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"FUNDAMENTAL OF FRACTIONAL DERIVATIVE OPERATORS"

Dr. Kamlesh Kumar Saini [Ph.D., M.Phil., M.Sc. (Maths)]

Head of Department Mathematics

Career Mahavidyalaya, Durana, Jhunjhunu (Raj.) 333041

Affiliated to Pandit Deendayal Upadhyaya Shekhawati University, Sikar (Raj.) India

ABSTRACT

This Research paper focuses on fractional calculus expands the idea of differentiation & integration to non-integer orders. Fractional derivatives provide extra modelling degrees of freedom while integer-order derivatives are well-known and often employed. Applications for fractional derivative operators can be found in many disciplines, including signal processing, image analysis, physics, and engineering. To fully utilise the potential of fractional derivative operators across a range of domains, it is crucial to comprehend their basic notions. The concept of fractional calculus was first introduced in a set of letters sent in 1695. Leibniz responded to L'Hopital's query about what would occur if the order of differentiation were assumed to be 1/2, and he said it seems that these contradictions will eventually have beneficial ramifications.

The symbol $\frac{d^n}{dx^n}$ f(x), created by Leibniz in the late seventeenth century, represents a function's n^{th} derivative, with the conclusion that n ∈ N. De l'Hospital was informed of this and in response, he questioned the importance of the operator if n = 1/ 2. Although n need not be restricted to Q, for the purposes of this paper, n ∈ R applies to all operators in the following text. This branch of mathematics is known as fractional calculus because of the specific questioning of Leibniz's operator in relation to n = 1/ 2 (a fraction).

Keywords : Fractional Order Differential integrals, Fraction Differintegrals, Riemann-Liouville fractional integral; Riemann-Liouville fractional derivative;

INTRODUCTION

I. Definitions of Fractional Order Differential integrals

Fractional operators come in a wide variety of forms today. The previous historical assessment mentioned the derivatives and integrals of Riemann-Liouville, Grunwald-Letnikov, Caputo, Weyl, and Erdely-Kober. Additionally, the majority of those operators can be specified as either the left or right fractional operators, giving rise to two alternative definitions. The "Riemann-Liouville (RL)" and "Grunwald-Letnikov (GL)" formulations are the two that are most usually used to define the general fractional differintegral, respectively. Additionally, the Riemann-Liouville differential operator is widely employed in combination with the Caputo derivative. Next, a brief description of these most popular operators is provided. The following is how Grunwald and Letnikov developed fractional derivative.

$$GL^D D_x^{\alpha} f(x) = \lim_{h\to 0} \frac{(\Delta_h^{\alpha} f(x))}{h^{\alpha}},$$
$$\Delta_h^{\alpha} f(x) = \sum_{0\le|j|<\infty} (-1)^j \binom{\alpha}{j} f(x-jh), h>0, \tag{1}$$

Equation 1 hold true for both fractional derivatives (α> 0) and integrals (α <0), which are typically combined into a single operator known as GL differintegral. If the functions that the GL derivatives and RL derivatives act on are sufficiently smooth, then they are identical. The generalised binomial coefficients are calculated as follows for $\alpha \in R$ and $0k \in N_0$,

$$\binom{\alpha}{j} = \frac{\alpha!}{j!\,(\alpha-j)!} = \frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)}, \binom{\alpha}{0} = 1 \tag{2}$$

Consider the expression $n = t - a/h$, where an is a real constant. Since the derivative operator has non-local features, this constant can be thought of as a lower terminal (an analogue of the lower integration limit). One way to express the GL differential is as a limit.

$$GLD_{a,t}^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh) \tag{3}$$

where [x] denotes the integer portion of x, a and t denote the operation's boundaries for the formula, $GLD_{a,t}^{\alpha}f(t)$. The following connection derived from the GL definition (3) can be used for the numerical estimation of "fractional-order derivatives." The following formula describes the relationship to the precise numerical estimate of the α-th derivative at the positions kh, $(k = 1,2, \ldots.)$

$$(x-L)^{D_x^{\pm\alpha} f(x) \approx h^{\pm\alpha} \sum_{j=0}^{N(x)} b_j^{\pm\alpha} f(x-jh)} \tag{4}$$

If h is the calculation's step size, L is the "memory length,"

$$N(t) = min\left\{\left[\frac{x}{h}\right], \left[\frac{L}{h}\right]\right\} \tag{5}$$

The integer component of [x] is x, while $b_j^{(\pm\alpha)}$ is the binomial coefficient specified by

$$b_0^{(\pm\alpha)} = 1,\ b_j^{(\pm\alpha)} = \left(1 - \frac{1 \pm \alpha}{j}\right) b_{j-1}^{(\pm\alpha)} \tag{6}$$

This approach relies on the finding that the top three definitions —GL, RL, and Caputo's—are identical for an extensive group of functions and under the assumption that every one of the initial circumstances are zero.

