \qquad (2.2)$$

Taking the limit as $n \to \infty$ in (2.1) and using the continuity of ψ, we have

$$\psi\,(r) \leq k\,\psi\,(r)$$

which is a contradiction unless $r = 0$.

Hence,

$$lim_{n\to\infty}\; d(x_n,x_{n+1}) = 0 \qquad (2.3)$$

Next we show that $\{\,x_n\}$ is a Cauchy sequence. If otherwise, there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{\;m(k)\;\}\;and\,\{\,n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $\left(x_{m(k)},x_{n(k)}\right) \geq \epsilon$.

Assume that n(k) is the smallest such positive integer, we get, $n(k) > m(k) > k$

$$d\left(x_{m(k)},x_{n(k)}\right) \geq \epsilon \text{ and } d\left(x_{m(k)},x_{n(k)-1}\right) < \epsilon.$$

Now,

$$\epsilon \leq d\left(x_{m(k)},x_{n(k)}\right) \leq d\left(x_{m(k)},x_{n(k)-1}\right) + d(x\_(n(k)-1),x_{n(k)})$$

that is,

$$\epsilon \leq d\left(x_{m(k)},x_{n(k)}\right) \leq \epsilon + d\left(x_{n(k)-1},x_{n(k)}\right)$$

Taking the limit as $k \to \infty$ in the above inequality and (2.3), we have

$$lim_{n\to\infty}\; d\left(x_{m(k)},x_{n(k)}\right) = \epsilon \qquad (2.4)$$

Again,

$$d\left(x_{m(k)}, x_{n(k)}\right) \leq d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{m(k)+1}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right)$$

and,

$$d\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq d\left(x_{m(k)+1}, x_{m(k)}\right) + d\left(x_{m(k)}, x_{n(k)}\right) + d\left(x_{n(k)}, x_{n(k)+1}\right)$$

Taking the limit as $k \to \infty$ in the above inequality and (2.3) and (2.4), we have,

$$lim_{n\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \epsilon \tag{2.5}$$

Again,

$$d\left(x_{m(k)}, x_{n(k)}\right) \leq d\left(x_{m(k)}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right)$$

and,

$$d\left(x_{m(k)}, x_{n(k)+1}\right) \leq d\left(x_{m(k)}, x_{n(k)}\right) + d\left(x_{n(k)}, x_{n(k)+1}\right)$$

Taking the limit as $k \to \infty$ in the above inequality and (2.3) and (2.4), we have,

$$lim_{n\to\infty} d\left(x_{m(k)}, x_{n(k)+1}\right) = \epsilon \tag{2.6}$$

Similarly we have that

$$lim_{n\to\infty} d\left(x_{n(k)}, x_{m(k)+1}\right) = \epsilon \tag{2.7}$$

For each positive integer k, $x_{m(k)}$ and $x_{n(k)}$ are comparable. Then using the monotone property of ψ and the condition (iv), we have

$$\psi\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right)\right) \leq \psi\left(\delta\left(Tx_{m(k)}, Tx_{n(k)}\right)\right)$$

$$\psi\left(\delta\left(Tx_{m(k)}, Tx_{n(k)}\right)\right) \leq \alpha\, \psi\left(max\left\{D\left(x_{m(k)}, Tx_{m(k)}\right), D\left(x_{n(k)}, Tx_{n(k)}\right)\right\}\right) + \beta\, \psi\left(max\left\{D\left(x_{m(k)}, Tx_{n(k)}\right), D\left(x_{n(k)}, Tx_{m(k)}\right)\right\}\right) + \gamma\, \psi\left(d\left(x_{m(k)}, x_{n(k)}\right)\right)$$

By using (iv) and on taking limit as $k \to \infty$ in the above inequality and (2.3) - (2.7), and using the continuity of ψ we have,

$$\psi(\epsilon) \leq k\, \psi(\epsilon)$$

which is contradiction by virtue of a property of ψ.

$Hence$ $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists a $z \in X$ such that

$$x_n \to z \quad as \quad n \to \infty \tag{2.8}$$

By the assumption (iii), $x_n \preccurlyeq z$, for all n.

