IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

Fixed Point Theorem Using Weakly Compatible Mapping In Metric Spaces

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Abstract

We prove a unique common fixed-point theorem for two pair of weakly compatible maps in a complete metric space, which generalizes the previous results.

Keywords: Common fixed point, Complete metric space, Weakly compatible maps.

1 Introduction

The notion of commutativity has been generalized in various ways. In this context, Sessa S [6] introduced the idea of weakly commuting, while Gerald Jungck [2] initiated the concept of compatibility. It is straightforward to verify that if two mappings are commuting, they are also compatible; however, the reverse is not necessarily true. In 1998, Jungck and Rhoades [4] presented the concept of weakly compatible mappings, demonstrating that compatible maps are indeed weakly compatible, but the converse does not hold. An important Common Fixed Point theorem was established by Brian Fisher [1].

The exploration of common fixed points of mappings that satisfy contractive type conditions has been a vibrant area of research over the past thirty years. In 1922, the Polish mathematician Banach proved a theorem that guarantees, under certain conditions, the existence and uniqueness of a fixed point. This result is known as the Banach fixed point theorem or the Banach contraction principle. This theorem offers a method for addressing a range of applied problems in mathematical science and engineering. Numerous authors have extended, generalized, and enhanced the Banach fixed point theorem in various manners. Jungck [2] introduced a more generalized form of commuting mappings, referred to as compatible mappings, which encompass both commuting and weakly commuting mappings.

The primary objective of this paper is to present fixed point results for two pairs of maps that satisfy a new contractive condition, utilizing the concept of weakly compatible maps within a complete metric space.

2 Preliminaries

We recall the definitions of complete metric space, the notion of convergence and other results that will be needed in the sequel.

Definition 2.1 Let f and g be two self-maps on a set X. Maps f and g are said to be commuting if $fgx = gfx$ for all $x \in X$.

Definition 2.2 Let f and g be two self-maps on a set X. If $fx = gx$, for some x in X then x is called coincidence point of f and g.

Definition 2.3 Let f and g be two self-maps defined on a set, then f and g are said to be weakly compatible if they commute at coincidence points. That is, if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

Lemma 2.4[2] Let f and g be weakly compatible self mappings of a set X. If f and g have a unique point of coincidence, that is, $w = fx = gx$, then w is the unique common fixed point of f and g.

Definition 2.5 A sequence $\{x_n\}$ in a metric space (X,d) is said to be convergent to a point $x \in X$, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition 2.6 A sequence $\{x_n\}$ in a metric space (X,d) is said to be Cauchy sequence if $\lim_{t\to\infty} d(x_n, x_m) = 0$ for all $n, m > t$.

Definition 2.7 A metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.8 A function $\phi:[0,\infty) \to [0,\infty)$ is said to be a contractive modules if $\phi:[0,\infty) \to [0,\infty)$ and $\phi(t) < t$ for $t > 0$.

Definition 2.9 A real valued function ϕ defined on $X \subseteq R$ is said to be upper semi continuous if $\lim_{n\to\infty} \sup \phi(t_n) \le \phi(t)$, for every sequence $\{t_n\} \in X$ with $t_n \to t$ as $n \to \infty$.

It is clear that every continuous function is upper semi continuous but converse may not true.

Theorem 2.10[1] Suppose S, P, T and Q are four self maps of a metric space (X,d) satisfying the following conditions.

1. $S(X) \subseteq Q(X)$ and $T(X) \subseteq P(X)$.
2. Pairs (S,P) and (T,Q) are commuting.
3. One of S, P, T and Q is continuous.
4. $d(Sx,Ty) \le c\lambda(x,y)$, where $\lambda(x,y) = \max\{d(Px,Qy), d(Px,Sx), d(Qy,Ty)\}$ for all $x, y \in X$ and $0 \le c < 1$.
   Further if
5. X is complete.

Then S, P, T and Q have a unique common fixed point $z \in X$. Also z is the unique common fixed point of (S,P) and (T,Q).

3 Main Result

In this section we prove a common fixed point theorem for two pairs of weakly compatible mappings in complete metric spaces using a contractive modules. This is the generalization of theorem 2.10 in the sense that instead of taking constant c, we take an upper semi continuous, contractive modulus.

