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On Statistical Euler Summability Methods For Sequences Of Fuzzy Numbers And Its Applications To Fuzzy Korovkin's Theory

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Abstract

Many scholars today are deeply interested in the concept of statistical convergence, primarily because it is stronger than regular convergence. A key concept in the convergence of positive linear operator sequences is the Korovkin-type approximation theorem. Additionally, this class of approximation theorems has been expanded to general sequence spaces using other statistical summability techniques. In this proposed research, we have proved a fuzzy Korovkin-type approximation theorem by introducing a new statistical Euler product summability mean for sequences of fuzzy integers. Additionally, under our suggested summability mean, we have proved another conclusion for the fuzzy rate of convergence, which is uniform in fuzzy Korovkin-type approximation theorem.

Key words: Sequences of fuzzy numbers; Euler summability method; Korovkin's theory; Statistical convergence

Classification of AMS Subjects (2010): 41A10, 41A30, 41G35, 41A25, 41A36.

1 Overview and Preliminary

The majority of a sequence's elements must be within an arbitrary small radius of the limit, according to the theory of convergence, which applies to both classical and fuzzy techniques. Yet, since statistics deals with big numbers, such as populations and samples, this need is somewhat loosened in statistical convergence. Furthermore, the requirement of convergence need only be met by the majority of the sequence's members. Fast [14] introduced and investigated statistical convergence for the first time in 1951.It was then investigated by Fridy [15], Steinhaus [34], and Conner [11]. Eventually, a number of scholars employed various summability strategies while working in this topic. Furthermore, a variety of mathematical fields including number theory, approximation theory, integration, differential equations, Fourier analysis, and functional analysis, rely heavily on statistical convergence and summability.

Let $K \subseteq N$ and $k_n = \{k : k \leq m \; and \; k \in K\}$, the natural (asymptotic) density of K is defined as

$$\delta(K) = lim_{m\to\infty} \frac{|k_m|}{m},$$

as long as the restriction is real.

For every $\varepsilon \geq 0$, a sequence $u = (u_k)$ is statistically convergent to u_0.

$$\{k : k \in N \; and \; |u_k - u_0| \geq \varepsilon \}$$

possesses zero natural density. In other words, for any $\varepsilon \geq 0$.

$$\delta(k : k \leq m \; and \; |u_k - u_0| \geq \varepsilon) = 0.$$

The fuzzy set theory has spread to almost every field of science and technology, including management and medical, and it has developed into a flexible multidisciplinary research field. Zedah [38] introduced and explored fuzzy sets for the first time. As such, numerous writers have created various facets of the Applications of fuzzy sets include fuzzy topological spaces, fuzzy differential equations, fuzzy graph theory, fuzzy logic, fuzzy mathematical programming, and more. For sequences with fuzzy numbers valued, a number of summability strategies have been defined.2010 saw the definition of Cesaro summability of order one for sequences of fuzzy real numbers by Altın et al. [2]. More recently, Yavuz [37] established Euler's summability method of sequences of fuzzy numbers, and Talo and Bal [35] established statistical summability (N, P) for sequences of fuzzy numbers. To further our investigation in this area, we introduced statistical versions of the Euler summability mean for sequences of fuzzy numbers. Specifically, we used the statistical (E, q) product summablity mean for sequences of fuzzy numbers to derive a fuzzy approximation theorem (Korovkin -type) based on fuzzy linear operators (positive). Furthermore, by defining the fuzzy rate of convergence using our suggested mean, we have also arrived to another conclusion.

Starting from the beginning, we will cover the fundamentals of fuzzy numbers, including their linear structure and algebraic features.

On the real axis $\mathbb{R}$, a fuzzy set is called a fuzzy real number. The fuzzy number $\tilde{u}: \mathbb{R} \to [0,1]$ is a mapping that meets the following requirements:

(i) $\tilde{u}$ is regular if there is a $t_0 \in \mathbb{R}$ such that $\tilde{u}(t_0) = 1$;

(ii) If $\tilde{u}(\beta t_0 + (1-\beta)t_1) \geq min\{t_0, t_1\}$ for any $t_0, t_1 \in \mathbb{R}$ and any $\beta \in [0,1]$, then ˎu is upper semi-continuous on R and fuzzy convex;

(iii) In the standard topology of $\mathbb{R}$, support is compact, as supp $[\tilde{u}] = \{t \in \mathbb{R}: (t) \geq 0\}$, and it is the closure of $\{t \in \mathbb{R}: \tilde{u}(t) \geq 0\}$.

