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Fg-Closed Sets And F-Normal Spaces

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Abstract: In this paper, a new kind of sets called F-generalized closed (briefly Fg-closed) sets are introduced, which is a generalization of F-closed as well as g-closed sets and also studied some basic properties of Fg-closed sets in topological spaces. Further by utilizing Fg-closed sets, we obtained a characterization of normal spaces. Moreover, we also introduced a new class of normal spaces is called F-normal spaces in topological spaces and investigated some properties of F-normal spaces in the terms of Fg-open sets.

Keyword:- F-closed set, Fg-closed set, Fg-open set, F-normal Space, almost Fg-closed function, almost Fg-continuous function etc.

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1. Introduction

The concept of closedness is fundamental with respect to study of topological spaces. Generalized closed sets play a very important role in topology and generalized closed sets are research topics of many topologists. In 1923, Tietze **[6]** first defined the notion of normal spaces and studied their properties. In 1937, M. Stone **[5]** introduced the notion of regular open sets. In 1970, Levine **[3]** introduced the notion of generalized closed sets and studied the properties of g-closed sets in topological spaces. In 1971, Crossley and Hildebrand **[2]** defined the concept of semi open sets and investigated their properties. In 2000, A. Pushpalatha **[4]** studied the concept of w-closed sets in topological spaces and obtained some basic properties of w-closed sets. In 2023, Mesfer H. Alqahtani **[1]** introduced the concept of F-open and F-closed sets in topological spaces. They studied the main properties of these sets and examine the relationships between F-open and F-closed sets with other kinds of closed and open sets such as regular open, regular closed, π-open, π-closed and open sets etc.

2. Preliminaries

In what follows, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated and $f : (X, \Im) \rightarrow (Y, \sigma)$ (or simply $f : X \rightarrow Y$) denotes a function f of a space $(X, \Im)$ into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by cl(A) and int(A), respectively.

2.1 Definition: A subset A of a topological space $(X, \Im)$ is said to be

(i) regular open [5] if $A = \mathrm{int}(\mathrm{cl}(A))$.

(ii) semi open [2] if $(A) \subset \mathrm{cl}(\mathrm{int}(A))$.

(iii) g-closed [3] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and $U \in \Im$.

(iv) w-closed [4] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is semi-open.

The collection of all semi open (resp. g-closed, w-closed) sets is denoted by S-O(X) (resp. g-C(X), w-C(X)).

2.2 Definition: An open subset A of the topological space $(X, \Im)$ is said to be **F-open [1]** set if $\mathrm{cl}(A) - A$ is finite set. The complement of the F-open set is called F-closed. The collection of all F-open (resp. F-closed) sets is denoted by F-O(X) (resp. F-C(X)).

3. Fg-closed set

3.1 Definition: A subset A of a topological space $(X, \Im)$ is said to be **Fg-closed** if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is F-open. The complement of the Fg-closed set is called Fg-open set. The collection of all Fg-open (resp. Fg-closed) sets is denoted by Fg-O(X) (resp. Fg-C(X)).

The intersection of all Fg-closed sets containing A, is called the Fg-closure of A and is denoted by Fg-cl(A). The Fg-interior of A, denoted by Fg-int(A) is defined to be the union of all Fg-open sets contained in A.

3.2 Remark. We summarize the fundamental relationships between several types of generalized closed sets in the following diagram.

$$\begin{tikzcd}
\text{F-closed} \arrow[r] \arrow[d] & \text{closed} \arrow[r] \arrow[d] \arrow[dl] \arrow[dr] & \text{semi closed} \\
\text{w-closed} \arrow[r] & \text{g-closed} \arrow[r] & \text{Fg-closed}
\end{tikzcd}$$

Where none of the implications is reversible can be seen from the following examples

3.3 Example. Let $X = \{a, b, c\}$ and $\Im = \{\phi, X, \{a\}, \{a, b\}\}$. Then

$C(X) = F\text{-}C(X) = w\text{-}C(X) = \{\phi, X, \{c\}, \{b, c\}\}$

$s\text{-}O(X) = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$

$s\text{-}C(X) = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}$

$g\text{-}C(X) = Fg\text{-}C(X) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\}$

Here {b} is semi closed set but not closed set, Fg-closed and g-closed also, the set {a, c} is Fg-closed and g-closed set but not semi closed set and closed. Hence it is clear that neither semi closed imply Fg-closed nor Fg-closed imply semi closed sets.