Theorem 3.1 Let (X,d) be a complete metric space. Suppose that the mapping P,Q,S and T are four self-mapping of X satisfying the following conditions:

1. $T(X) \subseteq P(X)$ and $S(X) \subseteq Q(X)$.
2. $d(Sx,Ty) \le \phi(\lambda(x,y))$
   where ϕ is an upper semi continuous, contractive modulus and $\lambda(x,y) = \max\{d(Px,Qy), d(Px,Sx), d(Qy,Ty), d(Px,Ty), d(Qy,Sx)\}$.
3. The pairs (S,P) and (T,Q) are weakly compatible.

Then P,Q,S and T have a unique common fixed point.

Proof: Suppose x_0 and y_0 are arbitrary points of X then we define the sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$y_n = Sx_n = Qx_{n+1}$$
$$y_{n+1} = Tx_{n+1} = Px_{n+2}$$

By (ii), we have

$$d(y_n, y_{n+1}) = d(Sx_n, Tx_{n+1}) \le \phi(\lambda(x_n, x_{n+1}))$$

where

$$\lambda(x_n, x_{n+1})$$

$$= \max\{d(Px_n, Qx_{n+1}), d(Px_n, Sx_n), d(Qx_{n+1}, Tx_{n+1}), d(Px_n, Tx_{n+1}), d(Qx_{n+1}, Sx_n)\}$$

$$= \max\{d(Tx_{n-1}, Sx_n), d(Tx_{n-1}, Sx_n), d(Sx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_{n+1}), d(Sx_n, Sx_n)\}$$

$$= \max\left\{d(Tx_{n-1}, Sx_n), d(Sx_n, Tx_{n+1}), \frac{1}{2}\left(d(Tx_{n-1}, Tx_{n+1}) + 0\right)\right\}$$

$$= \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1}), 0\}$$

$$\le \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}$$

Since ϕ is a contractive modulus, $\lambda(x_n, x_{n+1}) = d(y_n, y_{n+1})$ is not possible.

Thus

$$d(y_n, y_{n+1}) \le \phi(d(y_{n-1}, y_n)) \qquad (1)$$

Since ϕ is an upper semi continuous, contractive modulus, equation (1) implies that the sequence $\{d(y_{n+1}, y_n)\}$ is monotonic decreasing and continuous.

Hence there exists a real number, say $r \ge 0$ such that $\lim_{n\to\infty} d(y_{n+1}, y_n) = r$ Therefore as $n \to \infty$, equation (1) implies that

$$r \le \phi(r)$$

Which is possible only if $r = 0$ because ϕ is a contractive modulus.

Thus $\lim_{n\to\infty} d(y_{n+1}, y_n) = 0$

Now we show that $\{y_n\}$ is a Cauchy sequence.

Let if possible we assume that $\{y_n\}$ is not a Cauchy sequence.

Then there exists an $\varepsilon > 0$ and subsequences $\{n_i\}$ and $\{m_i\}$ such that $m_i < n_i < m_{i+1}$ and

$$d(y_{mi}, y_{ni}) \ge \varepsilon \quad \text{and} \quad d(y_{mi}, y_{ni-1}) < \varepsilon \qquad (2)$$

So that $\varepsilon \le d(y_{mi}, y_{ni}) \le d(y_{mi}, y_{ni-1}) + d(y_{ni-1}, y_{ni}) < \varepsilon + d(y_{ni-1}, y_{ni})$

Therefore $\lim_{i\to\infty} d(y_{mi}, y_{ni}) = \varepsilon$

Now $d(y_{mi-1}, y_{ni-1}) \le d(y_{mi-1}, y_{mi}) + d(y_{mi}, y_{ni}) + d(y_{ni}, y_{ni-1})$

By taking limit as $i \to \infty$, we get $\lim_{i\to\infty} d(y_{mi-1}, y_{ni-1}) = \varepsilon$

Now by (ii) and (2)

$$\varepsilon \le d(y_{mi}, y_{ni}) = d(Sx_{mi}, Tx_{ni}) \le \phi(\lambda(x_{mi}, x_{ni}))$$

$$\text{i.e.,} \quad \varepsilon \le \phi(\lambda(x_{mi}, x_{ni})) \qquad (3)$$

where

$\lambda(x_{mi}, x_{ni})$

$= \max\{d(Px_{mi}, Qx_{ni}), d(Px_{mi}, Sx_{mi}), d(Qx_{ni}, Tx_{ni}), d(Px_{mi}, Tx_{ni}), d(Qx_{ni}, Sx_{mi})\}$

