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A STUDY ON NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATION

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Abstract: This paper introduces a solution of partial differential equation using numerical mehods. The numerical results demonstrate that this method is able to provide more accurate numerical approximations in partial differential equation.
Keywords: Partial differential equation & Numerical methods

I. Introduction:

A real-life problem can be perspective as a system. It may agree inputs in a quantitative form (in some sense) and process them though one or more phases (or, using different subsystems), and ultimately produce a quantitative form of output (in the same sense or some other sense). In a biological system, an action accepts input in the form of some signals (such as light, temperature, sound, pressure, smell, touch, etc.) though the sensing devices (such as eye, ear, skin, nose, tongue, etc.). These devices are able to convert those signals to its appropriate (or, equivalent) amount of electrical signals. These then flow through the nervous system from the nerves into the brain.

II. Basic Definition and Terminology:

Definition: Partial differential equations are those which contain one or more partial derivatives, usually with respect to two or more independent variables.

Definition: The order of a partial differential is that of the partial derivatives of highest order in the partial differential equation.

Definition: The degree of a partial differential equation is that of highest degree of the partial derivatives in the partial differential equation, provided the equation can be written in some polynomial form.

Definition: A partial differential equation is said to be linear if its partial derivatives are linearly connected irrespective of the types of coefficients (constants and variables).

Definition: A linear partial differential equation is said to be homogeneous if the partial derivatives involved in the terms of the equation are all of the same order, and otherwise non-homogeneous.

III. Numerical Solutions of Partial Differential Equation:

A general linear second-order partial differential equation:
$a_1\frac{\partial^2 u}{\partial t^2} + a_2\frac{\partial^2 u}{\partial t\partial x} + a_3\frac{\partial^2 u}{\partial x^2} + D\left(t, x, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) = 0$ (3.1) or $a_1 u_{xx} + a_2 u_{xy} + a_3 u_{yy} + a_4 u_x + a_5 u_y + a_6 u + a_7 = 0$ (3.2) can be classified as an elliptic partial differential equation (EPDE) if it satisfies the following condition: $a_2^2 - 4a_1 a_3 < 0$ (3.3).

Illustration:

In the partial differential equation $u_{xx} + u_{yy} = 0$, $a_1 = 1, a_2 = 0, a_3 = 1$, we observe that $a_2^2 - 4a_1a_3 = -4 < 0$. Therefore, $u_{xx} + u_{yy} = 0$ is an elliptic partial differential equation. The widely used elliptic partial differential equations are Laplace equation, potential equation, Poisson's equation, Helmholtz equation etc. A two-dimensional parabolic heat equation is as follows: $\frac{\partial u}{\partial t} = \sigma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$, where $u = u(x, y, t)$ and $t > 0$ (3.4). This equation reduces to an elliptic equation known as Laplace's equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (3.5) If a steady state solution exists, i.e., $\frac{\partial u}{\partial t} = 0$. The Laplace's equation is also closely related to Poisson's equation given by: $\nabla^2 u = g(x, y)(3.6)$, where $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. Laplace's equation arises frequently in mathematical physics and engineering associated with 1. the steady state flow of heat or electricity, 2. the irrotational flow or an incompressible fluid, and potential problems in electricity and magnetism. 3. Elliptical partial differential equations are always boundary value problems. In order to determine a solution of the elliptic partial differential equation over the bounded region Γ, the following information is necessary:

(i) u, (ii) the normal derivative of u, and (iii) combination of u and normal derivation of u, at each part of the boundary $\partial\Gamma$.

Dirichlet problem : A boundary-value problem is said to be Dirichlet problem, if u is specified at all points of the boundary. There exists a unique solution of this problem provided the solution depends on all points of the boundary $\partial\Gamma$.

Neumann problem: A boundary-value problem is said to be Neumann problem, it the outward normal derivative is specified on the boundary $\partial\Gamma$. The solution is only unique to within a constant.

Robin problem: It is also known as third boundary value problem provided the solution of the differential equation exists subject to the following condition on $\partial\Gamma$: $\frac{\partial u}{\partial x} + a(x, y)u = gx, y)$ (3.6). Also, the equation has a unique solution only if $a > 0$. Few standard elliptic partial differential equations are (i) Laplace equation, (ii) Poisson's equation, (iii) Helmholtz equation, etc. described in basic books.

