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Total Chromatic Number Comb Product of Tadpole Graph

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Abstract:
The total chromatic number of a graph G is defined to be the minimum number of colors needed to color the vertices and edges of a graph in such a way that no two adjacent vertices, no two adjacent edges and no each edge and end vertices are given the same color. In this paper, we have obtained the total coloring and total chromatic number of comb product of tadpole graph with path, star, fan, and cycle.

Keywords: tadpole graph, path, star, cycle, fan, comb product, total chromatic number.

1.INTRODUCTION

Chromatic number of graphs is a special area in a Graph theory.(Behzad 1987) introduced the concept of total coloring and found the chromatic number. Coloring a graph G involves assigning colors to all of its vertices and edges such that no two adjacent vertices or edges have the same color, and each edge and end vertex is assigned a unique color. The total chromatic number of a graph G is the minimum number of colors required to produce a total coloring, denoted by χ(G) (Behzad 1987) conjectured that for any graph with a maximum degree Δ(G), the total chromatic number satisfies the condition $\Delta(G) + 1 \leq \chi_{tc}(G). \leq \Delta(G) + 2$ (Sudha. eat al.2017) have discussed the total colouring and total chromatic number of the central graph of a path a cycle, and a star. (Muthuramakrishnan eat al 2018) have discussed the total coloring of middle graph, total graph of path and sun let graph. Also, they have obtained the total chromatic number of those graph.Referring definition of comb graph and comb product by (Rohmatulloh et al 2021), (Suhadi Wido Saputro et al).Basic definition of star,cycle, path and fan by (J.A.Bondy et al)

In this paper, we investigate the total chromatic number of the comb product of various graphs including the tadpole graph with the path graph, star graph, fan graph, and cycle graph.

DEFINITIONS:

2.1 PATH

A path is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence and are nonadjacent otherwise.

2.2 CYCLE

A cycle graph or circular graph is a path P_{n+1} (n>3, if the graph is simple) whose end vertices are joined to form a closed chain. The cycle graph with *n* vertices is denoted by C_n.

2.3 STAR GRAPH

A Star graph is a complete bipartite graph $K_{1,n}$ for $n \geq 1$.

2.4 FAN GRAPH

The fan graph F_n is the join of P_n and K_1.

2.5COMB GRAPH

A comb graph is a graph obtained by joining a single pendant edge to each vertex of a path.

2.6 TADPOLE GRAPH

The (m,n)-tadpole graph is a special type of graph consisting of a cycle graph on m (at least 3) vertices and a path graph on n vertices, connected with a bridge. It is denoted by $\boldsymbol{T_{m,n}}$

2.7 COMB PRODUCT:

Let G and H be two connected graphs and o be a vertex of H. The comb product between G and H denoted by G $\rhd$ H, is a graph obtained by taking one copy of G and |V(G)| copies of H and grafting the vertex o of i-th copy of H with the i-th vertex of G.

3. MAIN RESULTS:

THEOREM: 3.1. The total chromatic number of the comb product of a tadpole graph with a path graph is $\chi_{tc}(\boldsymbol{T_{m,n}} \rhd P_n)$=6 for $n \geq 2$.

Proof:

Let $\{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\}$ be the vertices and $\{u_i u_{i+1}: 1 \leq i \leq m-1\}$ $\cup \{v_j v_{j+1}: 1 \leq j \leq n-1\} \cup \{u_m u_1\} \cup \{u_m v_1\}$ be an edges of tadpole graph Tm,n.

Let $\{w_i: 1 \leq i \leq n\}$ be the vertices and $\{w_i w_{i+1}: 1 \leq i \leq n-1\}$ be the edges of the path graph.

By definition of comb product of Tadpole graph with path graph, the vertex of path is recognized with each vertex of the tadpole graph $\boldsymbol{T_{m,n}}$.

Let the vertex set and the edge set of $(Tm,n \rhd P_n)$ as follows.

$$V(\boldsymbol{T_{m,n}} \rhd P_n) = \{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\} \cup \left\{ u_{ik} : \text{for} \begin{array}{l} 1 \leq i \leq m \\ 1 \leq k \leq n-1 \end{array} \right\} \cup \left\{ y_{jk} : \text{for} \begin{array}{l} 1 \leq j \leq n \\ 1 \leq k \leq n-1 \end{array} \right\}$$

$$E(\boldsymbol{T_{m,n}} \rhd P_n) = \{u_i u_{i+1}: 1 \leq i \leq m-1\} \cup \{v_j v_{j+1}: 1 \leq j \leq n-1\} \cup \{u_m u_1\} \cup \{u_m v_1\} \cup \{u_i u_{i1}: 1 \leq i \leq m\} \cup \{v_j v_{j1}: 1 \leq j \leq n\} \cup \{ v_{ik} v_{ik+1} : 1 \leq i \leq m, 1 \leq k \leq n-2\} \cup \{ y_{jk} y_{jk+1} :: 1 \leq j \leq n, 1 \leq k \leq n-2\}$$

The number of vertices and edges of the comb product of a tadpole graph with a path graph is (m+n)n.

Denote the vertices in the cycle as $u_1, u_2, u_3, \dots \dots \dots u_m$ and in the path as $v_1, v_2, v_3, \dots \dots \dots, v_n$ and here the bridge is $u_m v_1$. The vertices pendant paths in the comb product graph $\boldsymbol{T_{m,n}} \rhd P_n$ are denoted as $u_{i1}, u_{i2}, \dots \dots, u_{im-1}$ for i =1,2,……., m and $v_{j1}, v_{j2}, v_{j3}, \dots \dots \dots, v_{jn-1}$. For j =1,2,…….,n

deg(u_i)=deg(v_j)=3, for i =1,2,……..,m-1 , j =1,2,……….,n-1

deg(u_m)=4,deg(v_n)=2

$\deg(u_{ik}) = 2 = \deg(v_{jk})$ for $i = 1,2,\ldots,m$, $k = 1,2,\ldots\ldots,n-2$

$j = 1,2,\ldots\ldots,n$, $k = 1,2,\ldots\ldots,n-2$

$\deg(u_{in}) = 1 = \deg(v_{jn})$ for $i = 1,2,\ldots\ldots,m$ $j = 1,2,\ldots\ldots,n$

u_m is the only vertex having deg **Δ(G)**

In the comb product of tadpole graph with path graph, degree of u_m is 4.The vertices connected to the vertex u_m are v_1, u_{m-1}, u_1, and u_{m1}.These four vertices are non-adjacent vertices to each other. If assigned one color to u_m vertex and only one color to other 4 vertices, so u_m and its adjacent vertices can be colored by 2 different colors. The associated of 4 edges with the vertex u_m also required the 4 colors. Because they are connected to each other, and hence totally **6** colors required to color for this graph.