Then by the monotone property of ψ and the condition (iv), we have

$$\psi\left(d(x_{n+1},Tz)\right)\leq\ \psi\left(\delta\left(Tx_n,T(z)\right)\right)$$

By using (iv) and on taking limit as $k\to\infty$ in the above inequality from (2.3) and (2.8), and using the continuity of ψ we have,

$$\psi\big(\delta\,(z,Tz)\big)\leq\ k\,\psi\big((Dz,Tz)\big)\leq\ k\,\psi\left(\delta\,(z,Tz)\right),$$

which implies that, $\delta\,(z,Tz)\ =\ 0$ or that $\{\,z\,\}=\ Tz$. Moreover, z is a fixed point of T.

Corollary 2.2: Let $(X,\preccurlyeq)$ be a ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T:\ X\to B(X)$ be a multivalued mapping such that the following conditions are satisfied;

i. there exists $x_0\in X$ such that $\{\,x_0\}\preccurlyeq\ Tx_0$,
ii. for $x,y\in\ X,x\preccurlyeq\ y$ implies $Tx\preccurlyeq Ty$,
iii. if $x_n\to\ x$ is a non decreasing sequence in X, then $x_n\preccurlyeq\ x$ for all n,
iv. $\delta(Tx,Ty)\leq\alpha\max\{\,D(x,Tx),D(y,Ty)\}$
$+\beta\ max\,\{\,D(x,Ty),D(y,Tx)\}\ +\gamma\ d(x,y)$

For all comparable $x,y\in\ X$ where $\alpha\,,\beta\,,\gamma\in\ (0,1)$ such that $0\ <\alpha\ +\ 2\beta+\gamma\ <\ 1$ and $\backslash psi$ is an Generalized distance function. Then T has a fixed point.

Proof: On takeing an identity function in Theorem 2.1, then the above result is true and noting to prove.

The following corollary is a spacial case of Theorem 2.1 when T is a singlevalued mapping.

Corollary 2.3: Let $(X,\preccurlyeq)$ be a ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T:\ X\to\ X$ be a mapping such that the following conditions are satisfied;

i. there exists $x_0\in\ X$ such that $\{\,x_0\}\preccurlyeq Tx_0$,
ii. for $x,y\in\ X,x\preccurlyeq\ y\ implies\ Tx\preccurlyeq\ Ty,$
iii. if $x_n\to\ x$ is a non decreasing sequence in X, then $x_n\preccurlyeq\ x$ for all n,
iv. $\psi\big(d(Tx,Ty)\big)\leq\alpha\,\psi(\max\{\,d(x,Tx),d(y,Ty)\})$
$+\beta\,\psi\,(max\,\{d(x,Ty),d(y,Tx)\}\,)\ +\gamma\,\psi\left(d(x,y)\right)$

For all comparable $x,y\in X$ where $\alpha\,,\beta\,,\gamma\in\ (0,1)$ such that $0\ <\alpha\ +\ 2\beta\ +\gamma\ <\ 1$ and ψ is an Generalized distance function. Then T has a fixed point.

In the following theorem we replace condition (iii) of the above corollary by requiring T to be continuous.

Theorem 2.4: Let $(X,\preccurlyeq)$ be a ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T:\ X\to\ X$ be a mapping such that the following conditions are satisfied;

i. there exists $x_0\in\ X$ such that $\{\,x_0\}\preccurlyeq\ Tx_0$,
ii. for $x,y\in\ X,x\preccurlyeq\ y$ implies $Tx\preccurlyeq\ Ty,$
iii. $\psi\big(d(Tx,Ty)\big)\leq\alpha\,\psi(max\,\{\,d(x,Tx),d(y,Ty)\})$
$+\beta\,\psi\,(max\,\{\,d(x,Ty),d(y,Tx)\})\ +\gamma\,\psi\,(d(x,y))$

For all comparable $x, y \in X$ where $\alpha, \beta, \gamma \in (0,1)$ such that $0 < \alpha + 2\beta + \gamma < 1$ and ψ is an Generalized distance function. Then T has a fixed point.