We shall take into account the Riemann-Liouville n-fold integral for $n \in N, n > 0$ in order to describe the Riemann-Liouville definition! declared to be (this formula is commonly known as the Cauchy-like repeated integration formula)

$$\int_a^t \int_a^{t_n} \int_a^{t_{n-1}} \ldots \int_a^{t_3} \int_a^{t_2} f(t_1) dt_1 dt_2 \ldots dt_{n-1} dt_n = \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau)\, d\tau \tag{7}$$

For the function $f(t)$ for $\alpha, a \in R$, the fractional Riemann-Liouville integral of the order α can be written as follows. $RLI_a^{\alpha} f(t) \equiv RLD_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$ (8)

The fractional integral is shown as for the case of 0<$\alpha < 1, t > 0$ and f(t) being a causal function of t.

$$RLD_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)/(t-\tau)^{1-\alpha}\, d\tau, \qquad 0 < \alpha < 1, t > 0 \tag{9}$$

Additionally, the definitions of the left the appropriate "Riemann-Liouville fractional integral" are as follows: $RLI_a^{\alpha} f(t) \equiv RLD_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$

$$RLI_b^{\alpha} f(t) \equiv RLD_{b,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-\tau)^{\alpha-1} f(\tau)\, d\tau$$

(10)

(11)

wherever $\alpha > 0, n - 1 < \alpha < n$. It is possible to define the "Riemann-Liouville" fractional integral and the gamma function for any complex order, even orders with only positive real numbers. However, because stability problems, process control, signal processing, and modelling are the book's target application areas, only real-order processes are taken into account. Additionally, the definition of the left "Riemann-Liouville fractional derivative" is

$$RLD_{a,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\ d\tau \quad (12)$$

Hence the definition of the correct "Riemann-Liouville fractional derivative" is

$$RLD_{b,t}^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (t-\tau)^{n-\alpha-1} f(\tau)\ d\tau \quad (13)$$

Here $n - 1 \leq \alpha < n, a, b$ are the last points of the range [a,b] which may also be, $-\infty, \infty$. When the previously stated concept of the left "Riemann-Liouville fractional derivative" is restricted to the highly significant case of $\alpha \in (0,1)$

$$RLD_{a,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} f(\tau)\ d\tau \quad (14)$$

A crucial aspect is that the "Riemann-Liouville derivative" corresponds with the traditional, integer order one for integer values of order.

More specifically,

$$\lim_{\alpha \to (n-1)^+} RLD_{a,t}^{\alpha} f(t) = \frac{d^{n-1} f(t)}{dt^{n-1}} \quad (15)$$

And

$$\lim_{\alpha \to (n)^-} RLD_{a,t}^{\alpha} f(t) = \frac{d^n f(t)}{dt^n} \quad (16)$$

Because the derivative of fractions of an integer is not equal to zero is a highly intriguing characteristic of the fractional derivative. A constant C's RL fractional derivative has the following form:

$$RLD_{a,t}^{\alpha} C = C \frac{(t-\alpha)^{-\alpha}}{\Gamma(1-\alpha)} \neq 0 \quad (17)$$

The well-known and sophisticated theory of mathematics and appropriate needs, including the initial issue of the a fraction different calculation, and the nonzero problem associated with the Riemann-Liouville derivative of a constant, are at odds with definitions of the fractional differentiation of "Riemann-Liouville" type. Caputo made a suggestion to resolve this disagreement. The fractional derivative of the left Caputo is

$$CD_{a,t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\ d\tau \tag{18}$$

Hence the appropriate derivative of fractions of Caputo is

$$CD_{t,b}^{\alpha}f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\ d\tau \tag{19}$$

where $f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n}$ and $n-1 \le \alpha < n \in Z^+$ are present. The definition (19) makes it clear that the fractional Caputo derivatives of the constant is zero. The Caputo derivative fulfils the following restrictions with regard to continuation in relation to the differentiation order:

$$\lim_{\alpha \to (n-1)^+} CD_{a,t}^{\alpha}f(t) = \frac{d^{n-1}f(t)}{dt^{n-1}} = -D^{(n-1)}\,x(a) \tag{20}$$