Now consider the set of colors C= (1, 2, 3, 4, 5, 6) coloring

Let $S = V(\boldsymbol{T_{m,n}} \rhd P_n) \cup E(\boldsymbol{T_{m,n}} \rhd P_n)$

If m is even $f(u_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 2 \\ 2 & \text{if } i \equiv 2 \pmod 2 \end{cases}$,For $1 \le i \le m$

If m is odd $f(u_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 2 \\ 2 & \text{if } i \equiv 2 \pmod 2 \end{cases}$ For $1 \le i \le m$-1, $f(u_m)$={3}

For $1 \le j \le n$ $f(v_j) = \begin{cases} 1 & \text{if } j \equiv 1 (mod\ 2) \\ 2 & \text{if } j \equiv 2 \pmod 2 \end{cases}$

For $1 \le i \le m, 1 \le k \le n-1$

$$f(u_{ik}) = \begin{cases} 1 & \text{if } i \text{ is even,k is odd and if i is odd, k is even} \\ 2 & \text{if } i \text{ is even, k is even and if i is odd, k is odd} \end{cases}$$

For $1 \le j \le n, 1 \le k \le n-1$

$$f(v_{jk}) = \begin{cases} 1 & \text{if j is odd ,k is even and if j is even, k is odd} \\ 2 & \text{if j is odd, k is odd and if j is even, k is even} \end{cases}$$

For $1 \le i \le m-1$ $f(u_i u_{i+1}) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$

$f(u_m u_1)$ = {3}

$f(u_m v_1)$ = {6}

If m is even

For $1 \le i \le m-1$ $f(u_i u_{i+1}) = \begin{cases} 3 & \text{if } i \text{ is odd} \\ 4 & \text{if } i \text{ is even} \end{cases}$ f($u_m u_1$)={2},f($u_m v_1$)={6}

If m is odd

For $1 \le j \le n-1$ $f(v_j v_{j+1}) = \begin{cases} 3 & \text{if } j \text{ is odd} \\ 4 & \text{if } j \text{ is even} \end{cases}$

$F \quad or\ 1 \le i \le m$ $f(u_i u_{i1}) = \{5\}$

$For\ \ 1 \le j \le n$ $f(v_j v_{j1}) = \{5\}$

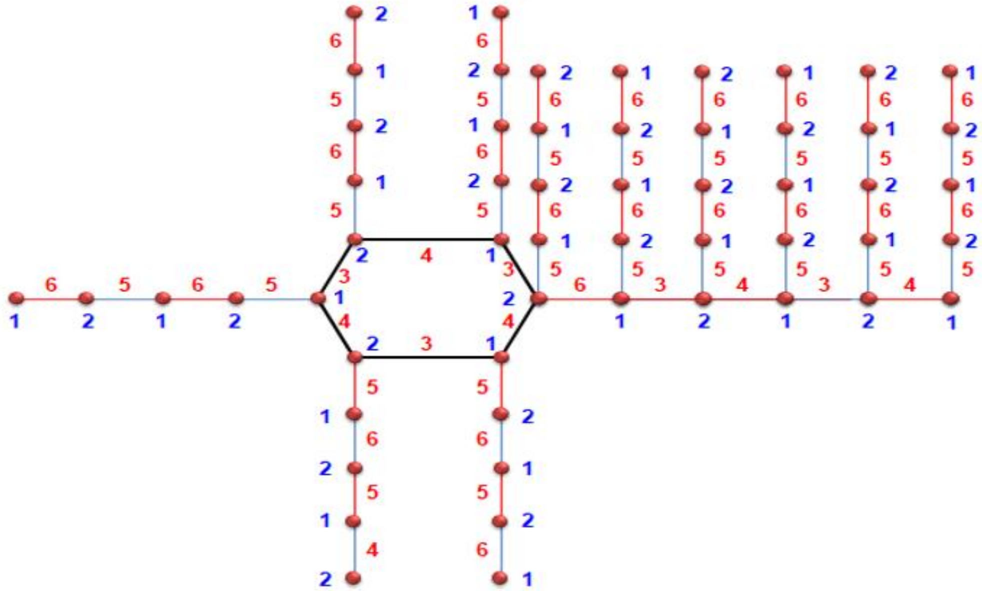
$For 1 \le i \le m, 1 \le k \le n-2$ $f(u_{ik} u_{ik_{+1}}) = \begin{cases} 6 & \text{if } i \text{ is odd ,k is odd and if i is even, k is odd} \\ 5 & \text{if } i \text{ is odd, k is even and if i is even, k is even} \end{cases}$

$For\ 1 \le j \le n, 1 \le k \le n-2$ $f(v_{jk} v_{jk_{+1}}) = \begin{cases} 6 & \text{if } j \text{ is odd ,k is odd and if j is even, k is odd} \\ 5 & \text{if } j \text{ is odd, k is even and if j is even, k is even} \end{cases}$

Hence $\chi_{tc}\left(T_{m,n} \rhd P_n\right) = 6$ for $n \geq 2$.