IV. Direct Method:

In this section we discuss different direct methods for the solution of elliptic partial differential equations.

Simple Numerical Methods for Laplace Equation

Solve the Laplace equation $u_{xx} + u_{yv} = 0, (x, y)$ in $\Gamma = \{(x, y): 0 \leq x, y \leq 1\}$ subject to the boundary conditions:

$$u(x, y) = \begin{cases} \psi_1(x), y = 0, & 0 \leq x \leq 1 \\ \psi_2(x), y = 1, & 0 \leq x \leq 1 \\ \phi_1(y), x = 0, & 0 \leq y \leq 1 \\ \phi_2(y), x = 1, & 0 \leq y \leq 1 \end{cases}$$

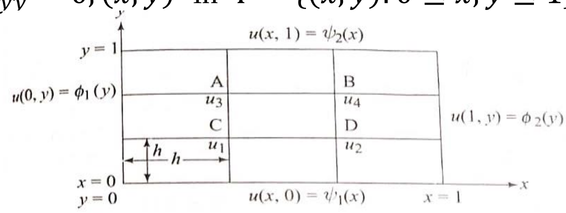


Fig 4.1 The region Γ where $h = 1/3$.

The region Γ is converted by a mesh with uniform spacing $h = 1/3$ (say). This region is shown in Fig. 4.1. By the central difference formula (approximate):

for the derivatives, we can write $u_{xx} + u_{yy} = 0$ as:

$$\frac{u(x + h, y) - 2u(x, y) + u(x - h, y)}{h^2} + \frac{u(x, y + h) - 2u(x, y) + u(x, y - h)}{h^2} = 0$$

Therefore, we get at A: $u_4 - 2u_3 + \phi_1(2h) + \psi_2(h) - 2u_3 + u_1 = 0$

or $u_1 - 4u_3 + u_4 = -\phi_1(2h) - \psi_2(h)$,Similarly,

At B: $u_2 + u_3 - 4u_4 = -\psi_2(2h) - \phi_2(2h)$

At C: $-4u_1 + u_2 + u_3 = -\phi_1(h) - \psi_1(h)$

At D: $u_1 - 4u_2 + u_4 = -\psi_1(2h) - \phi_2(h)$. This system of equations can be written in the following matrix form:

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -\phi_1(h) - \psi_1(h) \\ -\psi_1(2h) - \phi_2(h) \\ -\phi_1(2h) - \psi_2(h) \\ -\psi_2(2h) - \phi_2(2h) \end{bmatrix}$$

Algorithm:

Direct method:

Solve an elliptic partial differential equation by direct method.

Input: Parameters, initial and boundary conditions for the elliptic partial differential equation.

Output: $u(x, y)$ for all (x, y)

Step1: **[Input]** Read parameters for the equation, initial and boundary conditions.

Step 2: [**Generate a system of linear equations]** Construct a system of linear equations using the parameters of the elliptic partial differential equation, and given initial and boundary conditions for each unknown grid point.

Step 3: **[Solution]** Solve this system of linear algebraic equations.

Step 4: **[Output]** Print $u(x, y)$ for all (x, y).

Step 5: **[Termination]** Stop.

Finite Difference Methods:

In this section we discuss two finite difference methods such as
1. regular region with rectangular grids, 2. irregular region with rectangular grids.

Regular Region with Rectangular Grids:

I. Five-point Formula for Laplace 2D Equation:

The five-point approximation formula for the Laplace equation $\nabla^2 u(x, y) = 0$ can be written as: $\nabla^2 u = \frac{1}{h^2}\begin{Bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{Bmatrix} u_{ij}$ (4.1) Here, $\begin{Bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{Bmatrix} u_{ij}$ means $\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}$ convolved with $\begin{bmatrix} & u_{i-1,j} & \\ u_{i,j-1} & u_{i,j} & u_{i,j+1} \\ & u_{i+1,j} & \end{bmatrix}$,

i.e., $1 \times u_{i,j-1} + 1 \times u_{i-1,j} - 4 \times u_{i,j} + 1 \times u_{i+1,j} + 1 \times u_{i,j+1}$ produces
$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}$. Therefore, equation (4.8) is equivalent to
$(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j})/h^2 = 0$. This iterative formula for the computation of $u_{i,j}$ can be written as: $u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})$(4.2)
The iterative method in Equation (4.2) is known as Liebmann's iterative method.

