Total Chromatic Number Comb Product of Tadpole Graph

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Abstract:
The total chromatic number of a graph $G$ is defined to be the minimum number of colors needed to color the vertices and edges of a graph in such a way that no two adjacent vertices, no two adjacent edges and no each edge and end vertices are given the same color. In this paper, we have obtained the total coloring and total chromatic number of comb product of tadpole graph with path, star, fan, and cycle.

Keywords: tadpole graph, path, star, cycle, fan, comb product, total chromatic number.

1. INTRODUCTION
Chromatic number of graphs is a special area in a Graph theory. (Behzad 1987) introduced the concept of total coloring and found the chromatic number. Coloring a graph $G$ involves assigning colors to all of its vertices and edges such that no two adjacent vertices or edges have the same color, and each edge and end vertex is assigned a unique color. The total chromatic number of a graph $G$ is the minimum number of colors required to produce a total coloring, denoted by $\chi(G)$ (Behzad 1987) conjectured that for any graph with a maximum degree $\Delta(G)$, the total chromatic number satisfies the condition $\Delta(G) + 1 \leq \chi tc(G) \leq \Delta(G) + 2$ (Sudha. et al.2017) have discussed the total colouring and total chromatic number of the central graph of a path a cycle, and a star. (Muthuramakrishnan eat al.2018) have discussed the total coloring of middle graph, total graph of path and sun let graph. Also, they have obtained the total chromatic number of those graph. Referring definition of comb graph and comb product by (Rohmatulloh et al 2021), (Suhadi Wido Saputro et al.). Basic definition of star, cycle, path and fan by (J.A.Bondy et al) In this paper, we investigate the total chromatic number of the comb product of various graphs including the tadpole graph with the path graph, star graph, fan graph, and cycle graph.

DEFINITIONS:
2.1 PATH
A path is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence and are nonadjacent otherwise.

2.2 CYCLE
A cycle graph or circular graph is a path $P_{n+1}$ ($n>3$, if the graph is simple) whose end vertices are joined to form a closed chain. The cycle graph with $n$ vertices is denoted by $C_n$. 
2.3 STAR GRAPH
A Star graph is a complete bipartite graph $K_{1,n}$ for $n \geq 1$.

2.4 FAN GRAPH
The fan graph $F_n$ is the join of $P_n$ and $K_1$.

2.5 COMB GRAPH
A comb graph is a graph obtained by joining a single pendant edge to each vertex of a path.

2.6 TADPOLE GRAPH
The $(m,n)$-tadpole graph is a special type of graph consisting of a cycle graph on $m$ (at least 3) vertices and a path graph on $n$ vertices, connected with a bridge. It is denoted by $T_{m,n}$.

2.7 COMB PRODUCT:
Let $G$ and $H$ be two connected graphs and $o$ be a vertex of $H$. The comb product between $G$ and $H$ denoted by $G \bowtie H$ is a graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$ and grafting the vertex $o$ of $i$-th copy of $H$ with the $i$-th vertex of $G$.

3. MAIN RESULTS:

THEOREM: 3.1. The total chromatic number of the comb product of a tadpole graph with a path graph is $\chi_t(T_{m,n} \bowtie P_n) = 6$ for $n \geq 2$.

Proof: Let $\{u_i; 1 \leq i \leq m\} \cup \{v_j; 1 \leq j \leq n\}$ be the vertices and $\{u_iu_{i+1}; 1 \leq i \leq m-1\}$ $\cup \{v_jv_{j+1}; 1 \leq j \leq n-1\}$ be the edges of tadpole graph $T_{m,n}$.

Let $\{w_i; 1 \leq i \leq n\}$ be the vertices and $\{w_iw_{i+1}; 1 \leq i \leq n-1\}$ be the edges of the path graph.

By definition of comb product of Tadpole graph with path graph, the vertex of path is recognized with each vertex of the tadpole graph $T_{m,n}$.

Let the vertex set and the edge set of $(T_{m,n} \bowtie P_n)$ as follows.

$$V(T_{m,n} \bowtie P_n) = \{u_i; 1 \leq i \leq m\} \cup \{v_j; 1 \leq j \leq n\} \cup \{u_iu_{i+1}; 1 \leq i \leq m-1\} \cup$$

$$\{y_{jk}; 1 \leq j \leq n, 1 \leq k \leq n-1\}$$

$$E(T_{m,n} \bowtie P_n) = \{u_iu_{i+1}; 1 \leq i \leq m-1\} \cup \{v_jv_{j+1}; 1 \leq j \leq n-1\} \cup \{u_iu_1; 1 \leq i \leq m\} \cup \{v_jv_1; 1 \leq j \leq n\} \cup$$

$$\{v_iw_{i+1}; 1 \leq i \leq m, 1 \leq i \leq m-1\} \cup \{y_{jk}y_{jk+1}; 1 \leq j \leq n, 1 \leq k \leq n-2\}$$

The number of vertices and edges of the comb product of a tadpole graph with a path graph is $(m+n)n$.