First, let $\mathscr{F}_{\mathbb{R}}$ represent the set of all fuzzy numbers on $\mathbb{R}$. It is defined that the α-level set $[\tilde{u}]\beta \in \mathscr{F}_{\mathbb{R}}$

$$[\tilde{u}]\beta = \begin{cases} \{t \in \mathbb{R} : \tilde{u}(t) \geq \beta\}, if\ (0 < \beta \leq 1\} \\ \{t \in \mathbb{R}: \tilde{u}(t) > \beta\}, if\ (\beta = 0). \end{cases}$$

It should be noted that the aforementioned set is closed, bounded, and nonempty for any α, where $0 \leq \beta \leq 1$ and $[\tilde{u}]\lambda \subseteq [\tilde{u}]$α, where $0 \leq \beta \leq \lambda \leq 1$. Furthermore, $[\tilde{u}]\beta = [\tilde{u} - \beta, \tilde{u} + \beta]$ is true ($\tilde{u} - \beta \leq \tilde{u} + \alpha$ and $\tilde{u} - \beta, \tilde{u} + \beta \in R$). Put another way, the end points of the interval determine a fuzzy number $\tilde{u}$ in its entirety.

This lemma can be used to express a fuzzy integer in terms of an interval of $\mathbb{R}$.

Lemma 1. (Refer to [25]) If $[\tilde{u}]\beta = [\tilde{u} - \beta, \tilde{u} + \beta]$ and $\tilde{u} \in \mathscr{F}_{\mathbb{R}}$, in that case

(i) Over $(0,1], \tilde{u} - \beta$ is a left continuous function that increases monotonically;

(ii) Over $(0,1], \tilde{u} + \beta$ is a right continuous function that increases monotonically;

(iii) $\tilde{u} - \beta$ and $\tilde{u} + \beta$ are right continuous at $\beta = 0$;

(iv) $\tilde{u} - \alpha \leq \tilde{u} + \alpha$

Let $[\tilde{u}_\beta^-, \tilde{u}_\beta^+]$ and $[\tilde{v}_\beta^-, \tilde{v}_\beta^+]$ $(0 \leq \beta \leq 1)$ represent $\tilde{u} and\ \tilde{v} \in \mathscr{F}_{\mathbb{R}}$, respectively. Next, the following represents the scalar addition and multiplication on the fuzzy number set:

$$[\tilde{u} + \tilde{v}]_\beta = [\tilde{u}]_\beta + [\tilde{v}]_\beta = [\tilde{u}_\beta^- + \tilde{v}_\beta^-, \tilde{u}_\beta^+ + \tilde{v}_\beta^+]$$

$$[k\tilde{u}]_\beta = k[\tilde{u}]_\beta = \begin{cases} [k\tilde{u}_\beta^-, k\tilde{u}_\beta^+], k \geq 0; \\ [k\tilde{u}_\beta^+, k\tilde{u}_\beta^-]. k < 0. \end{cases}$$

$D : \mathcal{F}_{\mathbb{R}} \times \mathcal{F}_{\mathbb{R}} \rightarrow [\infty, 0)$ is a function that determines the Hausdorff distance (refer to [26]) between two fuzzy numbers.

$$D(\tilde{u}, \tilde{v}) = sup_{0\leq\beta\leq1}\, d([\tilde{u}]_\beta, [[\tilde{v}]_\beta)$$

$$= sup_{0\leq\beta\leq1} max\{|\tilde{u}_\beta^- - \tilde{u}_\beta^+ |, |\tilde{v}_\beta^- - \tilde{v}_\beta^+ |\},$$

with $\mathcal{D}$ being a Hausdorff metric. As we are aware

(i) A full metric space is $(\mathscr{F}_{\mathbb{R}}, \mathcal{D})$.