3.4 Example. Let $X = \{a, b, c, d\}$ and $\Im = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Then

$C(X) = F\text{-}C(X) = w\text{-}C(X) = \{\phi, X, \{d\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$

$S\text{-}O(X) = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$

$S\text{-}C(X) = \{\phi, X, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\} \}$

$g\text{-}C(X) = Fg\text{-}C(X) = \{\phi, X, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\} \{b, c, d\}\}$

Here {a, d} is g-closed set but not w-closed set.

3.5 Example. Let $X = \{a, b\}$ and $\mathfrak{I} = \{\phi, X\}$ Then $A = \{a\}$ is w-closed set but not closed.

3.6 Example. Let $(\mathbb{R}, \mathbf{U})$ be the usual topological space. The set of natural numbers $\mathbb{N}$ is a closed set but not F-closed set of $\mathbb{R}$ with respect to usual topology **U**, as $\mathbb{N} - \text{int}(\mathbb{N}) = \mathbb{N} - \phi = \mathbb{N}$ which is an infinite set.

3.7 Example. Let $(\mathbb{R}, \mathbf{U})$ be the usual topological space, $A_n = (n, n+1)\ \forall\ n \in \mathbb{Z}$ be the open sets of set of real numbers $\mathbb{R}$ in U. Now $\text{cl}(A_n) - A_n = [n, n+1] - (n, n+1) = \{n, n+1\}$ which is a finite set, i.e. A_n is a F-open subset of $\mathbb{R}$. Now define $A = \cup_{n\in\mathbb{Z}} A_n = \mathbb{R} - \mathbb{Z}$ which is an open set as it is countable union of open set. Also $\text{cl}(A) - A = \text{cl}(\mathbb{R} - \mathbb{Z}) - (\mathbb{R} - \mathbb{Z}) = \mathbb{R} - (\mathbb{R} - \mathbb{Z}) = \mathbb{Z}$ which is an infinite set hence A is not F-open set in $(\mathbb{R}, \mathbf{U})$. Now for A, $\mathbb{R}$ is the smallest F-open set containing A, also $\text{cl}(A) = \mathbb{R} \subset \mathbb{R}$ whenever $A \subset \mathbb{R}$ and $\mathbb{R}$ is F-open set, hence A is Fg-closed set. But A is not g-closed set as $A \subset A$ and A is open set but $\text{cl}(A) = \mathbb{R}$ is not subset of A.

3.8 Theorem: Union of two Fg-closed set is Fg-closed set.

Proof: Let A and B be two Fg-closed sets. Now A is Fg-closed set if $\text{cl}(A) \subset U_1$ whenever $A \subset U_1$ and U_1 is F-open set, also B is Fg-closed set if $\text{cl}(B) \subset U_2$ whenever $B \subset U_2$ and U_2 is F-open set. Now $U_1 \cup U_2$ is F-open set as U_1 and U_2 are F-open sets, and $A \cup B \subset U_1 \cup U_2$ as $A \subset U_1$ and $B \subset U_2$. Now $\text{cl}(A) \subset U_1$ and $\text{cl}(B) \subset U_2 \Rightarrow \text{cl}(A) \cup \text{cl}(B) \subset U_1 \cup U_2 \Rightarrow \text{cl}(A \cup B) \subset U_1 \cup U_2$ because by using the result $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$. Hence $\text{cl}(A \cup B) \subset U_1 \cup U_2$ whenever $A \cup B \subset U_1 \cup U_2$ and $U_1 \cup U_2$ is F-open set. Hence $A \cup B$ is Fg-closed set.

In general finite union of Fg-closed sets is Fg-closed set.