And

$$\lim_{\alpha \to (n)^-} CD_{a,t}^{\alpha}f(t) = \frac{d^n x(t)}{dt^n} \tag{21}$$

The Riemann-Liouville operator $\mathrm{RLD}_a^n, n \in (-\infty, +\infty)$ is obviously a continuous function of n. With the Caputo derivative, this is not the situation. Since the n-th order derivative must exist, the Caputo derivative is undoubtedly stricter than the "Riemann-Liouville derivative". On the other hand, the Caputo derivative is widely employed in engineering applications, and the starting points of fractional differential equations with Caputo derivative have a definite physical significance. The following formulas relate the left and right "Riemann-Liouville and Caputo fractional derivatives".

$$RLD_{a,t}^{\alpha}f(t) = CD_{a,t}^{\alpha}f(t) + \sum_{k=0}^{n-1}\frac{(-1)^k f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}. \tag{22}$$

$$RLD_{b,t}^{\alpha}f(t) = CD_{b,t}^{\alpha}f(t) + \sum_{k=0}^{n-1}\frac{(-1)^k f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha} \tag{23}$$

II. Properties of Fraction Differintegrals

As previously mentioned, the Riemann-Liouville definition and the "Grunwald-Letnikov" definition of the fractional derivative operator are equivalent for a large class of functions. As a result, only "Riemann-Liouville and Caputo derivatives" will be discussed in this section. Also, the next chapters mainly employ left-side operators. As a result, only this class of fractional operators will be taken into account for all of the features listed below. Accordingly, similar qualities can be developed and demonstrated for the right-sided operators.

The "Riemann-Liouville fractional integral" fulfils the semi-group property, which is true for any positive orders alpha and beta. This is similar to the traditional, integer-order integral.

$$RL\, I^{\alpha}_{ţa}\, RL\, I^{\beta}_{ţa}\, f(t) = RL\, I^{\beta}_{ţa}\, RL\, I^{\alpha}_{ţa}\, f(t) = RL\, I^{\beta+\alpha}_{ţa}\, f(t) \tag{24}$$

Surprisingly, the same is true for derivatives of integer order but not those of fractional order. Let's introduce the notation below.

$$f^{(n-j)}_{n-\alpha}(t) = \left(\frac{d}{dt}\right)^{n-j} RL\, I^{n-\alpha}_{a,t}\, f(t) \tag{25}$$

For instance, combining Riemann-Liouville derivatives yields the following expression.

$$RLD^{\alpha}_{a,t} RLD^{\beta}_{a,t} f(t) = RLD^{\alpha+\beta}_{a,t} f(t) - \sum_{j=1}^{n} \frac{f^{(n-j)}_{n-\beta}(a)}{\Gamma(1-j-\alpha)} (t-a)^{-j-\alpha} \tag{26}$$

n being the lowest integer greater than beta. Consequently, generally

$$RLD^{\alpha}_{a,t} RLD^{\beta}_{a,t} f(t) \neq RLD^{\beta}_{a,t} RLD^{\alpha}_{a,t} f(t) \neq RLD^{\alpha+\beta}_{a,t} f(t) \tag{27}$$

The converse, however, is untrue (for both of fractional and integer order).

$$RLI^{\alpha}_{a,t}\, RLD^{\alpha}_{a,t} f(t) = RLD^{\alpha+\beta}_{a,t} f(t) - \sum_{i=1}^{n} \frac{f^{(n-j)}_{n-\beta}(a)}{\Gamma(1-j-\alpha)} (t-a)^{-j-\alpha} \tag{28}$$

Similar equations linking the "Riemann-Liouville integral" and "derivative of Caputo type" can be generated using expression (22). In particular, the Caputo derivative is also the left inverse of the fractional integral if the integrand is continuous or, at the very least, essentially limited. It is crucial to note that when the initial conditions are zero, the Caputo and Riemann-Liouville formulations overlap. Additionally, the RL derivative makes sense when smoothness criteria are relaxed. In reality, a number of relationships between the "fractional order operators" are significantly simplified when assuming that all initial conditions are zero.