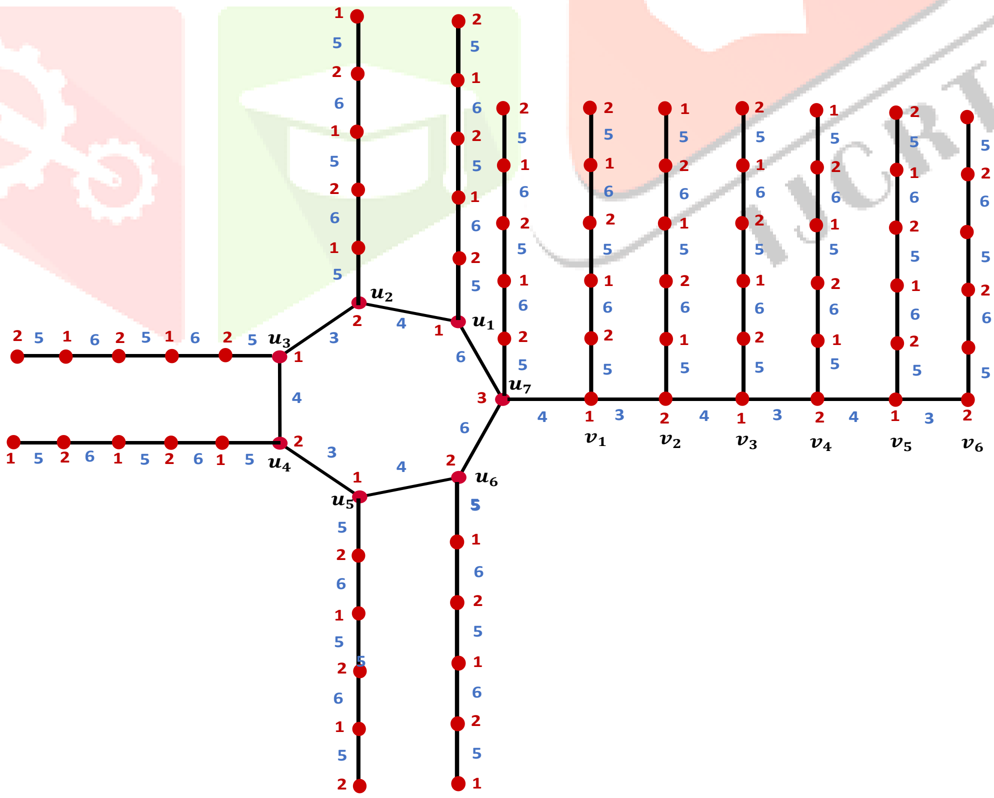
Example: 1 Consider the total chromatic number of the comb product of the tadpole with a pathgraph is 6.

If m is even and n is odd



Example: 2 consider the total chromatic number of the comb product of the tadpole with a pathgraph is 6.

If m is odd and n is even.



<u>THEOREM: 3.2</u>

The total chromatic number of Comb Product of Tadpole graph with fan graph is $\chi_{tc}\left(T_{m,n} \rhd F_n\right)$=n+5 for $n \geq 2$.

Proof:

Let $\{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\}$ be the vertices and $\{u_i u_{i+1}: 1 \leq i \leq m-1\} \cup \{v_j v_{j+1}: 1 \leq j \leq n-1\} \cup \{u_m u_1\} \cup \{u_m v_1\}$ be an edges of Tadpole graph $\boldsymbol{T_{m,n}}$.

Let $\{w_i: 1 \leq i \leq n\} \cup \{x\}$ be the vertices and $\{w_i w_{i+1}: 1 \leq i \leq n-1\} \cup \{xw_i: 1 \leq i \leq n\}$ be the edges of fan graph F_n. By definition the comb product of a tadpole graph with fan graph

The vertex x, of a star F_n *is* identified with each vertex of the tadpole graph $\boldsymbol{T_{m,n}}$

The vertex set and the edge set of $(\boldsymbol{T_{m,n}} \rhd F_n)$ as follows

$$V(\boldsymbol{T_{m,n}} \rhd F_n) = \{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\} \cup \left\{ v_{jk} : \text{for} \begin{array}{l} 1 \leq j \leq n \\ 1 \leq k \leq n-1 \end{array} \right\} \cup \left\{ u_{ik} : \text{for} \begin{array}{l} 1 \leq i \leq m \\ 1 \leq k \leq n-1 \end{array} \right\}$$

$$\left|E(\boldsymbol{T_{m,n}} \rhd F_n)\right| = \{u_i u_{i+1}: 1 \leq i \leq m-1\} \cup \{v_j v_{j+1}: 1 \leq j \leq n-1\} \cup \{u_m u_1\} \cup \{u_m v_1\}$$

$$\cup \left\{ u_i u_{ik} : \text{for} \begin{array}{l} 1 \leq i \leq m \\ 1 \leq k \leq n \end{array} \right\} \cup \left\{ v_j v_{jk}; \begin{array}{l} 1 \leq j \leq n \\ 1 \leq k \leq n \end{array} \right\} \cup \left\{ u_{ik} u_{ik+1}: \begin{array}{l} 1 \leq i \leq m \\ 1 \leq k \leq n-1 \end{array} \right\} \cup \left\{ v_{jk} v_{jk+1}; \begin{array}{l} 1 \leq j \leq n \\ 1 \leq k \leq n-1 \end{array} \right\}$$

The number of vertices and edges of the comb product of tadpole graph with fan graph is (m+n) (n+1) and (m+n)(2n). Denote the vertices in the cycle as $u_1, u_2, u_3, \ldots, u_m$ and in the path as $v_1, v_2, v_3, \ldots\ldots\ldots, v_n$ and here the bridge is $u_m v_1$. The vertices cycles the comb product graph $\boldsymbol{T_{m,n}} \rhd F_n$ are denoted as $u_{i1}, u_{i2}, \ldots\ldots\ldots, u_{in-1}$ and

For i =1,2,……., m and $v_{j1}, v_{j2}, v_{j3}, \ldots\ldots, v_{jn-1}$. For j =1,2,………,n

deg (u_i)=deg (v_j)=n+2, For i =1,2,……… ,m-1 , j =1,2,……. ,n-1

deg (u_m)=n+3,deg (v_n)=n+1

deg $(u_{ik}) = 2$=deg (v_{jk}) For i =1,2,…….,m, $j = 1,2,\ldots\ldots\ldots$, n $k = 1,2,\ldots\ldots\ldots, n-2$

deg ($u_{i1}) = 2$=deg (v_{j1}) For i =1,2,…,m For j =1,2,…….. ,n

deg $(u_{in}) = 2$=deg (v_{jn}) For i =1,2,…..,m For j =1,2,……..,n

The vertex u_m is the only vertex having Δ(G) as its deg (u_m)=n+3.

In the comb product of tadpole graph with star graph, degree of u_m is $n+3$. The vertices connected to the vertex u_m with $v_1, u_{m-1}, u_1, u_{m1}, u_{m2}, u_{m3}, \ldots\ldots, u_{m\,n}$. These $n+3$ vertices are non-adjacent vertices to each other. If assigned one color to u_m vertex and only one color to other vertices u_{m-1}, u_1 *and* v_1, so u_m and its adjacent vertices can be colored by 2 different colors. But the n vertices going from the u_m and the associated edges are requires $n+3$ color.