(ii) $\mathcal{D}(\rho\tilde{u}, \rho\ \tilde{v}) = |\rho|\mathcal{D}(\tilde{u}\ \tilde{v})$ additionally,

(iii) $D(\tilde{u} + \widetilde{w},\ \tilde{u} + \widetilde{w}) = \mathcal{D}(\tilde{u}\ \tilde{v}$, where $\rho \in \mathbb{R}, \tilde{u}, \tilde{v}, \widetilde{w} \ \in \mathscr{F}_{\mathbb{R}}$.

We then review the Hausdorff distance in relation to two functions. The Hausdorff distance, represented by $\mathcal{D}^*(f, g)$, between two fuzzy numbered valued functions, f, g: $[a, b] \rightarrow \mathscr{F}_{\mathbb{R}}$, is given by

$$\mathcal{D}^*(f, g) = sup_{0\leq y\leq1} sup_{0\leq\beta\leq1} max\left\{|\tilde{f}_\beta^- - \tilde{g}_\beta^- |, |\tilde{f}_\beta^+ - \tilde{g}_\beta^+ |\right\}.$$

Well known is the fact that $\mathscr{F}_{\mathbb{R}}$ can contain the set of all real numbers $\mathbb{R}$. The matching fuzzy number representation $\tilde{r}$ for every r in $\mathbb{R}$ is really provided by

$$\tilde{r}(t) = \begin{cases} 1, if\ t = r; \\ 0, if\ t \neq r. \end{cases}$$

We also review certain definitions related to fuzzy number sequence convergence. Matkola [24] presented the concept of convergent sequences of fuzzy numbers for the first time.

When every $n \in \mathbb{N}$ and $\varepsilon > 0$ for each fuzzy number $(\tilde{u}_n)$˜un, the sequence of fuzzy numbers is said to be convergent to $\tilde{u}_0$.

$$\mathcal{D}(\tilde{u}_n, \tilde{u}_0) < \varepsilon.$$

If there is a constant $M > 0$ such that a sequence of fuzzy numbers $(\tilde{u}_n)$ is bounded, then

$$\mathcal{D}(\tilde{u}_n, 0) \leq M\ (\forall\, n \in \mathbb{N}).$$

Nuray and Savas [26] developed the concept of statistical convergence for a sequence of fuzzy integers in 1995.

If for any$\varepsilon > 0$, a sequence $(\tilde{u}_k)$ of fuzzy numbers is statistically convergent to a fuzzy number $\tilde{u}_0$.

$$\delta(\{k \leq n : D(\tilde{u}_k, \tilde{u}_0) \geq \varepsilon\}) = 0.$$

This is where we write

$$stat_{\mathscr{F}} F\ lim_{k\to\infty}\,\tilde{u}_k \;=\; \tilde{u}_0.$$

If the sequence-to-sequence transformation, under (E, q)-summability, can statistically sum up a fuzzy sequence $(\tilde{u}_i)$ to a fuzzy number $(\tilde{u}_0)$,

$$t_m^E = \frac{1}{(q+1)^m}\sum_{i=0}^{m}\binom{m}{i} q^{m-i}\tilde{u}_i, \qquad for\ q > 0$$

is approaching $\tilde{u}_0$ statistically.

Stated otherwise, ($\tilde{u}_i$) can be statistically summable to a fuzzy number $\tilde{u}_0$through (E, q). The mean summability, if

$$stat_{\mathscr{F}} lim_{m\to\infty}\,\mathcal{D}(t_m^E, \tilde{u}_0) = 0.$$

Here, we write

$$\tilde{u}_0 = \text{(E, q)}\ stat_{\mathcal{F}}\_lim\tilde{u}_i.$$

Example 1.1. Let (u_m) be the series of fuzzy numbers in such a way that

$$u_m(y) = \begin{cases} y + (-1)^m, & (-1)^m < y \le (-1)^{m+1}; \\ y + (-1)^m + 3, & (-1)^m < y \le (-1)^{m+2}; \\ 0, otherwise. \end{cases}$$

Calculating the α-level set of $(u_m\ (\mathrm{x}))$

$$[u_m(y)]_\beta \;=\; [\beta + (-1)^m\,, 3 - \beta + (-1)^m\,]\,.$$

It is evident that $(u_m(y\,))$ is statistically Euler summable to the fuzzy number but not statistically convergent.