Denote the vertices in the cycle as $u_1, u_2, u_3, \ldots, u_m$ and in the path as $v_1, v_2, v_3, \ldots, v_n$ and here the bridge is $u_mv_1$. The vertices pendant paths in the comb product graph $T_{m,n} \bowtie P_n$ are denoted as $u_{i1}, u_{i2}, \ldots, u_{im-1}$ for $i = 1, 2, \ldots, m$ and $v_{j1}, v_{j2}, v_{j3}, \ldots, v_{jn-1}$. For $j = 1, 2, \ldots, n$.

$\deg(u_i) = \deg(v_j) = 3$, for $i = 1, 2, \ldots, m-1$, $j = 1, 2, \ldots, n-1$.

$\deg(u_m) = 4, \deg(v_n) = 2$. 


\[
\text{deg}(u_{ik}) = 2 = \text{deg}(v_{jk}) \text{ for } i = 1, 2, \ldots, m, k = 1, 2, \ldots, n - 2
\]
\[
j = 1, 2, \ldots, n, k = 1, 2, \ldots, n - 2
\]
\[
\text{deg}(u_{in}) = 1 = \text{deg}(v_{jn}) \text{ for } i = 1, 2, \ldots, m, j = 1, 2, \ldots, n
\]

\(u_m\) is the only vertex having \(\text{deg} \Delta (G)\)

In the comb product of tadpole graph with path graph, degree of \(u_m\) is 4. The vertices connected to the vertex \(u_m\) are \(v_1, u_{m-1}, u_1\), and \(u_{m1}\). These four vertices are non-adjacent vertices to each other. If assigned one color to \(u_m\) vertex and only one color to other 4 vertices, so \(u_m\) and its adjacent vertices can be colored by 2 different colors. The associated of 4 edges with the vertex \(u_m\) also required the 4 colors. Because they are connected to each other, and hence totally 6 colors required to color for this graph.

Now consider the set of colors \(C = \{1, 2, 3, 4, 5, 6\}\) coloring

Let \(S = V (T_{m,n} \rightarrow P_n) \cup E (T_{m,n} \rightarrow P_n)\)

<table>
<thead>
<tr>
<th>If (m) is even</th>
<th>(f(u_i) = \begin{cases} 1 &amp; \text{if } i \equiv 1 \text{ (mod 2)} \ 2 &amp; \text{if } i \equiv 2 \text{ (mod 2)} \end{cases}) for (1 \leq i \leq m)</th>
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<td>For (1 \leq i \leq m, 1 \leq k \leq n-1)</td>
<td>(f(u_{ik}) = \begin{cases} 1 &amp; \text{if } i \text{ is even}, k \text{ is odd and } i \text{ is odd, } k \text{ is even} \ 2 &amp; \text{if } i \text{ is even}, k \text{ is even and } i \text{ is odd, } k \text{ is odd} \end{cases})</td>
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<tr>
<td>(f(v_{jk}) = \begin{cases} 1 &amp; \text{if } j \text{ is odd}, k \text{ is even and } j \text{ is even, } k \text{ is odd} \ 2 &amp; \text{if } j \text{ is odd}, k \text{ is odd and } j \text{ is even, } k \text{ is even} \end{cases}) for (1 \leq j \leq n, 1 \leq k \leq n-1)</td>
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<tr>
<td>For (1 \leq i \leq m-1)</td>
<td>(f(u_i u_{i+1}) = \begin{cases} 3 &amp; \text{if } i \text{ is odd} \ 4 &amp; \text{if } i \text{ is even} \end{cases})</td>
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<tr>
<td>(f(u_m u_1) = {3})</td>
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<tr>
<td>(f(u_m v_1) = {6})</td>
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If \(m\) is even

For \(1 \leq i \leq m-1\) \(f(u_i u_{i+1}) = \begin{cases} 3 & \text{if } i \text{ is odd} \\ 4 & \text{if } i \text{ is even} \end{cases}\) \(f(u_m u_1) = \{2\}, f(u_m v_1) = \{6\}\)