In this scenario, the fractional derivation is both left & right opposite to the fractional integral of the identical order, the fractional integral has the semi-group property, and the operations of fractional differentiation and integration can freely swap places. For any $0 < \alpha < \beta$ in symbolic notation

$$\begin{gathered} RLD^{\alpha}_{a,t} RLD^{\beta}_{a,t} f(t) = RLD^{\beta}_{a,t} RLD^{\alpha}_{a,t} f(t) = RLD^{\alpha+\beta}_{a,t} f(t) \\ RLI^{\alpha}_{a,t}\, RLD^{\alpha}_{a,t} f = RLI^{\alpha}_{a,t}\, RLD^{\alpha}_{a,t} = f(t) \end{gathered} \tag{29}$$

(30)

Probably the most essential formal instruments in the fields of science and engineering is the laplace transform, particularly for modelling dynamical systems. Additionally, the Laplace transform is frequently utilised to resolve fractional integro-differential equations that are a part of many engineering challenges. The RL fractional derivative's Laplace transform L {,} is

$$L\{ RLD_{0,t}^{\alpha} f(t)\} = \int_0^{\infty} e^{-st}\, RLD_{0,t}^{\alpha} f(t)\, dt = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^k\, RLD_{0,t}^{\alpha-k-1} f(t) = 0 \quad (31)$$

The fractional integral (8) of the Riemann-Liouville formula for f(t) has a Laplace transform.

$$L\{ RLI_0^{\alpha} f(t)\} = \frac{1}{s^{\alpha}}\, F(s) \quad (32)$$

When solving fractional differential equations, the beginning conditions that are involved in the formula's terms in the total on the right side (31) must be given. The fractional derivative of Caputo's Laplace transform is

$$\int_0^{\infty} e^{-st}\, CD_{0,t}^{\alpha} f(t)\, dt = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}\, f(k)(0),\; n-1 < \alpha < n \quad (33)$$

It indicates that a set of solely “classical integer-order derivatives” can provide all the prime conditions needed by the fractional differential equation. Also keep in mind that when using fractal-order controls and filters, the presumption of zero beginning conditions is totally reasonable. However, the impact of starting points must be taken into account when trying to replicate a fractional order system. The distinction between numerous definitions of fractional operators cannot be ignored in such a scenario. In addition, Podlubny's work contains geometric & physical explanations of fractional integration & fractional differentiation. The partial differintegral may be exactly expressed through its transfer function if all beginning conditions are equal to zero.

$$G(s) = \frac{1}{s^{\alpha}} \quad (34)$$

This, for negative values of the exponent D, correlates to the fractional derivative and, for positive values, to the fractional integral. One can get the frequency characteristic for fractional operators by entering s = jω into (34) instead. Thus, the essential difference between fractional-order and integer-order systems is made clear. Integer order systems' amplitude characteristics' slope is always has an integer multiple of 20 dB/decade, which is a well-known fact. Contrary to popular belief, “fractional order systems” can, in most cases, have amplitude characteristics with any slope. Similar to this, a fractional order system can have any constant phase but an integer order structure can only have a constant phase if it is a multiple of “pi/2”. As a result, ideal Bode systems are sometimes used to refer to fractional systems.
Figures 1 and 2 show the magnitude and timing features of fraction differintegrals of various orders.

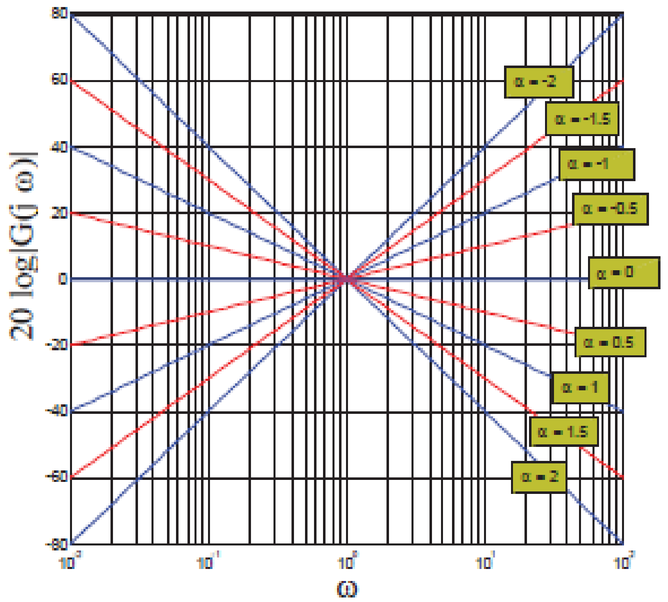


Fig. 1 Few fractional differintegrators' features of logarithmic amplitude (34)