$$u(x) \;=\; \begin{cases} x, & 0 < x \le 1; \\ -x + 3, & 1 < x \le 2;. \\ 0, & otherwise. \end{cases}$$

2 Utilizing Fuzzy Korovkin's Theory in practice.

When analyzing the convergence of sequences of positive linear operators, Korovkin's type approximation theory is crucial (both in classical and fuzzy techniques). By using other statistical summability techniques, these approximation theorems have been expanded to a more general space of sequences (see the recent studies $[12], [13], [27], [29] - [32]$. Furthermore, other mathematicians have explored the Korovkin-type theorems in a variety of ways for fuzzy number sequences using different test functions in a variety of setups in function spaces, abstract Banach lattices, Banach algebras, and other related spaces. For sequences of fuzzy numbers, the classical Korovkin-type theorems are provided in [7]–[10], and [16]. The purpose of this section is to demonstrate a Korovkin's type approximation theorem for sequences of fuzzy positive linear operators by introducing a new statistical product $(C,1)(E,\mu)$ −summability mean.

The fundamental fuzzy Korovkin theory was presented by Anastassiou [8] in 2005. The statistical fuzzy Korovkin approximation based on fuzzy positive linear operators was then examined by Anastassiou and Duman [10] in 2008. In order to demonstrate the Korovkin type theorem, Acar and Mohiuddine [1] proposed statistical $(C,1)(E,1)$ product summability very recently. Das et al. presented the statistical $(C,1)(E,\mu)$ −summability approach in 2020, which they used to demonstrate a fuzzy Korvokin-type approximation theorem for fuzzy positive linear operator sequences.

Driven primarily by the previously mentioned findings, we have presented the statistical (E,μ) −summability technique to demonstrate a fuzzy Korvokin-style approximation theorem for fuzzy positive linear operator sequences. Additionally, using our suggested mean, we defined the fuzzy rate of convergence that is uniform in the Korovkin type theorem and thereby proved another result.

Let f be a fuzzy number valued function such that $f : [a,b] \to \mathbb{R}_{\mathscr{F}}$. Remember that if $\mathcal{D}(x_n, x_0) < \varepsilon \ (n \to \infty)$ whenever $x_n \to x_0$, then f is fuzzy continuous at a point $x_0 \in [a,b]$. Additionally, f is fuzzy continuous in $[a,b]$ if it is fuzzy continuous at all points $x \in [a,b]$. $C_{\mathscr{F}}[a,b]$ is the collection of all fuzzy continuous functions that are defined within the interval $[a,b]$. It should be noted that $C_{\mathscr{F}}[a,b]$ is merely a scalar and not a vector space. Four Assuming that each $\mu_1, \mu_2 \in R$ and $f_1, f_2 \in C_{\mathscr{F}}[a,b]$ for every operator $\mathscr{L}: C_{\mathscr{F}}[a,b] \to C_{\mathscr{F}}[a,b]$ is fuzzy linear.

$$\mathcal{L}(\mu_1 \odot f_1 \oplus \mu_2 \odot f_1;\, x) = \mu_1 \odot \mathcal{L}(f_1) \oplus \mu_2 \odot \mathcal{L}(f_2).$$

Furthermore, a fuzzy linear operator L is said to be positive fuzzy linear, if it satisfies $\mathscr{L}(f_1;\, x) \ \ \mathscr{L}(f_2;\, x)$ such that $f_1, f_2 \in C_{\mathscr{F}}[a,b]$ and for all $x \in [a,b]$ with $f_1(x)\ f_2(x)$.

Theorem 1. Let $\{\mathscr{L}_o\}_{o\in N}$ be a sequence of linear operators (positive) from $C_{\mathscr{F}}[a,b]$ into itself. Suppose that, there exists a corresponding sequence $\{\overline{\mathscr{L}}_o\}_{o\in N}$ of positive linear operators from $C[a,b]$ into itself with the property

$$\{\mathscr{L}_o(f;y)\}_{\beta}^{\pm} = \overline{\mathscr{L}}(f_{\beta}^{\pm}, y). \tag{2.1}$$