Proof: We can treat T as a multivalued mapping in which case Tx is a singleton set for every $x \in X$. Then we consider the same sequence $\{x_n\}$ as in the proof of Theorem 2.1, Arguing exactly as in the proof of Theorem 2.1, we have that $\{x_n\}$ is a Cauchy sequence and $lim_{n\to\infty}(x_n) = z$. Then the continuity of T implies that,

$$z = lim_{n\to\infty}(x_{n+1}) = lim_{n\to\infty} T(x_n) = Tz$$

and this proves that z is a fixed point of T.

Theorem 2.5: Let $(X, \preccurlyeq)$ be a ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to B(X)$ be a multivalued mapping such that the following conditions are satisfied;

i. there exists $x_0 \in X$ such that $\{x_0\} \preccurlyeq Tx\_0$,
ii. for $x, y \in X, x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$,
iii. if $x_n \to x$ is a non decreasing sequence in X, then $x_n \preccurlyeq x$ for all n,
iv. $$\psi\big(\delta(Tx,Ty)\big) \le \psi(max\{D(x,Tx), D(y,Ty)\}) + \psi(max\{D(x,Ty), D(y,Tx)\}) + \psi\big(d(x,y)\big) - \phi(max\{\delta(x,Tx), \delta(y,Ty), \delta(x,Ty), \delta(y,Tx), d(x,y)\})$$

For all comparable $x, y \in X$ where ψ is an Generalized distance function $and\ \phi : [0,\infty) \to [0,\infty)$ is any continuous function $with\ \phi(t) = 0$ if and only $if\ t = 0$. Then T has a fixed point.

Proof: We take the same sequence $\{x_n\}$ as in the proof of Theorem 2.1. If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T. Hence we shall assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Using the monotone property of ψ and the condition (iv), we have for all $n \geq 0$,

$$\psi\big(d(x_{n+1}, x_{n+2})\big) \le \psi\big(\delta(Tx_n, Tx_{n+1})\big)$$

$$\psi\big(\delta(Tx_n, Tx_{n+1})\big) \le \psi(max\{D(x_n,Tx_n), D(x_{n+1},Tx_{n+1})\}) + \psi(max\{D(x_n,Tx_{n+1}), D(x_{n+1},Tx_n)\}) + \psi\big(d(x_n,x_{n+1})\big) - \phi(max\{\delta(x_n,Tx_n), \delta(x_{n+1},Tx_{n+1}), \delta(x_n,Tx_{n+1}), \delta(x_{n+1},Tx_n), d(x_n,x_{n+1})\})$$

$$\psi\big(d(x_{n+1},x_{n+2})\big) \le \psi(max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}) + \psi(max\{d(x_n,x_{n+2}), d(x_{n+1},x_{n+1})\}) + \psi\big(d(x_n,x_{n+1})\big)$$

$$-\phi(max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), d(x_n,x_{n+2}), d(x_{n+1},x_{n+1}), d(x_n,x_{n+1})\})$$

$$\psi d(x_{n+1},x_{n+2}) \le \psi(max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}) + \psi\big(d(x_n,x_{n+2})\big) + \psi\big(d(x_n,x_{n+1})\big)$$

$$-\phi(\max \{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), \})$$

Then from the above inequality we have,

$$\psi \left(d(x_{n+1}, x_{n+2})\right) \leq \psi \left(d(x_{n+1}, x_{n+2})\right) - \phi \left(d(x_{n+1}, x_{n+2})\right)$$

that is , $\phi \left(d(x_{n+1}, x_{n+2})\right) \leq 0$ which implies that $(d(x_{n+1}, x_{n+2})) = 0$, or that $x_{n+1} = x_{n+2}$, contradicting our assumption that is $x_n \neq x_{n+1}$ for each n.

Therefore, $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all $n \geq 0$ and $\{ d(x_n, x_{n+1})\}$ is monotone decreasing sequence of non negative real numbers. Hence there exists $an\ r \geq 0$ such that,

$$d(x_n, x_{n+1}) \to r \quad as \ n \to \infty \tag{2.9}$$

Taking the limit $as\ n \to \infty$ and using the continuity of ψ, we have

$$\psi (r) \leq \psi (r) - \phi (r)$$

which is a contradiction $unless\ r = 0$.