This method provides a diagonally dominant system of linear equation that can be solved by using Gauss-Seidel method.

NOTE:
1. Liebmann,s iterative method computes $u_{i,j}$, the average of the values of u at the four adjoining points of a rectangular region.
2. Initial values for this method may be obtained by any of the following ways: (a) by taking diagonal average or (b) by taking cross average.
3. Five-point stencil of Laplace 2D equation is shown in Fig 4.2.
4. The approximate local error inherent in this five-point formula is $O(h^2)$.
5. Convolve $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ with $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix}$, $i.e,\ \sum_{j=1}^{3}\sum_{i=1}^{3} a_{ij}u_{ij}$

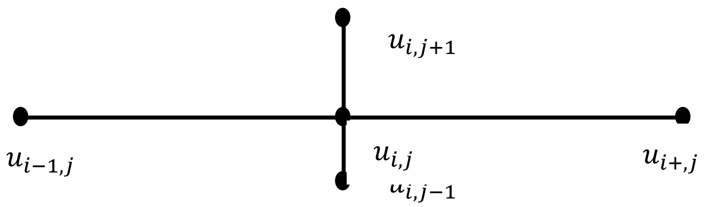


Fig 4.2 Laplace five-point stencil for 2D equation.

II. Sever-point Formula for Laplace 3D Equation

The sever-point approximation formula for the Laplace equation $\nabla^2 u(x, y, z) = 0$ can be written as $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{h^2}(u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}) + \frac{1}{h^2}(u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}) + \frac{1}{h^2}(u_{i,j,k+1} -$

$2u_{i,j,k} + u_{i,j,k-1}) \ \frac{1}{h^2}\begin{Bmatrix} & & 1 & & \\ & & & 1 & \\ 1 & & 6 & & 1 \\ & 1 & & & \\ & & 1 & & \end{Bmatrix} u_{i,j,k} = 0$ (4.10). Therefore, the iterative formula for the computation of $u_{i,j}$ can be written as:

$$u_{i,j,k} = \frac{1}{6}\left(u_{i+1,j,k} + u_{i-1,j,k} + u_{i,j+1,k} + u_{i,j-1,k} + u_{i,j,k+1} + u_{i,j,k-1}\right)(4.3)$$

Seven-point stencil of Laplace 3D equation is shown in Fig. 4.3.

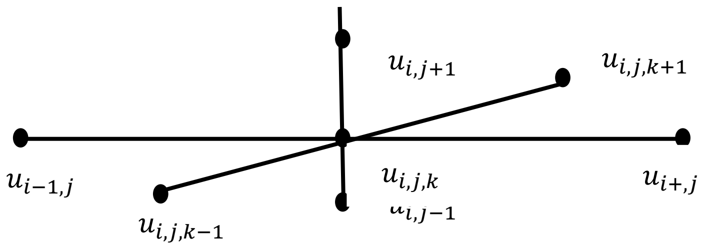


Fig. 4.3 Laplace seven-point stencil for 3D equation.

III. Nine-point Formula for Laplace 2D Equation

The nine-point approximation formula for the Laplace equations $\nabla^2 u(x,y) = 0$ can be written as $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{6h^2}\left(u_{i+1,j-1} + u_{i-1,j-1} + u_{i+1,j+1} + u_{i-1,j-1} + 4u_{i+1,j} + 4u_{i-1,j} + 4u_{i,j+1} + 4u_{i,j-1} - 20u_{i,j}\right) = \frac{1}{6h^2}\begin{Bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{Bmatrix} u_{i,j} = 0$ (4.4). Therefore, the iterative for the computation of $u_{i,j}$ can be written as:

$u_{i,j} = \frac{1}{20}\left(u_{i+1,j-1} + u_{i-1,j-1} + u_{i+1,j+1} + u_{i-1,j+1} + 4u_{i+1,j} + 4u_{i-1,j} + 4u_{i,j+1} + 4u_{i,j-1}\right)(4.5)$. The order of truncation error is $O(h^6)$ using this nine-point formula.

V. Problems on Partial Differential Equation Using Numerical Methods:

Problem: 5.1

Solve $u_{xx} + u_{yy} = 0$ over a unit subject to $u = 0$ at each of the boundary points except for the line segment $y = 0, 0 \leq x \leq 1$ where $u(x,0) = \sin^2 \pi x$.