Hence totally $n+5$ colors required to color for this graph.

Now consider the set of colors C=(1,2,3,…….,n+5) coloring.

Let $S = V(\boldsymbol{T_{m,n}} \rhd F_n) \cup E(\boldsymbol{T_{m,n}} \rhd F_n)$ *and* C = n+5

Case: 1 If m is an odd and n=1,2, 3,………

$For\ 1 \le i \le m-1 \quad f(u_i) = \begin{cases} n+5 & \text{if } i \text{ is odd} \\ n+4 & \text{if } i \text{ is even} \end{cases}$

$f(u_m)=\{n+3\}$

$For\ 1 \le j \le n \quad f(v_j) = \begin{cases} n+5 & \text{if } j \text{ is odd} \\ n+4 & \text{i } j \text{ is even} \end{cases}$

For $1 \le i \le m, 1 \le k \le n \quad f(u_{ik}) = k+1$

For $1 \le j \le n, 1 \le k \le n \quad f(v_{jk}) = k+1$

$For1 \le i \le m-1 \quad f(u_i u_{i+1}) = \begin{cases} n+3 & \text{if } i \text{ is odd} \\ n+2 & \text{if } i \text{ is even} \end{cases}$

$f(u_m u_1) = n+4$

$f(u_m v_1) = n+1$

$For1 \le j \le n-1 \quad f(v_j v_{j+1}) = \begin{cases} n+3 & \text{if } j \text{ is odd} \\ n+2 & \text{if } j \text{ is even} \end{cases}$

$For1 \le i \le m, 1 \le k \le n \quad f(u_i u_{ik}) = k$

$For1 \le j \le n, 1 \le k \le n \quad f(v_j v_{jk}) = k$

For $1 \le i \le m, 1 \le k \le n-1$ $f(u_{ik} u_{ik+1})$= n+k

For $1 \le i \le n, 1 \le k \le n-1$ $f(v_{jk} v_{jk+1})$= n+k

Case:2 If m is an even and n=1,2,3,……………….

$For1 \le i \le m \quad f(u_i) = \begin{cases} n+5 & \text{if } i \text{ is odd} \\ n+4 & \text{if } i \text{ is even} \end{cases}$

$For1 \le j \le n \quad f(v_j) = \begin{cases} n+5 & \text{if } j \text{ is odd} \\ n+4 & \text{i } j \text{ is even} \end{cases}$

For $1 \le i \le m, 1 \le k \le n \quad f(u_{ik}) = k+1$

For $1 \le j \le n, 1 \le k \le n \quad f(v_{jk}) = k+1$

$For1 \le i \le m-1 \quad f(u_i u_{i+1}) = \begin{cases} n+3 & \text{if } i \text{ is odd} \\ n+2 & \text{if } i \text{ is even} \end{cases}$

$For1 \le j \le n-1 \quad f(v_j v_{j+1}) = \begin{cases} n+2 & \text{if } j \text{ is odd} \\ n+3 & \text{if } j \text{ is even} \end{cases}$

$For1 \le i \le m, 1 \le k \le n \quad f(u_i u_{ik}) = k$

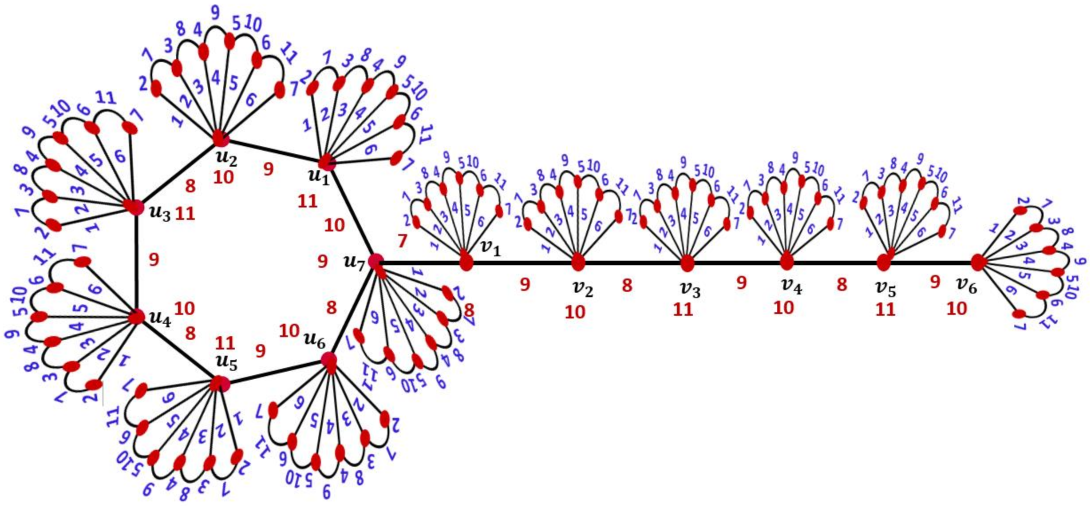
$For1 \le j \le n, 1 \le k \le n \quad f(v_j v_{jk}) = k$

For $1 \le i \le m, 1 \le k \le n-1$ $f(u_{ik} v_{ik+1})$= n+k

For $1 \le i \le n, 1 \le k \le n-1$ $f(v_{jk} v_{jk+1})$= n+k

Hence $\chi_{tc}(\boldsymbol{T_{m,n}} \rhd F_n) = n+5$ for $n \ge 2$.

Example 3: Consider the total chromatic number of the comb product of the tadpole graph with the fan graph is n+5. Hence $\chi_{tc}\left(\boldsymbol{T_{7,6}} \rhd F_6\right) = 11$



<u>THEOREM 3.3</u>

The total chromatic number of the comb product of the Tadpole graph with the Star graph is $\left(\boldsymbol{T_{m,n}} \rhd S_n\right)$= n+4 for $n \geq 2$

Proof

Let $\{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\}$ be the vertices and $\{u_i u_{i+1}: 1 \leq i \leq m-1\} \cup \{v_j v_{j+1}: 1 \leq j \leq n-1\} \cup \{u_\mathrm{m} u_1\} \cup \{u_m v_1\}$ be an edges of tadpole graph $\boldsymbol{T_{m,n}}$.