Hence,

$$lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{2.10}$$

Next we show that $\{ x_n\}$ is a Cauchy sequence. If not then using an argument to that given in Theorem 2.1, we can find two sequences of positive integers $\{ m(k)\}$ and $\{n(k)\}$ for which,

$$lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \epsilon \tag{2.11}$$

$$lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \epsilon \tag{2.12}$$

$$lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)+1}\right) = \epsilon \tag{2.13}$$

$$lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)+1}\right) = \epsilon \tag{2.14}$$

for each positive integer k, $x_{m(k)}, x_{n(k)}$ are comparable. Then using monotone property of ψ and the condition (iv), we have

$$\psi \left(d\left(x_{m(k)+1}, x_{n(k)+1}\right)\right) \leq \psi \left(\delta\left(Tx_{m(k)}, Tx_{n(k)}\right)\right)$$

$$\begin{aligned}\psi \left(\delta \left(Tx_{m(k)}, Tx_{n(k)}\right)\right) \leq\ & \psi(\max \{ D\left(x_{m(k)}, Tx_{m(k)}\right), D\left(x_{n(k)}, Tx_{n(k)}\right)\}) \\ & + \psi \left(\max \{D\left(x_{m(k)}, Tx_{n(k)}\right), D\left(x_{n(k)}, Tx_{m(k)}\right)\}\right) \\ & +\psi \left(d\left(x_{m(k)}, x_{n(k)}\right)\right) \\ & -\phi\left(\max\left\{\begin{matrix} \delta\left(x_{m(k)}, Tx_{m(k)}\right), \delta\left(x_{n(k)}, Tx_{n(k)}\right), \delta\left(x_{m(k)}, Tx_{n(k)}\right), \\ \delta\left(x_{n(k)}, Tx_{m(k)}\right), d\left(x_{m(k)}, x_{n(k)}\right)\end{matrix}\right\}\right)\end{aligned}$$

$$\begin{aligned}\psi \left(d\left(x_{m(k)+1}, x_{n(k)+1}\right)\right) \leq\ & \psi(\max \{ d\left(x_{m(k)}, x_{m(k)+1}\right), d\left(x_{n(k)}, x_{n(k)+1}\right)\}) \\ & + \psi \left(\max \{ d\left(x_{m(k)}, x_{n(k)+1}\right), d\left(x_{n(k)}, x_{m(k)+1}\right)\}\right) \\ & +\psi \left(d\left(x_{m(k)}, x_{n(k)}\right)\right)\end{aligned}$$

$$-\phi\left(max\begin{Bmatrix} d(x_{m(k)},x_{m(k)+1}),d(x_{n(k)},x_{n(k)+1}),d(x_{m(k)},x_{n(k)+1}),\\ d(x_{n(k)},x_{m(k)+1}),d(x_{m(k)},x_{n(k)})\end{Bmatrix}\right)$$

Letting $k \to \infty$ in the above inequality, using (2.10)-(2.14) and the continuous of ψ and ϕ, we have

$$\psi(\epsilon) \le \psi(\epsilon) - \phi(\epsilon)$$

which contradiction by virtue of the property of ϕ .

Hence $\{x_n\}$ is Cauchy sequence. From the completeness of X, there exists a $z \in X$ such that,

$$x_n \to z \ as \quad n \to \infty \tag{2.15}$$

by the assumption of (iii), $x_n \preccurlyeq z$, for all n,

Then by the monotone property of ψ and the condition (iv), we have

$$\psi\left(d(x_{n+1},Tz)\right) \le \psi\left(\delta\left(Tx_n,T(z)\right)\right)$$

$$\psi\left(\delta\left(Tx_n,T(z)\right)\right) \le \psi(max\{D(x_n,Tx_n),D((z),T(z))\}) +$$

$$\psi\left(max\{D(x_n,T(z)),D((z),Tx_n)\}\right)$$

$$+\psi\left(d(x_n,(z))\right) -$$

$$\phi\left(max\begin{Bmatrix}\delta(x_n,Tx_n),\delta((z),T(z)),\delta(x_n,T(z)),\\ \delta((z),Tx_n),d(x_n,(z))\end{Bmatrix}\right)$$