Furthermore assuming that,

$$stat_f \lim_{m\to\infty} \| \frac{1}{(1+\mu)^m} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \overline{\mathcal{L}}_j(e_{\bar{i}}) - e_{\bar{i}} \| = 0 \; for\; each\; \bar{i} = 0,1,2, \tag{2.2}$$

then for each and every

$$f \in C_{\mathscr{F}}[a,b],$$

$$stat_f \lim_{m\to\infty} \mathcal{D}^*\left(\frac{1}{(1+\mu)^m}\sum_{j=0}^{m}\binom{m}{j}\mu^{m-j}\bar{\mathcal{L}}_j(f)-f\right). \quad (2.3)$$

Proof. Let $z \in [a,b], \beta \in [0,1]$, and $f \in C_{\mathscr{F}}[a,b]$. Since $f_\beta^{\pm} \in C_{\mathscr{F}}[a,b]$, we may state the hypothesis as follows: for every $\varepsilon > 0$, there is an integer $\delta > 0$ such that, for every $z \in [a,b]$, meeting the constraint $|z-y| < \delta, |f_\beta^{\pm}(z) - f_\beta^{\pm}(y)| < \varepsilon$ holds. Next, we obtain for any $z \in [a,b]$,

$$\left|f_\beta^{\pm}(z) - f_\beta^{\pm}(y)\right| < \varepsilon\, \mathrm{h} + 2M_\beta^{\pm}\frac{(z-y)^2}{\delta^2}, \quad (2.4)$$

Whereas

$$M_\beta^{\pm} = sup\mathcal{D}\,(f(y),0)\; for\; all\; y \in [a,b].$$

In light of the operators $\overline{\mathscr{L}}_o$ positivity and linearity, we can obtain for each $m \in N$

$$\left|\frac{1}{(1+\mu)^m}\sum_{j=0}^{m}\binom{m}{j}\mu^{m-j}\overline{\mathcal{L}}_j\left(f_\beta^{\pm};y\right)-f_\beta^{\pm}(y)\right|$$

$$\leq \frac{1}{(1+\mu)^m}\sum_{j=0}^{m}\binom{m}{j}\mu^{m-j}\overline{\mathcal{L}}_j\left(|f_\beta^{\pm}(z)\right)-f_\beta^{\pm}(y)|;y)$$

$$+M_\beta^{\pm}\left|\frac{1}{(1+\mu)^m}\sum_{j=0}^{m}\binom{m}{j}\mu^{m-j}\overline{\mathcal{L}}_j(e_0;y)-e_0(y)\right|$$

$$\leq e+(e+M_\beta^{\pm})\left|\frac{1}{(1+\mu)^m}\sum_{j=0}^{m}\binom{m}{j}\mu^{m-j}\overline{\mathcal{L}}_j(e_0;y)-e_0(y)\right|$$

$$+\frac{2M_\beta^{\pm}}{\delta^2}\left|\frac{1}{(1+\mu)^m}\sum_{j=0}^{m}\binom{m}{j}\mu^{m-j}\overline{\mathcal{L}}_j(z-y)^2:y\right|. \quad (2.5)$$

Obviously,

$$\left|\frac{1}{(1+\mu)^m}\sum_{j=0}^{m}\binom{m}{j}\mu^{m-j}\overline{\mathcal{L}}_j\left(f_\beta^{\pm};y\right)-f_\beta^{\pm}(y)\right|$$

$$\leq e+(e+M_\beta^{\pm}+\frac{2h^2M_\beta^{\pm})}{\delta^2}\left|\frac{1}{(1+\mu)^m}\sum_{j=0}^{m}\binom{m}{j}\mu^{k-v}\overline{\mathcal{L}}_j(e_0;y)-e_0(y)\right|$$

$$+\frac{4hM}{\delta^2}\left|\frac{1}{(1+\mu)^m}\sum_{j=0}^{m}\binom{m}{j}\mu^{m-j}\overline{\mathcal{L}}_j(e_1;y)-e_1(y)\right|$$

$$+\frac{2M_\beta^{\pm}}{\delta^2}\left|\frac{1}{(1+\mu)^m}\sum_{j=0}^{m}\binom{m}{j}\mu^{m-j}\overline{\mathcal{L}}_j(e_2;y)-e_2(y)\right|. \quad (2.6)$$