$= \max\left\{d\left(Tx_{mi-1}, Sx_{ni-1}\right), d\left(Tx_{mi-1}, Sx_{mi}\right), d\left(Sx_{ni-1}, Tx_{ni}\right), d\left(Tx_{mi-1}, Tx_{ni}\right), d\left(Sx_{ni-1}, Sx_{mi}\right)\right\}$

$= \max\left\{d\left(y_{mi-1}, y_{ni-1}\right), d\left(y_{mi-1}, y_{mi}\right), d\left(y_{ni-1}, y_{ni}\right), d\left(y_{mi-1}, y_{ni}\right), d\left(y_{ni-1}, y_{mi}\right)\right\}$

By taking limit as $i \to \infty$, we get $\lim_{i\to\infty} \lambda\left(x_{mi}, x_{ni}\right) = \max\left\{\varepsilon, 0, 0, \left(\varepsilon, \varepsilon\right)\right\}$

Thus we have, $\lim_{i\to\infty} \lambda\left(x_{mi}, x_{ni}\right) = \varepsilon$

Therefore from (3) $\quad \varepsilon \le \phi\left(\varepsilon\right)$

This is a contradiction because $0 < \varepsilon$ and ϕ is contractive modulus.

Thus $\left\{y_n\right\}$ is a Cauchy sequence in X.

Since X is complete, there exists a point z in X such that $\lim_{n\to\infty} y_n = z$

Thus $\quad \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Qx_{n+1} = z$ and $\lim_{n\to\infty} Tx_{n+1} = \lim_{n\to\infty} Px_{n+2} = z$

i.e. $\quad \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Qx_{n+1} = \lim_{n\to\infty} Tx_{n+1} = \lim_{n\to\infty} Px_{n+2} = z$

Since $T\left(X\right) \subseteq P\left(X\right)$, there exists a point $u \in X$ such that $z = Pu$.

Then by (ii), we have

$$\begin{aligned} d\left(Su, z\right) &\le d\left(Su, Tx_{n+1}\right) + d\left(Tx_{n+1}, z\right) \\ &\le \phi\left(\lambda\left(u, x_{n+1}\right)\right) + d\left(Tx_{n+1}, z\right) \end{aligned}$$

Where $\lambda\left(u, x_{n+1}\right)$

$= \max\left\{d\left(Pu, Qx_{n+1}\right), d\left(Pu, Su\right), d\left(Qx_{n+1}, Tx_{n+1}\right), d\left(Pu, Tx_{n+1}\right), d\left(Qx_{n+1}, Su\right)\right\}$

$= \max\left\{d\left(z, Sx_n\right), d\left(z, Su\right), d\left(Sx_n, Tx_{n+1}\right), d\left(z, Tx_{n+1}\right), d\left(Sx_n, Su\right)\right\}$

Taking the limit as $n \to \infty$ yields,

$\lambda\left(u, x_{n+1}\right) = \max\left\{d\left(z, z\right), d\left(z, Su\right), d\left(z, z\right), d\left(z, Su\right)\right\} = d\left(Su, z\right)$

Thus as $n \to \infty$, $d\left(Su, z\right) \le \phi\left(d\left(Su, z\right)\right), d\left(z, z\right) = \phi\left(d\left(Su, z\right)\right)$

If $Su \ne z$ then $d\left(Su, z\right) > 0$ and hence as ϕ is contractive modulus

$\phi\left(d\left(Su, z\right)\right) < d\left(Su, z\right)$.

Therefore $d\left(Su, z\right) < d\left(Su, z\right)$, which is a contradiction.

Thus $Su = z$. So $Pu = Su = z$.

So u is a coincidence point of P and S.

Since the pair of maps S and P are weakly compatible, $SPu = PSu$, i.e. $Sz = Pz$.

Again Since

$S(X) \subseteq Q(X)$, there exists a point $v \in X$ such that $z = Qv$.