3.9 Theorem: Intersection of two Fg-closed set is Fg-closed set.

Proof: Let A and B be two Fg-closed set. Now A is Fg-closed set if $\text{cl}(A) \subset U_1$ whenever $A \subset U_1$ and U_1 is F-open set, also B is Fg-closed set if $\text{cl}(B) \subset U_2$ whenever $B \subset U_2$ and U_2 is F-open set. Now $U_1 \cap U_2$ is F-open set as U_1 and U_2 are F-open sets, and $A \cap B \subset U_1 \cap U_2$ as $A \subset U_1$ and $B \subset U_2$. Now $\text{cl}(A) \subset U_1$ and $\text{cl}(B) \subset U_2 \Rightarrow \text{cl}(A) \cap \text{cl}(B) \subset U_1 \cap U_2 \Rightarrow \text{cl}(A \cap B) \subset U_1 \cap U_2$ because by using the result $\text{cl}(A \cap B) \subset \text{cl}(A) \cap \text{cl}(B)$. Hence $\text{cl}(A \cap B) \subset U_1 \cap U_2$ whenever $A \cap B \subset U_1 \cap U_2$ and $U_1 \cap U_2$ is F-open set. Hence $A \cap B$ is Fg-closed set.

In general finite intersection of Fg-closed sets is Fg-closed set.

3.10 Theorem: Union of two Fg-open sets is Fg-open set.

Proof: Let A and B be two Fg-open subset of a topological space $(X, \mathfrak{I})$. Then $X - A$ and $X - B$ be two closed Fg-subset of X. Hence $(X - A) \cap (X - B)$ is Fg-closed subset of X by **Theorem 3.9**. Now $(X - A) \cap (X - B) = X - (A \cup B)$ be Fg-closed set $\Rightarrow A \cup B$ is Fg-open set. Hence union of two Fg-open sets is Fg-open set.

In general finite union of Fg-open sets is Fg-open set.

3.11 Theorem: Intersection of two Fg-open sets is Fg-open set.

Proof: Let A and B be two Fg-open subset of a topological space $(X, \mathfrak{I})$. Then $X - A$ and $X - B$ be two Fg-closed subsets of X. Hence $(X - A) \cup (X - B)$ be the Fg-closed subset of X by **Theorem 3.8**. Now $(X - A) \cup$

$(X - B) = X - (A \cap B)$ be Fg-closed set $\Rightarrow A \cap B$ is Fg-open set. Hence intersection of two Fg-open sets is Fg-open set.

In general finite intersection of Fg-open sets is Fg-open set.

3.12 Remark: Arbitrary union of Fg-closed sets is may not be Fg-closed set.

3.13 Example: Let $A_n = [1/n, n/n+1]$ $\forall$ $n \in \{2, 3, 4\ldots..\}$ be the closed set, hence Fg-closed subsets in the usual topological space ($\mathbb{R}$, **U**). Now let A be the countable union of A_n, i.e. $A = A_2 \cup A_3 \cup A_4 \cup \ldots. = (0, 1)$ which is not Fg-closed set as $A \subset (0, 1)$ and $(0, 1)$ is F-open set but $cl(A)= [0, 1]$ is not subset of $(0, 1)$. Hence arbitrary union of Fg-closed sets is may not be Fg-closed set.

3.14 Remark: Arbitrary intersection of Fg-open sets is may not be Fg-open set.

3.15 Example: Let $A_n = (-1/n, 1/n)$ $\forall$ $n \in \mathbb{N}$ be the open set, hence Fg-open sets in the usual topological space ($\mathbb{R}$, **U**). Now let A be the countable intersection of A_n, i. e. $A = \cap_{n\in\mathbb{N}} A_n = A_1 \cap A_2 \cap A_3 \cap A_4 \cap \ldots .= \{0\}$ which is not a Fg-open set as $\mathbb{R} - \{0\}$ is not Fg-closed set because $\mathbb{R} - \{0\} \subset \mathbb{R} - \{0\}$ and $\mathbb{R} - \{0\}$ is a F-open set but $cl(\mathbb{R} - \{0\})= \mathbb{R}$ is not subset of $\mathbb{R} - \{0\}$. Hence arbitrary intersection of Fg-open sets is may not be Fg-open set.