Let $\{w_i: 1 \leq i \leq n\} \cup$ {x} be the vertices and $\{xw_i: 1 \leq i \leq n\}$ be the edges of star **graph** S_n By the definition of the comb product of a tadpole graph with a star graph. The vertex x, of star S_n *is* identified with each vertex of the tadpole graph $\boldsymbol{T_{m,n}}$the vertex set and the edge set of $\left(\boldsymbol{T_{m,n}} \rhd S_n\right)$ as follows

$$V\left(\boldsymbol{T_{m,n}} \rhd S_n\right)| = \{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\} \cup \left\{v_{jk}:\text{for} \begin{array}{l} 1 \leq j \leq n \\ 1 \leq k \leq n-1 \end{array}\right\} \cup \left\{u_{ik}:\text{for} \begin{array}{l} 1 \leq i \leq m \\ 1 \leq k \leq n-1 \end{array}\right\}$$

$$\left|E\left(\boldsymbol{T_{m,n}} \rhd S_n\right)\right|$$

$$= \{u_i u_{i+1}: 1 \leq i \leq m-1\} \cup \{v_j v_{j+1}: 1 \leq j \leq n-1\} \cup \{u_\mathrm{m} u_1\} \cup \{u_m v_1\}$$

$$\cup \left\{u_i u_{ik}: \text{for} \begin{array}{l} 1 \leq i \leq m \\ 1 \leq k \leq n \end{array}\right\} \cup \left\{v_j v_{jk}; \begin{array}{l} 1 \leq j \leq n \\ 1 \leq k \leq n \end{array}\right\}$$

The number of vertices and edges of the comb product of tadpole graph with star graph is (m+n) (n+1) and (m+n) (n+1). Denote the vertices in the cycle, as $u_{1,} u_{2,} u_{3,} \dots \dots \dots u_m$ and in the path as $v_1, v_2, v_3, \dots \dots \dots ., v_n$ and here the bridge is $u_\mathrm{m} v_1$. The vertices of comb product of $\left(\boldsymbol{T_{m,n}} \rhd S_n\right)$ are denoted as $u_{i1}, u_{i2}, \dots \dots \dots \dots ., u_{in-1}$and

For

i =1,2,……., m and $v_{j1}, v_{j2}, v_{j3}, \dots \dots \dots, v_{jn-1}$.

deg $(u_i) = n + 2$=deg (v_j) i=1,2,………,m-1, j =1,2,……..,n-1

deg $(u_m) = n + 3$, deg (v_n)= $n + 1$,

deg $(u_{ik}) = 1 =$ deg (v_{jk}) =For i =1,2,…,m , $j = 1,2, \dots,$ n $k = 1,2, \dots \dots \dots, n$ is u_m the only vertex having Δ (G) as its deg (u_m)=n+3.

In the comb product of tadpole graph with star graph, degree of u_m is $n + 3$. $n + 3$ edges emerge from the point u_m.They are $u_m u_1, u_m v_1, u_m u_{m1}, u_m u_{m2}, u_m u_{m3}, \dots \dots, u_m u_{m\,n}$ and $u_m u_{m-1}$. These n+3 edges are related to each other. So point u_m should be given one color and these n+3 edges should be given a separate color.

Hence totally $n + 4$ colors require to color for this graph.

Now consider the set of colors C = (1, 2, 3,……….,n+4) coloring

Let $S = V(\boldsymbol{T_{m,n}} \rhd S_n) \cup E(\boldsymbol{T_{m,n}} \rhd S_n)$ and $C =$ n+4

Case:1 If m is an odd and n=1,2,3,………………..

$For 1 \le i \le m-1 \qquad f(u_i) = \begin{cases} n+4 & \text{if } i \text{ is odd} \\ n+3 & \text{if } i \text{ is even} \end{cases}$,

$f(u_m)$={n+1}

$For 1 \le j \le n \qquad f(v_j) = \begin{cases} n+2 & \text{if } j \text{ is odd} \\ n+1 & \text{i } j \text{ is even} \end{cases}$

For $1 \le i \le m, 1 \le k \le n \quad f(u_{ik}) = \text{n} + 2$

For $1 \le j \le n, 1 \le k \le n \quad f(v_{jk}) = \text{n} + 4$

$For 1 \le i \le m-1 \qquad f(u_i u_{i+1}) = \begin{cases} n+1 & \text{if } i \text{ is odd} \\ n+2 & \text{if } i \text{ is even} \end{cases}$

$f(u_m u_1) = n + 3$

$f(u_m v_1) = n + 4$

$For 1 \le j \le n-1 \qquad f(v_j v_{j+1}) = \begin{cases} n+3 & \text{if } j \text{ is odd} \\ n+4 & \text{if } j \text{ is even} \end{cases}$

$For 1 \le i \le m, 1 \le k \le n \quad f(u_i u_{ik}) = \text{k}$

$F\ or 1 \le j \le n, 1 \le k \le n \quad f(v_j v_{jk}) = \text{k}$

Case:2 If m is an even and n=1,2,3,………………..

$For 1 \le i \le m \qquad f(u_i) = \begin{cases} n+4 & \text{if } i \text{ is odd} \\ n+3 & \text{if } i \text{ is even} \end{cases}$

$For 1 \le j \le n \qquad f(v_j) = \begin{cases} n+1 & \text{if } j \text{ is odd} \\ n+2 & \text{i } j \text{ is even} \end{cases}$

For $1 \le i \le m, 1 \le k \le n \quad f(u_{ik}) = \text{n} + 1$

For $1 \le j \le n, 1 \le k \le n \quad f(v_{jk}) = \text{n} + 4$

$For 1 \le i \le m-1 \qquad f(u_i u_{i+1}) = \begin{cases} n+1 & \text{if } i \text{ is odd} \\ n+2 & \text{if } i \text{ is even} \end{cases}$

$f(u_m u_1) = n + 2$

$f(u_m v_1) = n + 4$

$For 1 \le j \le n-1 \qquad f(v_j v_{j+1}) = \begin{cases} n+3 & \text{if } j \text{ is odd} \\ n+4 & \text{if } j \text{ is even} \end{cases}$