If \(m\) is odd

For \(1 \leq j \leq n-1\) \(f(v_j v_{j+1}) = \begin{cases} 3 & \text{if } j \text{ is odd} \\ 4 & \text{if } j \text{ is even} \end{cases}\)

\(F\) or \(1 \leq i \leq m\) \(f(u_i u_{i+1}) = \{5\}\)

For \(1 \leq j \leq n\) \(f(v_j v_{j+1}) = \{5\}\)

For \(1 \leq i \leq m, 1 \leq k \leq n-2\) \(f(u_{ik} u_{ik+1}) = \begin{cases} 5 & \text{if } i \text{ is odd, } k \text{ is odd and } i \text{ is even, } k \text{ is odd} \\ 6 & \text{if } i \text{ is odd, } k \text{ is even and } i \text{ is even, } k \text{ is even} \end{cases}\)

For \(1 \leq j \leq n, 1 \leq k \leq n-2\) \(f(v_{jk} v_{jk+1}) = \begin{cases} 5 & \text{if } j \text{ is odd, } k \text{ is odd and } j \text{ is even, } k \text{ is odd} \\ 6 & \text{if } j \text{ is odd, } k \text{ is even and } j \text{ is even, } k \text{ is even} \end{cases}\)
Hence $\chi_{tc}(T_{m,n} \triangleright P_n) = 6$ for $n \geq 2$.

**Example: 1** Consider the total chromatic number of the comb product of the tadpole with a pathgraph is 6.
If $m$ is even and $n$ is odd

**Example: 2** consider the total chromatic number of the comb product of the tadpole with a pathgraph is 6.
If $m$ is odd and $n$ is even.

**THEOREM: 3.2**
The total chromatic number of Comb Product of Tadpole graph with fan graph $F_n$ is $\chi_{tc}(T_{m,n} \triangleright F_n) = n + 5$ for $n \geq 2$.

**Proof:**
Let \( \{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\} \) be the vertices and \( \{u_i u_{i+1}: 1 \leq i \leq m-1\} \cup \{v_j v_{j+1}: 1 \leq j \leq n-1\} \cup \{u_m u_1\} \cup \{u_m v_1\} \) be an edges of Tadpole graph \( T_{m,n} \).

Let \( \{w_i: 1 \leq i \leq n\} \cup \{x\} \) be the vertices and \( \{w_i w_{i+1}: 1 \leq i \leq n-1\} \cup \{x w_i: 1 \leq i \leq n\} \) be the edges of fan graph \( F_n \). By definition the comb product of a tadpole graph with fan graph

The vertex \( x \), of a star \( F_n \) is identified with each vertex of the tadpole graph \( T_{m,n} \)

The vertex set and the edge set of \( (T_{m,n} \setminus F_n) \) as follows

\[
V(T_{m,n} \setminus F_n) = \{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\} \cup \{v_{jk}: 1 \leq j \leq n, 1 \leq k \leq n-1\} \cup \{u_{ik}: 1 \leq i \leq m, 1 \leq k \leq n-1\}
\]

\[
E(T_{m,n} \setminus F_n) = \{u_i u_{i+1}: 1 \leq i \leq m-1\} \cup \{v_j v_{j+1}: 1 \leq j \leq n-1\} \cup \{u_m u_1\} \cup \{u_m v_1\}
\]

The number of vertices and edges of the comb product of tadpole graph with fan graph is \( (m+n) (n+1) \) and \( (m+n)(2n) \). Denote the vertices in the cycle as \( u_1, u_2, u_3, \ldots, u_m \) and in the path as \( v_1, v_2, v_3, \ldots, v_n \) and here the bridge is \( u_m v_1 \). The vertices cycles the comb product graph \((T_{m,n} \setminus F_n)\) are denoted as \( u_{i1}, u_{i2}, \ldots, u_{in-1} \) and

For \( i = 1,2, \ldots, m \) and \( j = 1,2, \ldots, n \),

\[
deg(u_i) = \deg(v_j) = n+2,
\]

\[
\deg(u_m) = n+3, \deg(v_n) = n+1
\]

\[
\deg(u_{ik}) = 2 = \deg(v_{jk}), \quad \text{for} \quad i = 1,2, \ldots, m, j = 1,2, \ldots, n
\]

\[
\deg(u_{in}) = 2 = \deg(v_{jn}), \quad \text{for} \quad i = 1,2, \ldots, m, j = 1,2, \ldots, n
\]

The vertex \( u_m \) is the only vertex having \( \Delta(G) \) as its \( \deg(u_m) = n+3 \).