Taking the limit as $n \to \infty$ in the above inequality and (2.10) and (2.15), we have,

$$\psi\left(\delta\left(z,T(z)\right)\right) \le \psi(D(z,Tz)) - \phi\left(\delta\left(z,T(z)\right)\right)$$

which implies that,

$$\psi\left(\delta(z,T(z))\right) \le \psi(\delta\left(z,T(z)\right)) - \phi\left(\delta(z,T(z))\right)$$

Which is contradiction unless $\delta\left(z,T(z)\right) = 0$ or that, $z = Tz$; that is Z is a fixed point of T.

On taking ψ an identity function in Theorem 2.5, we have the following result.

Corollary 2.6: Let $(X,\preccurlyeq)$ be a ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T : X \to B(X)$ be a multivalued mapping such that the following conditions are satisfied;

i. there exists $x_0 \in X$ such that $\{x_0\} \preccurlyeq Tx_0$,
ii. for $x,y \in X, x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$,
iii. if $x_n \to x$ is a non decreasing sequence in X, then $x\_n \backslash preceq\ x$ for all n,
iv. $\delta(Tx,Ty) \le \max\{D(x,Tx),D(y,Ty)\}$

$$+ \max\{ D(x,Ty), D(y,Tx)\} + (d(x,y)) -$$
$$\phi(max \{ \delta(x,Tx), \delta(y,Ty), \delta(x,Ty), \delta(y,Tx), d(x,y)\})$$

For all comparable $x, y \in X$ where ψ is an Generalized distance function and $\phi : [0,\infty) \to [0,\infty)$ is any continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a fixed point.

The following corollary is a special case of Theorem 2.5 when T is a singlevalued mapping.

Corollary 2.7: Let $(X, \preccurlyeq)$ be a ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a multivalued mapping such that the following conditions are satisfied;

i. there exists $x_0 \in X$ such that $\{x_0\} \preccurlyeq Tx_0$,
ii. for $x, y \in X, x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$,
iii. $if\ x_n \to x$ is a non decreasing sequence in X, then $x_n \preccurlyeq x$ for all n,
iv. $\psi(d(Tx,Ty)) \leq \psi(max\{d(x,Tx), d(y,Ty)\}) + \psi(max\{d(x,Ty), d(y,Tx)\})$
$+\psi(d(x,y)) - \phi(max\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), d(x,y)\})$

For all comparable $x, y \in X$ where ψ is an Generalized distance function and $\phi : [0,\infty) \to [0,\infty)$ is any continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a fixed point.

In the following theorem we replace condition (iii) of the above corollary by requiring T to be continuous.

Theorem 2.8: Let $(X, \preccurlyeq)$ be a ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a multivalued mapping such that the following conditions are satisfied;

i. there exists $x_0 \in X$ such that $\{x_0\} \preccurlyeq Tx\_0$,
ii. for $x, y \in X, x \preccurlyeq y$ implies $Tx \preccurlyeq Ty$,
iii. $\psi(d(Tx,Ty)) \leq \psi(max\{d(x,Tx), d(y,Ty)\}) + \psi(max\{d(x,Ty), d(y,Tx)\})$
$+\psi(d(x,y)) - \phi(max\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), d(x,y)\})$

For all comparable $x, y \in X$ where ψ is an Generalized distance function and $\phi : [0,\infty) \to [0,\infty)$ is any continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a fixed point.

Proof: We can treat T as a multivalued mapping in which $case\ Tx$ is a singleton set for every $x \in X$. Then we consider the same sequence $\{x_n\}$ as in the proof of Theorem 2.5, Arguing exactly as in the proof of Theorem 2.5, we have that $\{x_n\}$ is a Cauchy sequence and $lim_{n\to\infty}(x_n) = z$. Then the continuity of T implies that,

$$z = lim_{n\to\infty}(x_{n+1}) = lim_{n\to\infty} T(x_n) = Tz$$

and this proves that z is a fixed point of T.

Acknowledgments : Authors are heartly thanks to referee for their valuable suggestion to improve the quality of the article.

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