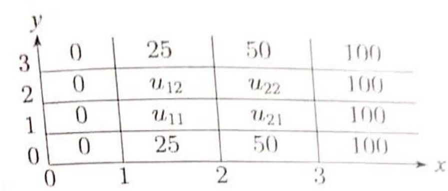
Solution: Here $h = 1/3$, $\phi_1(y) = 0, \phi_2(y) = 0, \psi_1(x) = \sin^2 \pi x, 0 \leq x \leq 1$ and $\psi_2(x) = 0$. Therefore, the system of equations can be written as:

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -\sin^2\left(\frac{\pi}{3}\right) \\ -\sin^2\left(\frac{2\pi}{3}\right) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ -\frac{3}{4} \\ 0 \\ 0 \end{bmatrix}$$

The solution of this system of equations is $u_1 = u_2 = 0.28125$, $u_3 = u_4 = 0.09375$.

Problem: 5.2

Solve $u_{xx} + u_{yy} = 0$ from the given boundary condition as shown below:



Solution:

By five-point formula, we get

At (1,1): $0 + u_{21} + u_{12} + 25 - 4u_{11} = 0$

i.e., $-4u_{11} + u_{12} + u_{21} = -25$ (5.1)

At (2,1): $u_{11} + 100 + u_{22} + 50 - 4u_{21} = 0$

i.e., $u_{11} - 4u_{21} + u_{22} = -150$ (5.2)

At (1,2): $0 + u_{22} + u_{11} + 25 - 4u_{12} = 0$

i.e., $u_{11} - 4u_{12} + u_{22} = -25$ (5.3)

At (2,2): $u_{12} + 100 + 50 + u_{21} - 4u_{22} = 0$

i.e., $u_{12} + u_{21} - 4u_{22} = -150$ (5.4)

After solving the linear equations from Equations (2.1)-(5.4),
we get $u_{11} = 28.125,\ u_{12} = 28.125,\ u_{21} = 59.375, u_{22} = 59.375.$

Therefore, the solution of the system is

0	**25**	**50**	**100**
0	28.125	59.375	100
0	28.125	59.375	100
0	25	50	100

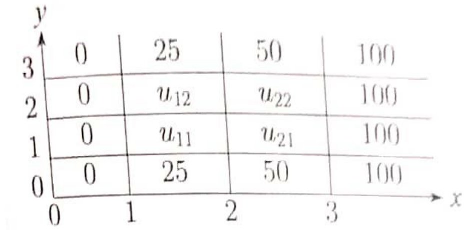
Problem: 5.3

Solve $u_{xx} + u_{yy} = 0$ from the given boundary condition as shown below:

Solution:

By applying five-point iterative formula, we get the solution: $u_{11} = 28.125,\ u_{12} = 28.125,\ u_{21} = 59.375,$ $u_{22} = 59.375.$

The solution of the system after 17 iterations is



0	25	50	100
0	28.125	59.375	100
0	28.125	59.375	100
0	25	50	100

The computational results for 17 iterations are shown in Table 5.1.

TABLE: 5.1

Solution for Problem 5.3.

Itr.	u_{11}	u_{12}	u_{21}	u_{22}
0	0.000	0.000	0.000	0.000
1	6.250	37.500	6.250	37.500
2	17.188	48.438	17.188	48.438
3	22.656	53.906	22.656	53.906
4	25.391	56.641	25.391	56.641
5	26.758	58.008	26.758	58.008
6	27.441	58.691	27.441	58.691
7	27.783	59.033	27.783	59.033
8	27.954	59.204	27.954	59.204
9	28.040	59.290	28.040	59.290
10	28.082	59.332	28.082	59.332
11	28.104	59.354	28.104	59.354
12	28.114	59.364	28.114	59.364

13	28.120	59.370	28.120	59.370
14	28.122	59.372	28.122	59.372
15	28.124	59.374	28.124	59.374
16	28.124	59.374	28.124	59.374
17	28.125	59.374	28.125	59.375

VI. Conclusion:

We have derived the solution of the Laplace's differential equation using numerical methods for direct methods and finite difference method and have explained how the equation is solved by the above mentioned methods. Finally, my future study for this solution of Laplace equation with irregular region.

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