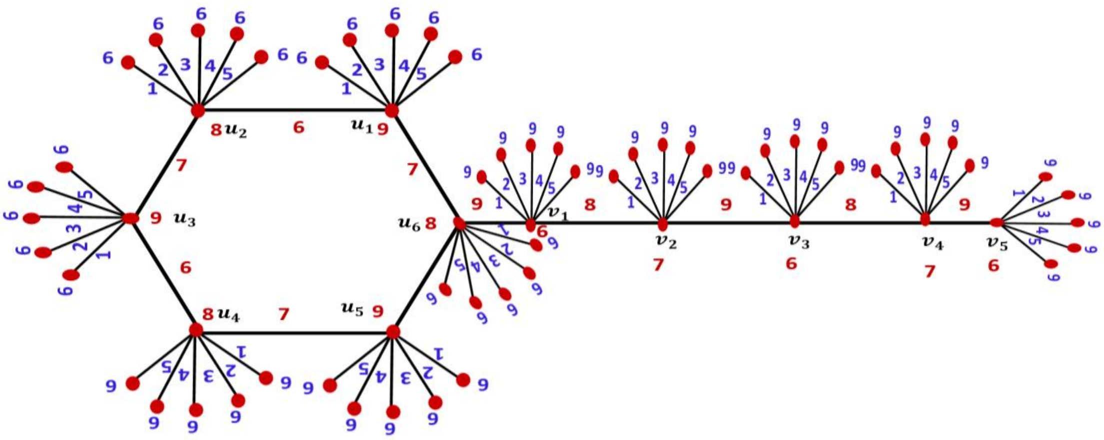
$For 1 \leq i \leq m, 1 \leq k \leq n \;\; f(u_i u_{ik}) = \mathrm{k}$,

$For 1 \leq j \leq n, 1 \leq k \leq n \;\; f(v_j v_{jk}) = \mathrm{k}$

The above defined function f gives the total coloring for the graph.

Hence $\chi_{tc} n(\boldsymbol{T_{m,n}} \rhd S_n) = n + 4$ for $n \geq 2$.

Example 4: Consider the Total chromatic number of comb product of tadpole graph with star graph is n+4. If m is even, n is odd

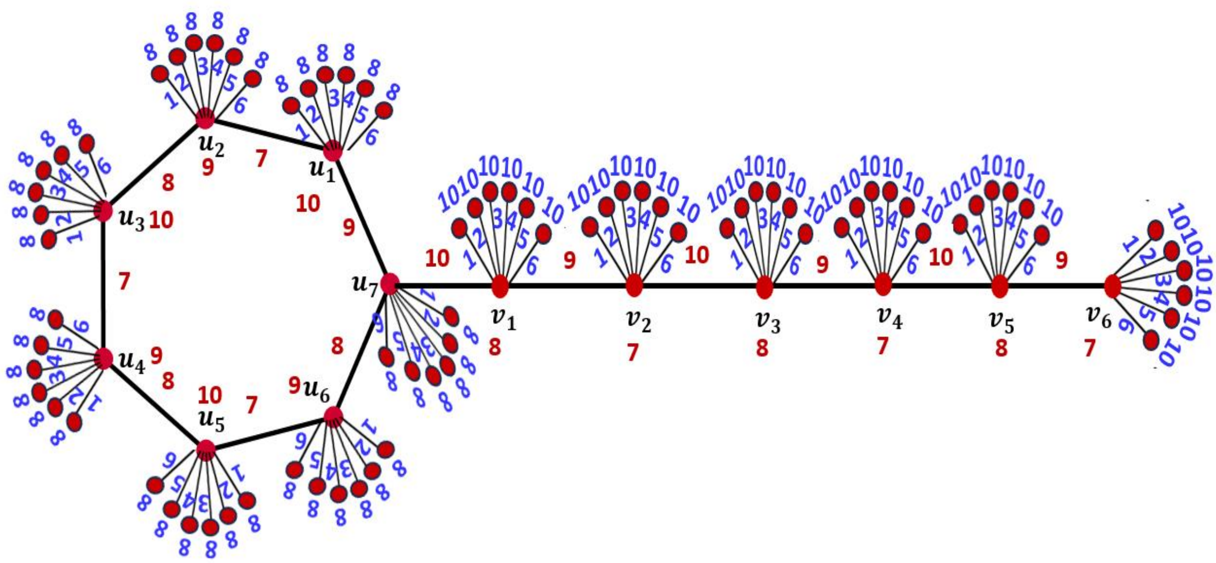


$\chi_{tc}(T_{6,5} \rhd S_5) = n + 4 = 9$

Example 5:

Consider the total chromatic number of the comb product of the tadpole graph with the star graph is n+4..

If m is even, n is odd



$\chi_{tc}(\boldsymbol{T_{7,6}} \rhd S_6) = n + 4 = \mathbf{10}$

<u>THEOREM: 4</u>

The total chromatic number of Comb Product of the Tadpole graph with the cycle graph is $(\boldsymbol{T_{m,n}} \rhd C_n)$=7 for $n \geq 2$.

<u>Proof:</u>

Let $\{u_i: 1 \leq i \leq m\} \cup \{v_j: 1 \leq j \leq n\}$ be the vertices and $\{u_i u_{i+1}: 1 \leq i \leq m-1\} \cup$

$\{v_j v_{j+1}: 1 \le j \le n-1\} \cup \{u_m u_1\} \cup \{u_m v_1\}$ be an edges of tadpole graph $T_{m,n}$.

Let $\{w_i: 1 \le i \le n\}$ be the vertices and $\{w_i w_{i+1}: 1 \le i \le n\}$ be the edges of cycle graph C_n.