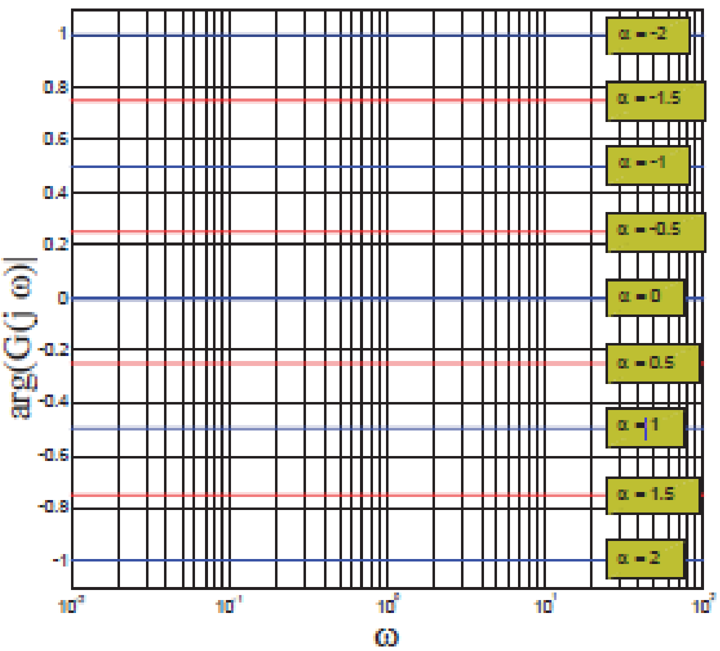
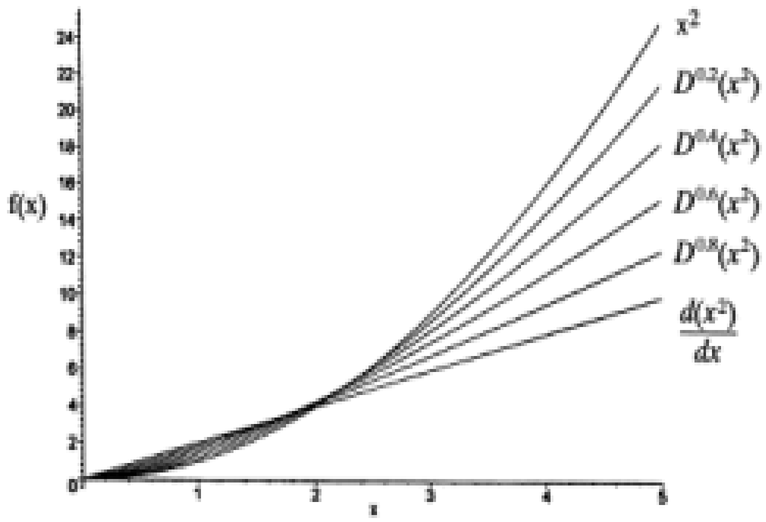


Fig. 2 Few fractional differ integrators' phase characteristics (34).