In the comb product of tadpole graph with star graph, degree of \( u_m \) is \( n+3 \). The vertices connected to the vertex \( u_m \) with \( v_1, u_{m-1}, u_1, u_{m1}, u_{m2}, u_{m3}, \ldots, u_{mn} \). These \( n+3 \) vertices are non-adjacent vertices to each other. If assigned one color to \( u_m \) vertex and only one color to other vertices \( u_{m-1}, u_1 \) and \( v_1 \), so \( u_m \) and its adjacent vertices can be colored by 2 different colors. But the \( n \) vertices going from the \( u_m \) and the associated edges are requires \( n+3 \) color.

Hence totally \( n+5 \) colors required to color this graph.

Now consider the set of colors \( C = \{1,2,3, \ldots, n+5\} \) coloring.

Let \( S = V(T_{m,n} \setminus F_n) \cup E(T_{m,n} \setminus F_n) \) and \( C = n+5 \)

Case: 1 If \( m \) is an odd and \( n = 1,2,3, \ldots \)
For 1 ≤ i ≤ m - 1
\[ f(u_i) = \begin{cases} n + 5 & \text{if } i \text{ is odd} \\ n + 4 & \text{if } i \text{ is even} \end{cases} \]
\[ f(u_m) = \{n+3\} \]

For 1 ≤ j ≤ n
\[ f(v_j) = \begin{cases} n + 5 & \text{if } j \text{ is odd} \\ n + 4 & \text{if } j \text{ is even} \end{cases} \]

For 1 ≤ i ≤ m, 1 ≤ k ≤ n
\[ f(u_{ik}) = k + 1 \]

For 1 ≤ j ≤ n, 1 ≤ k ≤ n
\[ f(v_{jk}) = k + 1 \]

For 1 ≤ i ≤ m - 1
\[ f(u_{i+1}) = \begin{cases} n + 3 & \text{if } i \text{ is odd} \\ n + 2 & \text{if } i \text{ is even} \end{cases} \]
\[ f(u_m u_1) = n + 4 \]
\[ f(u_m v_1) = n + 1 \]

For 1 ≤ j ≤ n - 1
\[ f(v_{j+1}) = \begin{cases} n + 3 & \text{if } j \text{ is odd} \\ n + 2 & \text{if } j \text{ is even} \end{cases} \]

Case: 2 If m is an even and n = 1, 2, 3,...............

For 1 ≤ i ≤ m
\[ f(u_i) = \begin{cases} n + 5 & \text{if } i \text{ is odd} \\ n + 4 & \text{if } i \text{ is even} \end{cases} \]

For 1 ≤ j ≤ n
\[ f(v_j) = \begin{cases} n + 5 & \text{if } j \text{ is odd} \\ n + 4 & \text{if } j \text{ is even} \end{cases} \]

For 1 ≤ i ≤ m, 1 ≤ k ≤ n
\[ f(u_{ik}) = k + 1 \]

For 1 ≤ j ≤ n, 1 ≤ k ≤ n
\[ f(v_{jk}) = k + 1 \]

For 1 ≤ i ≤ m - 1
\[ f(u_{i+1}) = \begin{cases} n + 3 & \text{if } i \text{ is odd} \\ n + 2 & \text{if } i \text{ is even} \end{cases} \]

For 1 ≤ j ≤ n - 1
\[ f(v_{j+1}) = \begin{cases} n + 2 & \text{if } j \text{ is odd} \\ n + 3 & \text{if } j \text{ is even} \end{cases} \]

For 1 ≤ i ≤ m, 1 ≤ k ≤ n
\[ f(u_{ik}) = k \]

For 1 ≤ j ≤ n, 1 ≤ k ≤ n
\[ f(v_{jk}) = k \]

For 1 ≤ i ≤ m, 1 ≤ k ≤ n - 1
\[ f(u_{ik} u_{i+1}) = n + k \]

For 1 ≤ i ≤ n, 1 ≤ k ≤ n - 1
\[ f(v_{jk} v_{jk+1}) = n + k \]

Hence \( \chi_c(T_{m,n} \triangleright F_n) = n + 5 \) for \( n \geq 2 \).
Example 3: Consider the total chromatic number of the comb product of the tadpole graph with the fan graph is $n+5$. Hence $\chi_{tc}(T_7 \triangleright F_6) = 11$

**THEOREM 3.3**

The total chromatic number of the comb product of the Tadpole graph with the Star graph is $(T_{m,n} \triangleright S_n) = n+4$ for $n \geq 2$

**Proof**

Let $\{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\}$ be the vertices and $\{u_i u_{i+1}: 1 \leq i \leq m-1\} \cup \{v_j v_{j+1}: 1 \leq j \leq n-1\} \cup \{u_m u_1\} \cup \{u_m v_1\}$ be an edges of tadpole graph $T_{m,n}$.