By the definition of the comb product of a tadpole graph with a cycle graph

The vertex w_1,of cycle C_n *is* identified with each vertex of the tadpole graph $T_{m,n}$ vertex set and the edge set of $(T_{m,n} \rhd C_n)$ as follows

$$V(T_{m,n} \rhd C_n) = \{u_i: 1 \le i \le m\} \cup \left\{ y_{jk}: \text{for} \quad \begin{matrix} 1 \le j \le n \\ 1 \le k \le n-1 \end{matrix} \right\} \cup$$

$$\{v_j: 1 \le j \le n\} \cup \{u_{ik}: 1 \le i \le m, 1 \le k \le n-1\}$$

$$\begin{aligned} E(T_{m,n} \rhd C_n) = \ & \{u_m u_1\} \cup \{u_m v_1\} \\ & \{u_i u_{i+1}; 1 \le i \le m-1\} \cup \\ & \{v_j v_{j+1}: 1 \le j \le n-1\} \cup \\ & \{u_i u_{i1}: 1 \le i \le m\} \cup \{v_j v_{j1}: 1 \le j \le n\} \cup \{u_i\, u_{in-1}: 1 \le \mathrm{i} \le m\} \\ & \{v_j y_{jn-1}, 1 \le j \le n\} \cup \{u_{ik} u_{ik+1}; \quad 1 \le i \le m, 1 \le k \le n-2\} \\ & \{v_{jk} v_{jk+1}; 1 \le j \le n, 1 \le k \le n-2\} \end{aligned}$$

The number of vertices and edges of the comb product of a tadpole graph with a cycle graph is (m+n)n and (m+n)(n+1). Denote the vertices in the cycle as $u_{1,} u_{2,} u_{3,} \ldots \ldots \ldots, u_m$ and in the path as $v_1, v_2, v_3, \ldots \ldots \ldots., v_n$ and here the bridge is $u_m v_{1.}$

The vertices cycles in the comb product graph $\boldsymbol{T_{m,n}} \rhd C_n$ are denoted as $u_{i1}, u_{i2}, \ldots \ldots \ldots \ldots., u_{in-1}$ for i =1,2,… …,m and $v_{j1}, v_{j2}, v_{j3}, \ldots \ldots., v_{jn-1}$ For j =1,2, … … .,n

deg (u_i)=deg(v_j)=4, for i =1,2, , … … . ., ,m-1 , j =1,2, , … … .,n-1

deg (u_m)=5,d(v_n)=3

deg $(u_{ik}) = 2 = \deg(v_{jk})$ for i =1,2, … ..,m , $k = 1,2, \ldots \ldots., n-1$

j =1,2,… …,n, $\mathrm{k} = 1,2, \ldots \ldots, \mathrm{n} - 1$

u_m is the only vertex having deg Δ**(G)**

In the comb product of the tadpole graph with cycle graph is the degree of u_m is 5.The vertices connected to the vertex u_m are v_1, u_{m-1}, u_1 , $u_{m\,n-1}$ and u_{m1} .These five vertices are non-adjacent to each other. If assigned one color to u_m vertex and only one color to other five vertices. so u_m and its adjacent vertices can be colored by 2 different colors. The five associated edges with the vertex u_m also require the 5 colors and hence totally **7** colors requires coloring for this graph.

Now consider the set of colors $C = \{1,2,3,4,5,6,7\}$

Let$S = V(\boldsymbol{T_{m,n}} \rhd C_n) \cup E(\boldsymbol{T_{m,n}} \rhd C_n)$

Case:1 If m is even and n=1,2,3,4,

For $1 \le i \le \mathrm{m}$ $\quad f(u_i) = \begin{cases} 1 & \text{if } i \equiv 1 \text{ (mod 2)} \\ 2 & \text{if } i \equiv 2 \text{(mod 2)} \end{cases}$

For $1 \le j \le n$ $\; f(v_j) = \begin{cases} 1 & \text{if } j \equiv 1 (mod\ 2) \\ 2 & \text{if } j \equiv \ 2 \text{ (mod 2)} \end{cases}$

For $1 \le i \le m, 1 \le k \le n-1$

$$f\ (u_{ik}) = \begin{cases} 1 & \text{if } i \text{ is odd, k is even and if is } i \text{ even, k is odd} \\ 2 & \text{if } i \text{ is even, k is even and if } i \text{ is odd, k is odd} \end{cases}$$

For $1 \le j \le n, 1 \le k \le n-1$

$f(v_{jk}) = \begin{matrix} 1 & \text{if } j \text{ is odd ,k is even and if is even, k is odd} \\ 2 & \text{if } j \text{ is odd, k is odd and if is even, k is even} \end{matrix}$

$f(v_{jk}) = \begin{cases} 1 & \text{if } j \text{ is odd, k is even and if } j \text{ is even, k is odd} \\ 2 & \text{if } j \text{ is even, k is even and if } j \text{ is odd, k is odd} \end{cases}$

$f(u_m u_1) = \{3\}, f(u_m v_1) = \{6\}$

For $1 \leq i \leq m-1$ $\quad f(u_i u_{i+1}) = \begin{cases} 4 & \text{if } i \text{ is odd} \\ 3 & \text{if } i \text{ is even} \end{cases}$

For $1 \leq i \leq n-1$ $\quad f(v_j v_{j+1}) = \begin{cases} 3 & \text{if } j \text{ is odd} \\ 4 & \text{if } j \text{ is even} \end{cases}$

For $1 \leq i \leq m$ $\quad f(u_i u_{i1}) = \{5\}$

For $1 \leq i \leq m$ $\quad f(u_i u_{im-1}) = \{7\}$

For $1 \leq j \leq n$ $\quad f(v_j v_{j1}) = \{5\}$

For $1 \leq j \leq n$ $\quad f(v_j v_{jn-1}) = \{7\}$

For $1 \leq i \leq m, 1 \leq k \leq n-2$

$f(u_{ik} u_{ik+1}) = \begin{cases} 5 & \text{if } i \text{ is odd ,k iseven and if i is even, k is even} \\ 6 & \text{f } i \text{ is odd, k is odd,and if i is even, k is odd} \end{cases}$

For $1 \leq j \leq n, 1 \leq k \leq n-2$

$f(v_{jk} v_{jk+1}) = \begin{cases} 5 & \text{if } j \text{ is odd ,k is even and if j is even, k is even} \\ 6 & \text{f } j \text{ is odd, k is odd and if j is even, k is odd} \end{cases}$

Case:2 If m is odd and n=1,2,3, 4……………..