4. F-NORMAL SPACES

4.1 Definition: A space X is said to be **F-normal** (resp. **normal** [**10**]) if for every pair of disjoint F-closed (resp. closed) sets A and B in X, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

4.2 Remark: Every normal space is F-normal but not conversely.

4.3 Theorem: For a topological space X, the following properties are equivalent:
(1) X is F-normal;
(2) for any disjoint $H, K \in F\text{-}C(X)$, there exist disjoint Fg-open sets U, V such that $H \subset U$ and $K \subset V$;
(3) for any $H \in F\text{-}C(X)$ and any $V \in F\text{-}O(X)$ containing H, there exists a Fg-open set U of X such that $H \subset U \subset Fg\text{-}cl(U) \subset V$;
(4) for any $H \in F\text{-}C(X)$ and any $V \in F\text{-}O(X)$ containing H, there exists an open set U of X such that $H \subset U \subset cl(U) \subset V$;
(5) for any disjoint $H, K \in F\text{-}C(X)$, there exist disjoint regular open sets U, V such that $H \subset U$ and $K \subset V$.
Proof: (1) ⇒ (2): Since every open set is Fg-open, the proof is obvious.

(2) ⇒ (3): Let $H \in F\text{-}C(X)$ and V be any F-open set containing H. Then $H, X - V \in F\text{-}C(X)$ and $H \cap (X - V) = \phi$. By (2), there exist Fg-open sets U, G such that $H \subset U$, $X - V \subset G$ and $U \cap G = \phi$. Therefore, we have $H \subset U \subset (X - G) \subset V$. Since U is Fg-open and $X - G$ is Fg-closed, we obtain $H \subset U \subset Fg\text{-}cl(U) \subset (X - G) \subset V$.

(3) ⇒ (4): Let $H \in F\text{-}C(X)$ and $H \subset V \in F\text{-}O(X)$. By (3), there exists a Fg-open set U_0 of X such that $H \subset U_0 \subset Fg\text{-}cl(U_0) \subset V$. Since $Fg\text{-}cl(U_0)$ is Fg-closed and $V \in F\text{-}O(X)$, $cl(Fg\text{-}cl(U_0)) \subset V$. Put $int(U_0) = U$, then U is open and $H \subset U \subset cl(U) \subset V$.

(4) ⇒ (5): Let H, K be disjoint F-closed sets of X. Then $H \subset (X - K) \in FO(X)$ and by (4) there exists an open set U_0 such that $H \subset U_0 \subset cl(U_0) \subset (X - K)$. Therefore, $V_0 = (X - cl(U_0))$ is an open set such that $H \subset U_0$, $K \subset V_0$ and $U_0 \cap V_0 = \phi$. Moreover, put $U = int(cl(U_0))$ and $V = int\ (cl(V_0))$, then U, V are regular open sets such that $H \subset U$, $K \subset V$ and $U \cap V = \phi$.

(5) ⇒ (1): This is obvious.

By using Fg-open sets, we obtain a characterization of normal spaces.

4.4 Theorem: For a topological space X, the following properties are equivalent:

(1) X is normal;
(2) for any disjoint closed sets A and B, there exist disjoint Fg-open sets U and V such that $A \subset U$ and $B \subset V$;
(3) for any closed set A and any open set V containing A, there exists a Fg-open set U of X such that $A \subset U \subset cl(U) \subset V$.

Proof: (1) ⇒ (2): This is obvious since every open set is Fg-open.

(2) ⇒ (3): Let A be a closed set and V be any open set containing A. Then A and (X – V) are disjoint closed sets. There exist disjoint Fg-open sets U and W such that $A \subset U$ and $(X - V) \subset W$. Since X – V is closed, we have $(X - V) \subset int(W)$ and $U \cap int(W) = \phi$. Therefore, we obtain $cl(U) \cap int(W) = \phi$ and hence $A \subset U \subset cl(U) \subset (X - int(W)) \subset V$.

(3) ⇒ (1): Let A, B be disjoint closed sets of X. Then $A \subset (X - B)$ and (X – B) is open. By (3), there exists a Fg-open set G of X such that $A \subset G \subset cl(G) \subset (X - B)$. Since A is closed, we have $A \subset int(G)$. Put $U = int(G)$ and $V = (X - cl(G))$. Then U and V are disjoint open sets of X such that $A \subset U$ and $B \subset V$. Therefore, X is normal.