Let $\{w_i: 1 \leq i \leq n\} \cup \{x\}$ be the vertices and $\{xw_i: 1 \leq i \leq n\}$ be the edges of star graph $S_n$. By the definition of the comb product of a tadpole graph with a star graph, the vertex $x$, of star $S_n$ is identified with each vertex of the tadpole graph $T_{m,n}$, the vertex set and the edge set of $(T_{m,n} \triangleright S_n)$ as follows

$V(T_{m,n} \triangleright S_n) = \{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\} \cup \{u_{ik}: 1 \leq i \leq m, 1 \leq k \leq n-1\}$

$E(T_{m,n} \triangleright S_n) = \{u_i u_{i+1}: 1 \leq i \leq m-1\} \cup \{v_j v_{j+1}: 1 \leq j \leq n-1\} \cup \{u_m u_1\} \cup \{u_m v_1\}$

$\cup \{u_{ik}: 1 \leq i \leq m, 1 \leq k \leq n\} \cup \{v_{jk}: 1 \leq j \leq n, 1 \leq k \leq n-1\}$

The number of vertices and edges of the comb product of tadpole graph with star graph is $(m+n)(n+1)$ and $(m+n)(n+1)$. Denote the vertices in the cycle, as $u_1, u_2, u_3, \ldots, u_m$ and in the path as $v_1, v_2, v_3, \ldots, v_n$ and here the bridge is $u_m v_1$. The vertices of comb product of $(T_{m,n} \triangleright S_n)$ are denoted as $u_{i_1}, u_{i_2}, \ldots, u_{i_{m-1}}$ and

For
\[ i = 1, 2, \ldots, m \text{ and } v_{j1}, v_{j2}, v_{j3}, \ldots, v_{jn-1}. \]
\[ \deg (u_i) = n + 2 = \deg (v_j) i = 1, 2, \ldots, m-1, j = 1, 2, \ldots, n-1 \]
\[ \deg (u_m) = n + 3, \deg (v_n) = n + 1, \]
\[ \deg (u_{ik}) = 1 = \deg (v_{jk}) = \text{For } i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \]
\[ k = 1, 2, \ldots, n \text{ is } u_m \text{ the only vertex having } \Delta (G) \text{ as its } \deg (u_m) = n+3. \]

In the comb product of tadpole graph with star graph, degree of \( u_m \) is \( n + 3, n + 3 \) edges emerge from the point \( u_m \). They are \( u_mu_1, u_mv_1, u_mu_m1, u_mu_{m2}, u_mu_{m3}, \ldots, u_mu_{mn} \) and \( u_mu_{m-1} \). These \( n+3 \) edges are related to each other. So point \( u_m \) should be given one color and these \( n+3 \) edges should be given a separate color.

Hence totally \( n + 4 \) colors require to color for this graph.

Now consider the set of colors \( C = \{1, 2, 3, \ldots, n+4\} \) coloring

Let \( S = V(T_{m,n} \cup S_n) \cup E(T_{m,n} \cup S_n) \) and \( C = n+4 \)

**Case:1 If \( m \) is an odd and \( n = 1, 2, 3, \ldots \)**

\[ For 1 \leq i \leq m - 1 \quad \begin{cases} n + 4 & \text{if } i \text{ is odd} \\ n + 3 & \text{if } i \text{ is even} \end{cases} \]
\[ f(u_i) = \begin{cases} n + 1 & \text{if } i \text{ is even} \end{cases} \]
\[ f(u_m) = \begin{cases} n + 1 & \text{if } i \text{ is even} \end{cases} \]
\[ f(v_j) = \begin{cases} n + 2 & \text{if } j \text{ is odd} \\ n + 1 & \text{if } j \text{ is even} \end{cases} \]
\[ 1 \leq j \leq n \]
\[ f(u_{ik}) = n + 2 \]
\[ f(v_{jk}) = n + 4 \]
\[ 1 \leq i \leq m - 1 \]
\[ f(u_iu_{i+1}) = \begin{cases} n + 1 & \text{if } i \text{ is odd} \\ n + 2 & \text{if } i \text{ is even} \end{cases} \]
\[ f(u_{m}u_1) = n + 3 \]
\[ f(u_{m}v_1) = n + 4 \]
\[ 1 \leq j \leq n - 1 \]
\[ f(v_jv_{j+1}) = \begin{cases} n + 3 & \text{if } j \text{ is odd} \\ n + 4 & \text{if } j \text{ is even} \end{cases} \]
\[ 1 \leq i \leq m, 1 \leq k \leq n \]
\[ f(u_{ik}) = k \]
\[ f(v_{jk}) = k \]