Then by (ii), we have

$$d(z,Tv) = d(Su,Tv)$$
$$\leq \phi(\lambda(u,v))$$

where

$$\lambda(u,v) = \max\{d(Pu,Qv), d(Pu,Su), d(Qv,Tv), d(Pu,Tv), d(Qv,Su)\}$$

$$= \max\{d(z,z), d(z,z), d(z,Tv), d(z,Tv), d(z,z)\}$$

$$= d(z,Tv)$$

Thus $d(z,Tv) \leq \phi(d(z,Tv))$

If $Tv \neq z$ then $d(z,Tv) > 0$ and hence as ϕ is contractive modulus $\phi(d(z,Tv)) < d(z,Tv)$.

Therefore $d(z,Tv) < d(z,Tv)$, which is a contradiction.

Therefore $Tv = Qv = z$.

So v is a coincidence point of Q and T.

Since the pair of maps Q and T are weakly compatible, $QTv = TQv$, i.e. $Qz = Tz$.

Now we show that z is a fixed point of S.

By (ii), we have

$$d(Sz,z) = d(Sz,Tv)$$

$$\leq \phi(\lambda(z,v))$$

where

$$\lambda(z,v) = \max\{d(Pz,Qv), d(Pz,Sz), d(Qv,Tv), d(Pz,Tv), d(Qv,Sz)\}$$

$$= \max\{d(Sz,z), d(Sz,Sz), d(z,z), d(Sz,z), d(z,Sz)\}$$

$$= d(Sz, z)$$

Thus $d(Sz, z) \le \phi(d(Sz, z))$

If $Sz \ne z$ then $d(Sz, z) > 0$ and hence as ϕ is contractive modulus $\phi(d(Sz, z)) < d(Sz, z)$.

Therefore $d(Sz, z) < d(Sz, z)$, which is a contradiction.

Therefore $Sz = z$.

Hence $Sz = Pz = z$.

Now, we show that z is a fixed point of T.

By (ii), we have

$$d(z, Tz) = d(Sz, Tz)$$

$$\le \phi(\lambda(z, z))$$

where

$$\lambda(z, z) = \max\{d(Pz, Qz), d(Pz, Sz), d(Qz, Tz), d(Pz, Tz), d(Qz, Sz)\}$$

$$= \max\{d(z, Tz), d(z, z), d(Tz, Tz), d(z, Tz), d(Tz, z)\}$$

$$= d(z, Tz)$$

Thus $d(z, Tz) \le \phi(d(z, Tz))$

If $z \ne Tz$ then $d(z, Tz) > 0$ and hence as ϕ is contractive modulus $\phi(d(z, Tz)) < d(z, Tz)$.

Therefore $d(z, Tz) < d(z, Tz)$, which is a contradiction.

Hence $z = Tz$.

Therefore $Tz = Qz = z$.

Therefore $Sz = Pz = Tz = Qz = z$ i.e. z is a common fixed point of P, Q, S and T.

Uniqueness: For uniqueness of z let if possible, we assume that z and $w, (z \ne w)$ are common fixed points of P,Q,S and T.

By (ii), we have

$$d(z, w) = d(Sz, Tw)$$

$$\le \phi(\lambda(z, w))$$

where

$$(z,w)=\max\{d(Pz,Qw),d(Pz,Sz),d(Qw,Tw),d(Pz,Tw),d(Qw,Sz)\}$$

$$=\max\{d(z,w),d(z,z),d(w,w),d(z,w),d(w,z)\}$$

$$=d(z,w)$$

Thus $d(z,w)\le\phi(d(z,w))$

Since $z\ne w$ then $d(z,w)>0$ and hence as ϕ is contractive modulus $\phi(d(z,w))<d(z,w)$.

Therefore $d(z,w)<d(z,w)$, which is a contradiction.

Therefore $z=w$.

Thus z is the unique common fixed point of P,Q,S and T.

Hence the theorem.

Corollary 3.2 Let (X,d) be a complete metric space. Suppose that the mappings P,S and T are self-maps of X satisfying the following conditions:

1. $T(X)\subseteq P(X)$ and $S(X)\subseteq P(X)$;
2. $d(Sx,Ty)\le\phi(\lambda(x,y))$

   where ϕ is an upper semi continuous, contractive modulus and

   $\lambda(x,y)=\max\{d(Px,Py),d(Px,Sx),d(Py,Sy),d(Px,Sy),d(Py,Sx)\}$.
3. The pair (S,P) is weakly compatible.

Then P and S have a unique common fixed point.

Proof: By taking $P=Q$ and $S=T$ in theorem 3.1 we get the proof.

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