4.5 Theorem: Let X be a F-normal space. Then a semi-regular subspace Y of X is also F-normal.

Proof: Let X be a F-normal space and Y be a semi-regular subspace of X. Let $A \in F\text{-}C(Y)$ and $B \in F\text{-}O(Y)$ containing A. Since Y is semi regular, so $A \in F\text{-}C(X)$ and $B \in F\text{-}O(X)$. Hence by **Theorem 4.3(4)**, there exists an open set U in X such that $A \subset U \subset cl_X(U) \subset B$. This gives $A \subset (U \cap Y) \subset cl_Y(U \cap Y) \subset B$, where $U \cap Y$ is open in Y and hence Y is F-normal.

5. FUNCTIONS AND F-NORMAL SPACES

5.1 Definition: A function $f : X \to Y$ is said to be:
(1) almost Fg-continuous if for any regular open set V of Y, $f^{-1}(V) \in Fg\text{-}O(X)$;
(2) almost Fg-closed if for any regular closed set F of X, $f(F) \in Fg\text{-}C(Y)$.

5.2 Definition: A function $f : X\ Y \to$ is said to be:
(1) **F-irresolute (resp. F-continuous)** if for any F-open (resp. open) set V of Y, $f^{-1}(V)$ is F-open in X;
(2) pre-F-closed (resp. F-closed [1]) if for any F-closed (resp. closed) set F of X, f(F) is F-closed in Y.

5.3 Theorem: A function $f : X \to Y$ is an almost Fg-closed surjection if and only if for each subset S of Y and each regular open set U containing $f^{-1}(S)$, there exists a Fg-open set V such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof: Necessity. Suppose that f is almost Fg-closed. Let S be a subset of Y and U be a regular open set of X containing $f^{-1}(S)$. Put $V = Y – f(X – U)$, then V is a Fg-open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency: Let F be any regular closed set of X. Then $f^{-1}(Y - f(F)) \subset (X – F)$ and $X - F$ is regular open. There exists a Fg-open set V of Y such that $(Y - f(F)) \subset V$ and $f^{-1}(V) \subset (X – F)$. Therefore, we have $f(F) \supset (Y – V)$ and $F \subset f^{-1}(Y – V)$. Hence, we obtain $f(F) = Y - V$ and f(F) is Fg-closed in Y. This shows that f is almost Fg-closed.

5.4 Theorem: If $f : X \to Y$ is an almost Fg-closed F-irresolute (resp. F-continuous) surjection and X is F-normal, then Y is F-normal (resp. normal).

Proof: Let A and B be any disjoint F-closed (resp. closed) sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint F-closed sets of X. Since X is F-normal, there exist disjoint open sets U and V of X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Put G = int(cl(U)) and H = int(cl(V)), then G and H are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. By **Theorem 5.3**, there exist Fg-open sets K and L of Y such that $A \subset K$, $B \subset L$. $f^{-1}(K) \subset G$ and $f^{-1}(L) \subset H$. Since G and H are disjoint, so K and L are also disjoint. It follows from **Theorem 4.3** (resp. **Theorem 4.4**) that Y is F- normal (resp. normal).

5.5 Theorem: If $f : X \to Y$ is a continuous almost Fg-closed surjection and X is a normal space, then Y is normal.

Proof: The proof is similar to that of **Theorem 5.4.**

5.6 Theorem: If $f : X \to Y$ is an almost Fg-continuous pre-F-closed (resp. F-closed) injection and Y is F-normal, then X is F-normal (resp. normal).

Proof: Let H and K be disjoint F-closed (resp. closed) sets of X. Since f is a pre-F-closed (resp. F-closed) injection, f(H) and f(K) are disjoint F-closed sets of Y. Since Y is F-normal, there exist disjoint open sets P and Q such that $f(H) \subset P$ and $f(K) \subset Q$. Now, put U = int(cl(P)) and V = int(cl(Q)), then U and V are disjoint regular open sets such that $f(H) \subset U$ and $f(K) \subset V$. Since f is almost Fg-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint Fg-open sets such that $H \subset f^{-1}(U)$ and $K \subset f^{-1}(V)$. It follows from **Theorem 4.3** (resp. **Theorem 4.4**) that X is F-normal (resp. normal).

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