**Case:2 If \( m \) is an even and \( n = 1, 2, 3, \ldots \)**

\[ For 1 \leq i \leq m \quad \begin{cases} n + 4 & \text{if } i \text{ is odd} \\ n + 3 & \text{if } i \text{ is even} \end{cases} \]
\[ f(u_i) = \begin{cases} n + 1 & \text{if } i \text{ is even} \end{cases} \]
\[ 1 \leq j \leq n \]
\[ f(v_j) = \begin{cases} n + 1 & \text{if } j \text{ is odd} \\ n + 2 & \text{if } j \text{ is even} \end{cases} \]
\[ 1 \leq j \leq n \]
\[ f(u_{ik}) = n + 1 \]
\[ f(v_{jk}) = n + 4 \]
\[ 1 \leq i \leq m - 1 \]
\[ f(u_iu_{i+1}) = \begin{cases} n + 1 & \text{if } i \text{ is odd} \\ n + 2 & \text{if } i \text{ is even} \end{cases} \]
\[ f(u_{m}u_1) = n + 2 \]
\[ f(u_{m}v_1) = n + 4 \]
\[ 1 \leq j \leq n - 1 \]
\[ f(v_jv_{j+1}) = \begin{cases} n + 3 & \text{if } j \text{ is odd} \\ n + 4 & \text{if } j \text{ is even} \end{cases} \]
For \(1 \leq i \leq m, 1 \leq k \leq n\) \(f(u_iu_{ik}) = k\),
For \(1 \leq j \leq n, 1 \leq k \leq n\) \(f(v_jv_{jk}) = k\)

The above defined function \(f\) gives the total coloring for the graph.

Hence \(\chi_{\text{tc}}(T_{m,n} \triangleright S_n) = n + 4\) for \(n \geq 2\).

**Example 4:** Consider the Total chromatic number of comb product of tadpole graph with star graph is \(n+4\).

If \(m\) is even, \(n\) is odd

\[\chi_{\text{tc}}(T_6,5 \triangleright S_5) = n + 4 = 9\]

**Example 5:**
Consider the total chromatic number of the comb product of the tadpole graph with the star graph is \(n+4\).
If \(m\) is even, \(n\) is odd

\[\chi_{\text{tc}}(T_7,6 \triangleright S_6) = n + 4 = 10\]

**THEOREM: 4**
The total chromatic number of Comb Product of the Tadpole graph with the cycle graph \((T_{m,n} \triangleright C_n)\) is 7 for \(n \geq 2\).

**Proof:**
Let \(\{u_i; 1 \leq i \leq m\} \cup \{v_j; 1 \leq j \leq n\}\) be the vertices and \(\{u_iu_{i+1}; 1 \leq i \leq m - 1\} \cup \)
\[ \{ v_jv_{j+1} : 1 \leq j \leq n - 1 \} \cup \{ u_mu_1 \} \cup \{ u_mv_1 \} \text{ be an edges of tadpole graph } T_{m,n}. \]

Let \( \{ w_i : 1 \leq i \leq n \} \) be the vertices and \( \{ w_dw_{i+1} : 1 \leq i \leq n \} \) be the edges of cycle graph \( C_n. \)

By the definition of the comb product of a tadpole graph with a cycle graph

The vertex \( w_1 \) of cycle \( C_n \) is identified with each vertex of the tadpole graph \( T_{m,n} \) vertex set and the edge set of \( (T_{m,n} \triangleright C_n) \) as follows

\[
V(T_{m,n} \triangleright C_n) = \{ u_i : 1 \leq i \leq m \} \cup \{ v_j : 1 \leq j \leq n \} \cup \{ u_{ik} : 1 \leq i \leq m, 1 \leq k \leq n - 1 \}
\]

\[
E(T_{m,n} \triangleright C_n) = \{ u_mu_1 \} \cup \{ u_mv_1 \}
\]

\[
\{ u_iu_{i+1} : 1 \leq i \leq m - 1 \} \cup \{ v_jv_{j+1} : 1 \leq j \leq n - 1 \} \cup \{ v_jv_{j+1} : 1 \leq j \leq n \}
\]

\[
\{ u_iu_1 : 1 \leq i \leq m \} \cup \{ u_iu_{i-1} : 1 \leq i \leq m \} \cup \{ u_iu_{i-1} : 1 \leq i \leq m \}
\]

\[
\{ v_jv_{j+1} : 1 \leq j \leq n \} \cup \{ v_{j+1}v_j : 1 \leq j \leq n \}
\]

\[
\{ v_{j+1}v_j : 1 \leq j \leq n \} \cup \{ u_{ik}u_{ik+1} : 1 \leq i \leq m, 1 \leq k \leq n - 2 \}
\]