Whereas

$$h = max\{|a|,|b|\}.$$

Taking

$$K = \max\left\{e + \left(e + M_\beta^{\pm} + \frac{2h^2 M_\beta^{\pm}}{\delta^2}\right), \frac{4hM_\beta^{\pm}}{\delta^2}, \frac{2M_\beta^{\pm}}{\delta^2}\right\}$$

and taking supremum over all $y \in [a,b]$, the inequality (2.6) above can be expressed like this

$$||\frac{1}{(1+\mu)^m} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j(f_\beta^{\pm}; y) - f_\beta^{\pm}(y)||$$

$$\leq e + K\Big\{||\frac{1}{o+1}\sum_{i=0}^{o} \frac{1}{(1+\mu)^i} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j(e_0; y) - e_0(y)||$$

$$+||\frac{1}{(1+\mu)^m} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j(e_1; y) - e_1(y)||$$

$$+||\frac{1}{(1+\mu)^m} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j(e_2; y) - e_2(y)|| \qquad (2.7)$$

Currently, by using (2.1), we have

$$\mathcal{D}^*\left(\frac{1}{o+1}\sum_{i=0}^{o} \frac{1}{(1+\mu)^i} \sum_{j=0}^{m} \binom{m}{j} q^{m-j} \mathcal{L}_j(f), f\right)$$

$$= sup_{y\in[a,b]}\mathcal{D}\left(\frac{1}{(1+\mu)^m} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j(f; y) - f(y)\right)$$

$$= sup_{y\in[a,b]} sup_{\beta\in[a,b]} \max\{\left|\frac{1}{(1+\mu)^m}\sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j(f_\beta^-; y) - f_\beta^-(y)\right|,$$

$$\left|\frac{1}{(1+\mu)^m}\sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j\left(f_\beta^+; y\right) - f_\beta^+(y)\right|\}$$

$$= sup_{\beta\in[a,b]} \max\{||\frac{1}{(1+\mu)^m}\sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j(f_\beta^-; y) - f_\beta^-(y)||$$

$$||\frac{1}{(1+\mu)^m}\sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j\left(f_\beta^+; y\right) - f_\beta^+(y)||\} \qquad (2.8)$$

adding (2.7) and (2.8), we have

$$\mathcal{D}^*\left(\frac{1}{(1+\mu)^m} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \mathcal{L}_j(f), f\right)$$

$$\leq e + K\Big\{||\frac{1}{(1+\mu)^m} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j(e_0; y) - e_0(y)||$$

$$+||\frac{1}{o+1} \sum_{i=0}^{o} \frac{1}{(1+\mu)^m} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j(e_1; y) - e_1(y)||$$

$$+||\frac{1}{(1+\mu)^m} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j(e_2; y) - e_2(y)|| \qquad (2.9)$$

Now, given a ε'> 0, select such that $0 < \varepsilon < \varepsilon'$ and let's define the subsequent sets

$$E = \left\{o \in \mathbb{N}: \mathcal{D}^*\left(\frac{1}{(1+\mu)^m} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \mathcal{L}_j(f), f\right) \geq \varepsilon\right\},$$

$$E_1 = \left\{o \in \mathbb{N}: ||\frac{1}{(1+\mu)^m} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j(e_0; y) - e_0(y)|| \geq \frac{\varepsilon'-\varepsilon}{3K}\right\},$$

$$E_2 = \left\{o \in \mathbb{N}: ||\frac{1}{(1+\mu)^m} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j(e_1; y) - e_1(y)|| \geq \frac{\varepsilon'-\varepsilon}{3K}\right\}$$

$$E_3 = \left\{o \in \mathbb{N}: ||\frac{1}{(1+\mu)^m} \sum_{j=0}^{m} \binom{m}{j} \mu^{m-j} \bar{\mathcal{L}}_j(e_2; y) - e_2(y)|| \geq \frac{\varepsilon'-\varepsilon}{3K}\right\}$$

Obviously,

$$E \subseteq E_1 \cup E_2 \cup E_3 \qquad (2.10)$$

and this suggests that

$$\delta(E) \subseteq \delta(E_1) \cup \delta(E_2) \cup \delta(E_3). \quad (2.11)$$

The proof is now finished using the hypotheses (2.2) and (2.3).

Example 2. Assume the following for the fuzzy Bernstein-type operators $B_n^{\mathscr{F}}(f;\ y)$

$$\mathcal{B}_m^{\mathcal{F}}(f;\ y) = f_m \odot \oplus_{k=0}^{m}\binom{m}{k}x^k(1-y)^{m-k}\odot f\left(\frac{m}{k}\right),$$

The sequence of functions considered is $f_m(y)$, where $f \in C_{\mathscr{F}}[0,1], y \in [0,1], m \in \mathbb{N}$. In this instance, we compose

$$\left\{\mathcal{B}_m^{\mathcal{F}}(f;\ y)\right\}_{\pm}^{r} = \mathcal{B}_m^{*}\left(f_{\pm}^{r};\ y\right) = f_m \sum_{k=0}^{m}\binom{m}{k}y^k(1-y)^{m-k} f_{\pm}^{r}\binom{m}{k},$$

where $C[0,1]$ contains $f_{\pm}^{r}$. Furthermore, note that

$$\mathcal{B}_m^{*}(f_0, y) = 1, \mathcal{B}_m^{*}(f_1, y) = y \text{ and } \mathcal{B}_m^{*}(f_2, y) = \left(y^2 + \frac{y(1-y)}{m}\right).$$

Additionally, we view $\overline{\mathscr{L}}_m: C[0,1] \to C[0,1]]$ as the positive linear sequence of operators of the following form:

$$\bar{\mathcal{L}}_m\left(f_{\pm}^{r}; y\right) = [1 + u_m(y)]y\left(1 + y\frac{d}{dy}\right)\mathcal{B}_m^{*}\left(f_{\pm}^{r};\ y\right) \quad (f \in C[0,1]). \quad (2.12)$$

In addition, AI-Salam already employed a portion of our operator, namely $y(1 + y\frac{d}{dy})$ [3]. We refer to [33] and [36] for a few families of operators of this type. After selecting the function sequence $u_m(y)$ as in example 1, we have

$$\bar{\mathcal{L}}_m(f_0, y) = [1 + u_m(y)]y\left(1 + y\frac{d}{dy}\right)\mathcal{B}_m^{*}(f_0; y)$$

$$= [1 + u_m(y)]y\left(1 + y\frac{d}{dy}\right).1 = [1 + f_m(y)]y,$$

$$\bar{\mathcal{L}}_o(f_1, y) = [1 + u_m(y)]y\left(1 + y\frac{d}{dy}\right).\mathcal{B}_m^{*}(f_1; y)$$

$$= [1 + u_m(y)]y\left(1 + y\frac{d}{dy}\right).$$

$$= [1 + u_m(y)]y(1 + x),$$

And

$$\bar{\mathcal{L}}_o(f_2, y) = [1 + u_m(y)]y\left(1 + y\frac{d}{dy}\right)(1 + y\frac{d}{dy}).\mathcal{B}_m^{*}(f_2; y)$$

$$= [1 + u_m(y)]y\left(1 + y\frac{d}{dy}\right).\left(y^2 + \frac{y(1-y)}{y}\right)$$

$$= [1 + u_m(y)]\left\{y^2\left[3y + \frac{1-y}{m} + \frac{1}{m}(1-2y)\right]\right\}.$$

In order for us to have,

$$stat_{\mathcal{F}}lim_{o\to\infty}||\frac{1}{(1+\mu)^m}\sum_{j=0}^{m}\binom{m}{j}\mu^{m-j}\,\overline{\mathcal{L}}_j(e_{\bar{\imath}}) - e_{\bar{\imath}}|| = 0 \text{ for every } \bar{\imath}=0,1,2,$$

The sequence $\overline{\mathscr{L}}_m(f;y)$ meets the requirements, in other words (2.2). In light of this, theorem 1 gives us

$$stat_{\mathcal{F}}lim_{o\to\infty}\mathcal{D}^*\left(\frac{1}{(1+\mu)^m}\sum_{j=0}^{m}\binom{m}{j}\mu^{m-j}\,\overline{\mathcal{L}}_j(f), f\right) = 0.$$

As such, it is $(E,\mu)-$ summable statistically. But since(u_n) is not statistically convergent, we deduce that our Theorem 1 still hold true for the operator defined by (2.12). While the earlier results in [8] and [10] are invalid for the operators defined by (2.12).

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