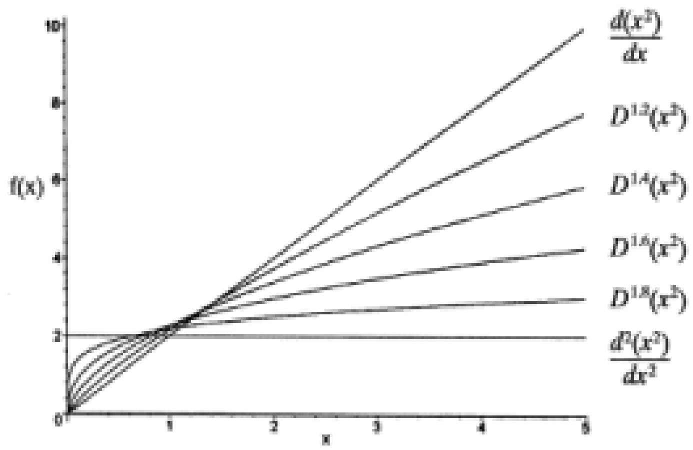
III. Example

The function $f(x) = x^2$ and its derivatives D_{RL}^{α}, which are instances of fractional-order derivatives, are Fig. 3 (By illustrates this)
(a) α = 0.2, 0.4, 0.6, 0.8, 1.0.
(b) α = 1, 1.2, 1.4, 1.6, 1.8, 2.0.

(a)



(b)



CONCLUSIONS

This paper introduced the concept of fractional-order derivative theory, including physics, fluid dynamics, physiological science, health care research, and epidemic illnesses. In several disciplines, fractional operators have developed incomparable benefits over integral operators as information science has advanced. In parameter identification, a fractional derivative of a continuous neural network can significantly enhance estimation accuracy. Theoretical foundations for government can be found in complex behaviours in fractional-order financial systems. Fractional-order control systems outperform classical systems in terms of accuracy and elegance. The properties of partial differential operators, such as "nonlocality," "memorability," and "weak derivatives," are also used in signal processing. These qualities can increase an image's high frequency while keeping its low & medium frequency performance.

As a result of the revolutionary research reported in this paper and the transformational capabilities of fractional derivative operators, picture denoising will undergo a major revolution. The results unmistakably show the advantage of fractional calculus-based techniques in noise reduction while keeping the small details that characterise an image's core.

This study goes beyond the limitations of conventional picture denoising techniques by utilising the enormous potential of fractional order differentiation. The suggested algorithms demonstrate an unmatched capacity to remove noise from a wide range of picture datasets, including extremely complex digital images, bright natural sceneries, and complicated medical imagery.

The findings of this study reevaluate the boundaries of what is feasible in the field of "image denoising".

REFERENCES

1. A.N. Kochubei, Distributed order calculus and equations of ultraslow diffusion. J. Math. Anal. Appl. 340 (2008), 252–281.10.1016/j.jmaa.2007.08.024
2. A.N. Kochubei, General fractional calculus, evolution equations, and renewal processes. Integr. Equa. Operator Theory71 (2011), 583–600.10.1007/s00020-011-1918
3. A.N. Kochubei, General fractional calculus. Chapter in: A. Kochubei, Yu. Luchko (Eds.), Handbook of Fractional Calculus with Applications. Vol.1: Basic Theory, Walter de Gruyter, Berlin/Boston, 2019, 111–126.
4. A.N. Kochubei and Yu. Luchko, Basic FC operators and their properties. Chapter in: A. Kochubei, Yu. Luchko (Eds.), Handbook of Fractional Calculus with Applications. Vol.1: Basic Theory, Walter de Gruyter, Berlin/Boston, 2019, 23–46.
5. M. Kwaśnicki, Fractional Laplace operator and its properties. Chapter in: A. Kochubei, Yu. Luchko (Eds.), Handbook of Fractional Calculus with Applications. Vol.1: Basic Theory, Walter de Gruyter, Berlin/Boston, 2019, 159–194.
6. A.V. Letnikov, Theory of differentiation of arbitrary order. Mat. Sbornik3 (1868), 1–68 (in Russian).
7. Yu. Luchko, J.J. Trujillo, Caputo-type modification of the Erdélyi-Kober fractional derivative. Fract. Calc. Appl. Anal. 10, No 3 (2007), 249–267.
8. S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach, New York (1993).
9. M. Stynes, Fractional-order derivatives defined by continuous kernels are too restrictive. Appl. Math. Lett. 85 (2018), 22–26; 10.1016/j.aml.2018.05.013.
10. V.E. Tarasov, No violation of the Leibniz rule. No fractional derivative. Commun. Nonlinear Sci. Numer. Simul. 18 (2013), 2945–2948; 10.1016/j.cnsns.2013.04.001.
11. V.E. Tarasov, Caputo-Fabrizio operator in terms of integer derivatives: memory or distributed lag? Comput. Appl. Math. 38 (2019), Art. 113, 15; 10.1007/s40314-019-0883-8.