The number of vertices and edges of the comb product of a tadpole graph with a cycle graph is \((m+n)n\) and \((m+n)(n+1)\). Denote the vertices in the cycle as \( u_1u_2u_3 \ldots \ldots \ldots u_m \) and in the path as \( v_1, v_2, v_3 \ldots \ldots \ldots v_n \) and here the bridge is \( u_mv_1. \)

The vertices cycles in the comb product graph \( T_{m,n} \triangleright C_n \) are denoted as \( u_{i1}, u_{i2}, \ldots \ldots \ldots u_{in-1} \) for \( i=1,2, \ldots \ldots \ldots m \) and \( v_{j1}, v_{j2}, v_{j3}, \ldots \ldots \ldots v_{jn-1} \) for \( j=1,2, \ldots \ldots \ldots n \)

\[
\deg(u_i) = \deg(v_j) = 4, \text{ for } i=1,2, \ldots \ldots \ldots m \quad j=1,2, \ldots \ldots \ldots n-1
\]

\[
\deg(u_m) = 5, \text{ for } v_1 \quad d(v_1) = 3
\]

\[
\deg(u_{ik}) = 2 = \deg(v_{jk}) \text{ for } i=1,2, \ldots \ldots \ldots m \quad k = 1,2, \ldots \ldots n-1
\]

\[
j=1,2, \ldots \ldots \ldots n \quad k = 1,2, \ldots \ldots n-1
\]

\( u_m \) is the only vertex having \( \deg \Delta(G) \)

In the comb product of the tadpole graph with cycle graph is the degree of \( u_m \) is 5.The vertices connected to the vertex \( u_m \) are \( v_1, u_{m-1}, u_1, u_{m-n}, u_{m-1} \) and \( u_{m1} \). These five vertices are non-adjacent to each other. If assigned one color to \( u_m \) vertex and only one color to other five vertices. so \( u_m \) and its adjacent vertices can be colored by 2 different colors. The five associated edges with the vertex \( u_m \) also require the 5 colors and hence totally 7 colors requires coloring for this graph.

Now consider the set of colors \( C = \{1,2,3,4,5,6,7\} \)

Let \( S = V(T_{m,n} \triangleright C_n) \cup E(T_{m,n} \triangleright C_n) \)

**Case:1 If \( m \) is even and \( n=1,2,3,4 \),**

For \( 1 \leq i \leq m \)
\[
f(u_i) = \begin{cases} 
1 & \text{if } i \equiv 1 \text{ (mod 2)} \\
2 & \text{if } i \equiv 2 \text{ (mod 2)}
\end{cases}
\]

For \( 1 \leq j \leq n \)
\[
f(v_j) = \begin{cases} 
1 & \text{if } j \equiv 1 \text{ (mod 2)} \\
2 & \text{if } j \equiv 2 \text{ (mod 2)}
\end{cases}
\]

For \( 1 \leq i \leq m, 1 \leq k \leq n - 1 \)
\[
f(u_{ik}) = \begin{cases} 
1 & \text{if } i \text{ is odd, } k \text{ is even and if } i \text{ is even, } k \text{ is odd} \\
2 & \text{if } i \text{ is even, } k \text{ is even and if } i \text{ is odd, } k \text{ is odd}
\end{cases}
\]

For \( 1 \leq j \leq n, 1 \leq k \leq n - 1 \)
\[
f(v_{jk}) = \begin{cases} 
1 & \text{if } j \text{ is odd, } k \text{ is even and if } j \text{ is even, } k \text{ is odd} \\
2 & \text{if } j \text{ is odd, } k \text{ is odd and if } j \text{ is even, } k \text{ is even} 
\end{cases}
\]

\[
f(v_{jk}) = \begin{cases} 
1 & \text{if } j \text{ is odd, } k \text{ is even and if } j \text{ is even, } k \text{ is odd} \\
2 & \text{if } j \text{ is even, } k \text{ is even and if } j \text{ is odd, } k \text{ is odd} 
\end{cases}
\]

\[
f(u_m u_1) = \{3\}, f(u_m v_1) = \{6\}
\]