For $1 \leq i \leq m\text{-}1$ $\quad f(u_i) = \begin{cases} 2 & \text{if } i \equiv 1 \text{ (mod 2)} \\ 1 & \text{if } i \equiv 2 \text{(mod 2)} \end{cases}$

For $1 \leq j \leq n$ $\quad f(v_j) = \begin{cases} 1 & \text{if } j \equiv 1 (mod\ 2) \\ 2 & \text{if } j \equiv \text{ 2 (mod 2)} \end{cases}$

$f(u_m) = \{3\}$

For $1 \leq i \leq m, 1 \leq k \leq n-2$

$f(u_{ik}) = \begin{cases} 1 & \text{if } i \text{ is odd, k is even and if i is even, k is odd} \\ 2 & \text{if } i \text{ is even, k is even and if i is odd, k is odd} \end{cases}$

For $1 \leq j \leq n, 1 \leq k \leq n-2$

$f(v_{jk}) = \begin{matrix} 1 & \text{if } j \text{ is odd ,k is even and if j is even, k is odd} \\ 2 & \text{if } j \text{ is odd, k is odd and if j is even, k is even} \end{matrix}$

$f(u_m u_1) = \{4\},\ f(u_m v_1) = \{6\},\ f(u_{m-1} u_m) = \{4\}$

For $1 \leq i \leq m$ $\quad f(u_i u_{m-1}) = \{3\}$

For $1 \leq j \leq n$ $\quad f(v_j v_{n-1}) = \{3\}$

For $1 \leq i \leq m-2$ $\quad f(u_i u_{i+1}) = \begin{cases} 3 & \text{if } i \text{ is odd} \\ 4 & \text{if } i \text{ is even} \end{cases}$

For $1 \leq j \leq n-1$ $\quad f(v_j v_{j+1}) = \begin{cases} 3 & \text{if } j \text{ is odd} \\ 4 & \text{if } j \text{ is even} \end{cases}$

For $1 \leq i \leq m$ $\quad f(u_i u_{i1}) = \{5\}$

For $1 \leq i \leq m$ $\quad f(u_i u_{in}) = \{7\}$

For $1 \leq j \leq n$ $\quad f(v_j v_{j1}) = \{5\}$

For $1 \leq j \leq n$ $\quad f(v_j v_{jn}) = \{7\}$

For $1 \leq i \leq m, 1 \leq k \leq n-2$

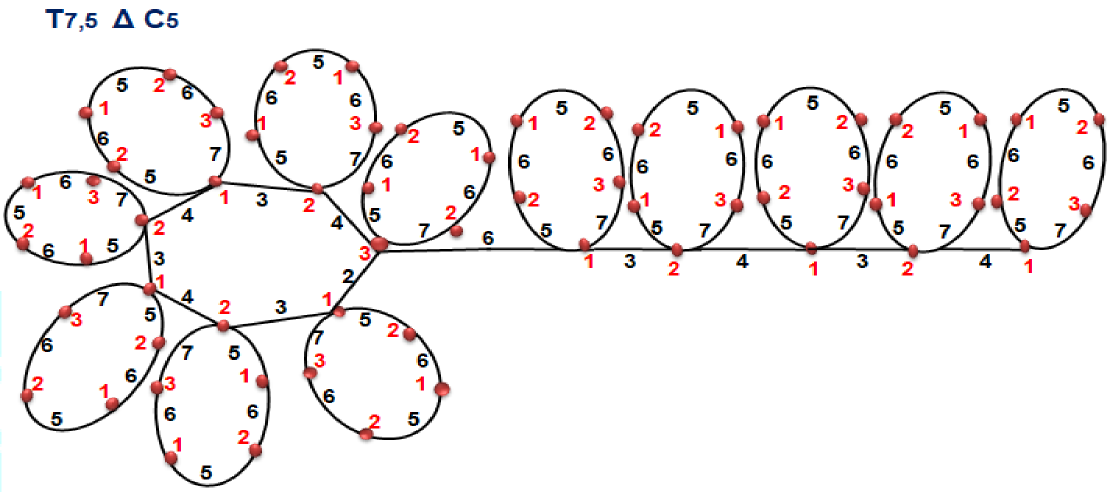
$$f(u_{ik}u_{ik+1}) = \begin{cases} 5 & \text{if } i \text{ is odd ,k iseven and if i is even, k is even} \\ 6 & \text{f } i \text{ is odd, k is odd,and if i is even, k is odd} \end{cases}$$

$For\ 1 \leq j \leq n\ , 1 \leq k \leq n-2$

$$f\left(v_{jk}v_{jk+1}\right) = \begin{cases} 5 & \text{if } j \text{ is odd ,k is even and if j is even, k is even} \\ 6 & \text{f } j \text{ is odd, k is odd and if j is even, k is odd} \end{cases}$$

Hence $\chi_{tc}\left(\boldsymbol{T}_{\boldsymbol{m,n}} \rhd C_n\right)=7$ for $n \geq 2$.

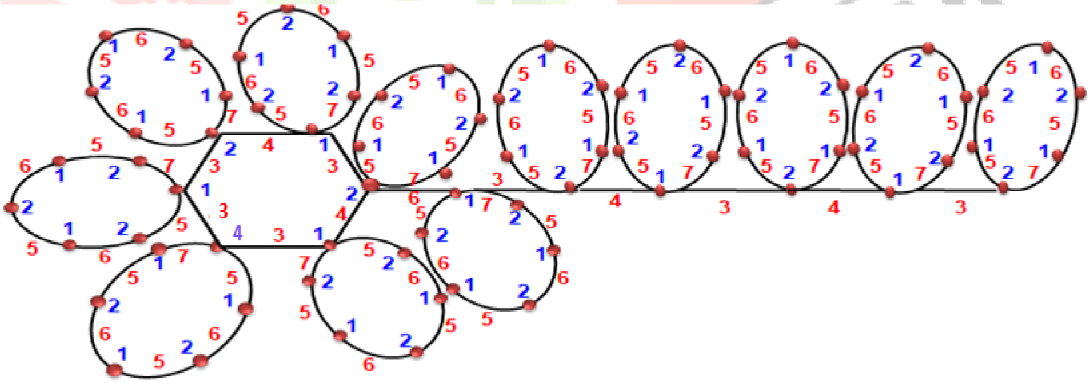
Example:6 Consider the Total chromatic number comb product of Tadpole graph with cycle graph is 7. If m is odd and n is odd



$\chi_{tc}\left(\boldsymbol{T}_{\boldsymbol{7,5}} \rhd C_5\right)=7$

Example: 7

Consider the Total chromatic number comb product of Tadpole graph with cycle graph is 7. If m is even and n is even



$\chi_{tc}\left(\boldsymbol{T}_{\boldsymbol{6,6}} C_6\right)=7$

CONCLUSION:

We obtained the total chromatic number of the Tadpole graph with path, star, cycle, and fan graphs.

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