For \(1 \leq i \leq m\) \(f(u_i u_{i+1}) = \begin{cases} 
4 & \text{if } i \text{ is odd} \\
3 & \text{if } i \text{ is even} 
\end{cases}
\]

For \(1 \leq i \leq n\) \(f(v_j v_{j+1}) = \begin{cases} 
3 & \text{if } j \text{ is odd} \\
4 & \text{if } j \text{ is even} 
\end{cases}
\]

Case 2: If \(m\) is odd and \(n = 1,2,3,4\)

For \(1 \leq i \leq m-1\) \(f(u_i) = \begin{cases} 
2 & \text{if } i = 1 \mod 2 \\
1 & \text{if } i = 2 \mod 2 
\end{cases}
\]

For \(1 \leq j \leq n\) \(f(v_j) = \begin{cases} 
1 & \text{if } j = 1 \mod 2 \\
2 & \text{if } j = 2 \mod 2 
\end{cases}
\]

\(f(u_m) = \{3\}\)

For \(1 \leq i \leq m, 1 \leq k \leq n-2\)

\(f(u_{ik} u_{ik+1}) = \begin{cases} 
5 & \text{if } i \text{ is odd, } k \text{ is even and if } i \text{ is even, } k \text{ is odd} \\
6 & \text{if } i \text{ is odd, } k \text{ is odd and if } i \text{ is even, } k \text{ is odd} 
\end{cases}
\]

For \(1 \leq j \leq n, 1 \leq k \leq n-2\)

\(f(v_{jk} v_{jk+1}) = \begin{cases} 
5 & \text{if } j \text{ is odd, } k \text{ is even and if } j \text{ is even, } k \text{ is odd} \\
6 & \text{if } j \text{ is odd, } k \text{ is odd and if } j \text{ is even, } k \text{ is odd} 
\end{cases}
\]

\(f(u_m u_1) = \{4\}, f(u_m v_1) = \{6\}, f(u_{m-1} u_m) = \{4\}\)

For \(1 \leq i \leq m\) \(f(u_i u_{i-1}) = \{3\}\)

For \(1 \leq j \leq n\) \(f(v_j v_{j-1}) = \{3\}\)

For \(1 \leq i \leq m-2\) \(f(u_i u_{i+1}) = \begin{cases} 
3 & \text{if } i \text{ is odd} \\
4 & \text{if } i \text{ is even} 
\end{cases}
\]

For \(1 \leq j \leq n-1\) \(f(v_j v_{j+1}) = \begin{cases} 
3 & \text{if } j \text{ is odd} \\
4 & \text{if } j \text{ is even} 
\end{cases}
\]

For \(1 \leq i \leq m\) \(f(u_i u_{i+1}) = \{5\}\)

For \(1 \leq i \leq m\) \(f(u_i u_{i-1}) = \{7\}\)

For \(1 \leq j \leq n\) \(f(v_j v_{j+1}) = \{5\}\)

For \(1 \leq j \leq n\) \(f(v_j v_{j-1}) = \{7\}\)

For \(1 \leq i \leq m, 1 \leq k \leq n-2\)
\[
    f(u_i u_{i+1}) = \begin{cases} 
      5 & \text{if } i \text{ is odd, } k \text{ is even and if } i \text{ is even, } k \text{ is even} \\
      6 & \text{if } i \text{ is odd, } k \text{ is odd and if } i \text{ is even, } k \text{ is odd} 
    \end{cases}
\]

For \(1 \leq j \leq n, 1 \leq k \leq n - 2\)

\[
    f(v_{jk} v_{jk+1}) = \begin{cases} 
      5 & \text{if } j \text{ is odd, } k \text{ is even and if } j \text{ is even, } k \text{ is even} \\
      6 & \text{if } j \text{ is odd, } k \text{ is odd and if } j \text{ is even, } k \text{ is odd} 
    \end{cases}
\]

Hence \(\chi_{tc}(T_{mn} \Delta C_n) = 7\) for \(n \geq 2\).

Example: 6 Consider the Total chromatic number comb product of Tadpole graph with cycle graph is 7. If \(m\) is odd and \(n\) is odd

Example: 7

Consider the Total chromatic number comb product of Tadpole graph with cycle graph is 7. If \(m\) is even and \(n\) is even

CONCLUSION:

We obtained the total chromatic number of the Tadpole graph with path, star, cycle, and fan graphs.
References:


