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A STUDY ON APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATION USING CAUCHY INTEGRAL FORMULA WITH LAPLACE TRANSFORM

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Abstract: In this paper we present the possibility of applying the Cauchy integral formula using Laplace transforms in partial differential equations (PDE's).

Keywords: Partial differential equation, Laplace transform & Cauchy Integral Formula.

I. Introduction:

In this paper, we discussed the idea about the solution of partial differential equation using Laplace transform with the help Cauchy integral formula. There are numerous integral transforms that have been developed over the years, many of which are highly specialized. The most versatile of all integral transforms, including the Fourier transform, is the Laplace transform. Laplace transforms date back to the French mathematician Laplace who made use of the transform integral in his work on probability theory in the 1780s. S. D. Poisson (1781-1840) also knew of the Laplace transform integral in the 1820s and it occurred in Fourier's famous 1811 paper on heat conduction. Nonetheless, it was Oliver Heaviside who popularized the use of the Laplace transform as a computational tool in elementary differential equations and electrical engineering. We can define the Laplace transform outright, but it is instructive to formally derive it and its inversion formula directly from the Fourier integral theorem.

II. Basic Definitions & Facts:

Definition: Let $f(t)$ be a piecewise continuous function. where the function $f(t)$ is said to be of exponential order α, if there exists a real and finite positive number M such that
$\lim_{t\to\infty}|f(t)|e^{-\alpha t} \le M$ and we write $|f(t)| = O(e^{\alpha t})$

Definition: Let $f(t)$ be a continuous and single-valued function of the real variable t defined for all t, $0 < t < \infty$, and is of exponential order. Then the Laplace transform of $f(t)$ is defined as $\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$, (2.1)

Fact 2.1: The application of an integral transform to a Partial Differential Equations reduces the independent variables by one.

Fact 2.2: Let $f(z)$ be analytic for $Re(z) \geq \gamma$ where γ is a real constant greater than zero. Then, for $Re(z_0) > \gamma,\ f(z_0) = -\frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} \frac{f(z)dz}{z - z_0}$.

III. Properties of Laplace transform:

Linearity Property: If $f_1(t)$ and $f_2(t)$ are any two functions of t and c is any constant then, $\mathcal{L}[f_1(t) \pm f_2(t)] = \mathcal{L}[f_1(t)] \pm \mathcal{L}[f_2(t)]$ & $\mathcal{L}[cf_1(t)] = c\mathcal{L}[f_1(t)]$.
Shifting Property: If $\mathcal{L}[f(t)] = \bar{f}(s)$,then$\mathcal{L}[e^{-at}f(t)] = \bar{f}(s+a)$&$\mathcal{L}[e^{at}f(t)] = \bar{f}(s-a)$

Change of Scale Property: If $\mathcal{L}[f(t)] = \bar{f}(s)$ then $\mathcal{L}[f(at)] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Laplace transform of derivations: If $\mathcal{L}[f(t)] = \bar{f}(s)$ then
(i) $\mathcal{L}[f'(t)] = s\bar{f}(s) - f(0)$ (ii) $\mathcal{L}[f''(t)] = s^2\bar{f}(s) - sf(0) - f'(0)$

Laplace transforms of integrals: If $\mathcal{L}[f(t)] = \bar{f}(s)$ then $\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{f(s)}{s}$.

Translation Property: If $F(P)$ is the Laplace transform of $f(t)$, then
$\mathcal{L}\{f(t-a)h(t-a); P\} = e^{-aP}F(P), \qquad a > 0$

Periodic Property: Let f be piecewise continuous on $t \geq 0$ and of $O(e^{c_0 t})$. If f is also periodic with period T, then $\mathcal{L}\{f(t); P\} = \left[\frac{1}{1-e^{-PT}}\right] \int_0^T e^{-PT} f(t)\,dt$

IV. Theorems on Laplace Transform:

Initial value theorem: If $f(t)$ is piecewise continuous on $t \geq 0$ and is $O(e^{c_0 t})$. Then its Laplace transform $F(P)$ satisfies $\lim_{|P| \to \infty} PF(P) = \lim_{t \to 0} f(t) = f(0)$.

Final value theorem: If $f(t)$ is piecewise continuous on $t \geq 0$ and is $O(e^{c_0 t})$ and if the integral $\int_0^\infty f'(t)dt$ exists, then the transform $F(P)$ satisfies $\lim_{p \to 0} PF(P) = \lim_{t \to \infty} f(t) = f(\infty)$

Watson's Lemma: If $f(t)$ is $O(e^{c_0 t})$ and if, in some neighborhood of $t = 0$, the function $f(t)$ has the Maclaurin series expansion $f(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n,\ |t| < R$, then the transform function $F(P)$ has the asymptotic series $F(P) \sim \sum_{n=0}^{\infty} \frac{a_n}{pn+1},\ |P| \to \infty$.

Theorems related to property: If $L[f(t); s] = F(s)$, then
$$L^{-1}[F(s+a); t] = e^{-at} L^{-1}[F(s); t].$$

Theorems related to property: If $f(t)$ is a piecewise continuous function and satisfies the condition of exponential order c_0 such that $\operatorname*{Lt}_{t \to 0} f(t)/t$ exists, then for $s > c_0$.
$L^{-1}\left[\frac{F(s)}{s}; t\right] = \int_0^t f(x)\,dx$.

Change of scale property: If $L^{-1}[F(s); t] = f(t)$, then $L^{-1}[F(\alpha s); t] = \frac{1}{\alpha} f\left(\frac{t}{\alpha}\right)$

Theorems related to property: If $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$ respectively, then $F(s)G(s)$ is the Laplace transform of $\int_0^t f(t-u)g(u)\,du$
This integral is called the convolution of f and g and is denoted by the symbol $f * g$.

V. Problems on PDE Using Cauchy Integral Formula with Laplace Transform:

Problem: 5.1

Use Laplace transform method to solve the initial value problem $k\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\ 0<x<1,\ 0<t<\infty$ Subject to the conditions $u(0,t)=0;\ u(l,t)=g(t),\ 0<t<\infty$ and $u(x,0)=0,\ \ 0<x<1.$

Solution:

Applying Laplace transform with respect to t to the given partial differential equation, we get $k\mathcal{L}\left[\frac{\partial u}{\partial t}\right]=\mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right],\ (or)\ \ k[s\bar{u}(x,s)-u(x,0)]=\frac{d^2\bar{u}}{du^2},\ (or)$
$\frac{d^2u}{dx^2}-ks\bar{u}=0,$ as $\bar{u}(x,0)=0$ (5.1). $\bar{u}(x,s)=A\cosh(\sqrt{k}sx)+B\sinh(\sqrt{k}sx)$, (5.2)
Using Laplace transform of BC's, we get $\bar{u}(0,s)=0, \bar{u}(l,s)=\bar{g}(s),$ (5.3)
Using equation (5.3) in (5.2) we get $A=0$. Thus, $\bar{g}(s)=B\sinh(\sqrt{k}sl)$, (or)
$B=\frac{\bar{g}(s)}{\sinh(\sqrt{k}sl)}$. Taking the inverse Laplace transform, we obtain
$u(x,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\bar{g}(s)e^{st}\frac{\sinh(\sqrt{k}sx)}{\sinh(\sqrt{k}sl)}ds$, (5.4). To evaluate the integral on the right-hand side of equation (5.4), we use the method of residues. The poles of the integral are given by $\sinh(\sqrt{k}sl)=0,\ \ (or)e^{\sqrt{k}sl}-e^{-\sqrt{k}sl}=0\ (or),\ \ e^{2\sqrt{k}sl}=1=e^{2n\pi i}$
$(or),\ s_n=-\frac{n^2\pi^2}{kl^2},\ n=0,\pm1,\pm2,\pm3,\dots$. Now, by Cauchy Residue theorem, we have
$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\bar{g}(s)e^{st}\frac{\sinh(\sqrt{k}sx)}{\sinh(\sqrt{k}sl)}ds=\sum R_i$, where $R_i=$ Residue at the i^{th} pole. Here residue at $s=0$, is zero, the residue at $s=s_n$ is

$$\lim_{s\to s_n}\bar{g}(s)\frac{e^{st}\sinh(\sqrt{k}sx)}{\frac{d}{ds}[\sinh\sqrt{k}sl]}=\lim_{s\to s_n}\frac{1\sqrt{s}\bar{g}(s)}{l\sqrt{k}[\cosh(\sqrt{k}sl)]}e^{st}\sinh(\sqrt{k}sx)$$

$$=\frac{2\sqrt{\frac{-n^2\pi^2}{kl^2}}}{l\sqrt{k}\cosh\left(\frac{-n^2\pi^2}{kl^2}l\right)}e^{\frac{n^2\pi^2}{kl^2}}\sinh\left(\sqrt{\left(\frac{-n^2\pi^2}{kl^2}\right)}x\right),\qquad n=1,2,3\dots$$

$$=\frac{2in\pi\bar{g}\left(\frac{-n^2\pi^2}{kl^2}\right)}{kl^2\cos(in\pi)}\sinh\left(\frac{in\pi}{l}x\right)e^{\frac{-n^2\pi^2t}{kl^2}},\qquad n=1,2,3,\dots$$

$\cosh(in\pi)=\cosh(n\pi)$ and $\sinh\left(\frac{in\pi}{l}x\right)=i\sin\left(\frac{n\pi}{l}x\right)$

Thus, the required solution is $u(x,t)=\frac{2\pi}{kl^2}\sum_{n=0}^{\infty}(-1)^n n\bar{g}\left(\frac{n^2\pi^2}{kl^2}\right)\sin\left(\frac{n\pi}{l}x\right)e^{\frac{-n^2\pi^2}{kl^2}t}$.

Problem: 5.2

Find the solution of the boundary value problem given by $\frac{\partial^2 u}{\partial x^2}=\frac{1}{k}\frac{\partial u}{\partial t},\ 0\le x\le a,\ t>0$ Subject to the initial and boundary conditions $u(x,0)=0, u(0,t)=f(t), u(a,t)=0$ by using the Laplace transform method.

Solution:

Taking the Laplace transform of the given partial differential equation, we have
$\frac{d^2\bar{u}}{dx^2}=\frac{1}{k}[s\bar{u}(x,s)-u(x,0)]$. Using the initial condition $u(x,0)=0$, we get
$\frac{d^2\bar{u}}{dx^2}-\frac{s}{k}\bar{u}(x,s)=0$. Its general solution is found to be
$\bar{u}(x,s)=Ae^{\sqrt{(s/k)x}}+Be^{-\sqrt{(S/G)x}}$, (5.5). Now taking Laplace transform of the boundary conditions, we have $\bar{u}(0,s)=\bar{f}(s),$ (5.6), $\bar{u}(a,s)=0,$ (5.7). Using equations (5.6) and (5.7) into (5.5) we get, $\bar{f}(s)=A+B,$ (5.8) and
$\bar{u}(a,s)=A\exp\left[\sqrt{\left(\frac{s}{k}\right)}a\right]+B\exp\left[-\sqrt{\left(\frac{s}{k}\right)}a\right]$, (5.9). Combining equation (5.8) and (5.9), we have
$\bar{f}(s)=A\left[1-e^{2a\sqrt{\frac{s}{k}}}\right]$. $\therefore A=\frac{\bar{f}(s)}{\left(1-e^{2a\sqrt{s/k}}\right)}$ and $B=\bar{f}(s)-\frac{\bar{f}(s)}{1-e^{2a\sqrt{s/k}}}$
$=-\frac{\bar{f}(s)e^{2a\sqrt{s/k}}}{1-e^{2a\sqrt{s/k}}}$ which implies that $A=\bar{f}(s)e^{-a\sqrt{s/k}}\left[e^{-a\sqrt{s/k}}-e^{a\sqrt{s/k}}\right]$

$B = \bar{f}(s)e^{a\sqrt{s/k}}\left[e^{a\sqrt{s/k}} - e^{-a\sqrt{s/k}}\right]$ Therefore, $\bar{u}(x,s) = \bar{f}(s)\frac{\sinh\left[\sqrt{\frac{s}{k}}(a-x)\right]}{\sinh\left[\sqrt{\frac{s}{k}}a\right]}$

Taking inverse Laplace transform, we obtain $u(x,t) = \mathcal{L}^{-1}\left[\bar{f}(s)\frac{\sinh\left[\sqrt{\frac{s}{k}}(a-x)\right]}{\sinh\left[\sqrt{\frac{s}{k}}a\right]}\right]$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}\bar{f}(s)\sinh\left[\sqrt{\frac{s}{k}}(a-x)\right]}{\sinh\left[\sqrt{\frac{s}{k}}a\right]}ds.$$

Problem: 5.3

Using Laplace transform method, solve the initial boundary value problem $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \cos wt$, $0 \le x \le \infty,\ 0 \le t \le \infty$. Subject to the initial and boundary conditions $u(0,t) = 0,\ u$ is bounded as $x \to \infty\ \frac{\partial u}{\partial t}(x,0) = u(x,0) = 0$.

Solution:

Taking the Laplace transform of the partial differential equation, we obtain $\frac{d^2\bar{u}}{dx^2} = \frac{1}{c^2}\left[s^2\bar{u}(x,s) - su(x,0) - \frac{\partial u(x,0)}{\partial t}\right] - \frac{s}{s^2+w^2}$. Using the initial conditions, we get $\frac{d^2\bar{u}}{dx^2} - \frac{s^2}{c^2}\bar{u}(x,s) = -\frac{s}{s^2+w^2}$, (5.10). The general solution of (5.10) is found to be $\bar{u}(x,s) = Ae^{(s/c)x} + Be^{-(s/c)x} + \frac{c^2}{s(s^2+w^2)}$, (5.11). As $x \to \infty$, the transform should also be bounded which is possible if $A = 0$, thus $\bar{u}(x,s) = Be^{-sx/c} + \frac{c^2}{s(s^2+w^2)}$. Taking Laplace transform of the boundary conditions, we get $\bar{u}(0,s) = 0$. Using this result in equation (5.11),We have $B = -\frac{c^2}{s(s^\wedge+w^2)}$. Hence, $\bar{u}(x,s) = \frac{c^2}{s(s^2+w^2)}\left[1 - e^{-(s/c)x}\right]$

Taking its inverse Laplace transform, we get $u(x,t) = c^2\mathcal{L}^{-1}\left[\frac{1}{s(s^2+w^2)}\right] - c^2\mathcal{L}^{-1}\left[\frac{e^{(s/c)x}}{s(s^2+w^2)}\right]$ $= \frac{c^2}{w^2}(1-\cos wt) - \frac{c^2}{w^2}\left[\left\{1 - \cos w\left(t - \frac{x}{c}\right)\right\}H\left(t - \frac{x}{c}\right)\right]$. where H is the Heaviside unit function.

Problem: 5.4

Solve the initial boundary value problem $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},\ 0 < x < 1,\ t > 0$

Subject to the initial and boundary conditions $u(x,0) = \sin\pi x, \frac{\partial u}{\partial t}(x,0) = -\sin\pi x,\ 0 < x < 1$ and $u(0,t) = u(1,t) = 0,\ t > 0$.

Solution:

Taking the Laplace transform of the partial differential equation, we get $\frac{d^2\bar{u}}{dx^2} = s^2\bar{u}(x,s) - su(x,0) - \frac{\partial u}{\partial t}(x,0)$. Using the initial conditions, this equation becomes, $\frac{d^2\bar{u}}{dx^2} - s^2\bar{u} = (1-s)\sin\pi x$, its general solution is given by $\bar{u}(x,s) = Ae^{sx} + Be^{-sx} + \frac{(s-1)\sin\pi x}{\pi^2+s^2}$, (5.12). The Laplace transform of the boundary conditions gives $\bar{u}(0,s) = 0 = \bar{u}(1,s)$ using these equations (5.12), we find that $A = 0 = B$. Hence we obtain $\bar{u}(x,s) = \frac{(s-1)\sin\pi x}{\pi^2+s^2}$. Taking the inverse Laplace transform, we get $u(x,t) = \sin\pi x\,\mathcal{L}^{-1}\left(\frac{s-1}{\pi^2+s^2}\right) = \sin\pi x\left(\cos\pi x - \frac{\sin\pi x}{\pi}\right)$. Hence the required solution of the given boundary value problem is $u(x,t) = \sin\pi x\left(\cos\pi x - \frac{\sin\pi x}{\pi}\right)$

Problem: 5.5

A string is stretched and fixed between two points $(0,0)$ and $(l,0)$. Motion is initiated by displaying the string in the form $u = \sin\left(\frac{\pi x}{l}\right)$ and released from rest at time $t = 0$. Find the displacement of any point on the string at any time t.

Solution:

The displacement $u(x,t)$ of the string is governed by
$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < l,\ t > 0$. Subject to the conditions
$u(x,0) = \sin\left(\frac{\pi x}{l}\right), \frac{\partial u}{\partial t}(x,0) = 0\ u(0,t) = u(l,t) = 0$. Taking the Laplace transform of the given partial differential equation we get
$s^2\bar{u}(x,s) - su(x,0) - \frac{\partial u}{\partial t}(x,0) = c^2 \frac{d^2\bar{u}}{dx^2}$. Using the initial conditions, we get
$\frac{d^2\bar{u}}{dx^2} - \frac{s^2}{c^2}\bar{u} = -\frac{s}{c^2}\sin\frac{\pi x}{l}$. Its general solution is given by
$\bar{u}(x,s) = Ae^{sx/c} + Be^{-sx/c} + \frac{s\sin\frac{\pi x}{l}}{s^2+\pi^2c^2/l^2}$, (5.13). The Laplace transform of the boundary conditions is given by $\bar{u}(0,s) = 0, \bar{u}(l,s) = 0$. Applying these conditions in equation (5.13), we get $A + B = 0$. $A^{sl/c} + B^{-sl/c} = 0$. Solving these equations, we obtain $A = B = 0$. Thus $\bar{u}(x,s) = \frac{s\sin(\pi x/l)}{s^2+\pi^2c^2/l^2}$. Taking inverse of the Laplace transform we get
$u(x,t) = \cos\left(\frac{\pi c}{l}t\right)\sin\frac{\pi x}{l}$.

Problem: 5.6

Find the solution of the partial differential equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},\ 0 < x < \infty$, $t > 0$. Subject to the initial and boundary conditions $u(x,0) = 0, u(x,t) \to 0,\ \text{as} x \to \infty$
$u(0,t) = g(t)$.

Solution:

Taking the Laplace transform of the partial differential equation, we have
$k\frac{d^2\bar{u}}{dx^2} = s\bar{u}(x,s) - \bar{u}(x,0)$. Using the initial condition $u(x,0) = 0$, we get
$k\frac{d^2\bar{u}}{dx^2} - \frac{s}{k}\bar{u} = 0$. Its general solution is given by
$\bar{u}(x,s) = A\exp\left(\sqrt{\frac{s}{k}}x\right) + B\exp\left(-\sqrt{\frac{s}{k}}x\right)$. The Laplace transform of the first boundary conditions gives, $\bar{u} \to 0,\ as\ x \to \infty$ and $\bar{u}(0,s) = \bar{g}(s)$. Using these in the solution, we get $A = 0, B = \bar{g}(s)$.therefore, $\bar{u}(x,s) = \left\{\bar{g}(s)\exp\left(-\sqrt{\frac{s}{k}}x\right)\right\}$. Taking the inverse Laplace transform, we obtain $u(x,t) = \mathcal{L}^{-1}\left\{\bar{g}(s)\exp\left(-\sqrt{\frac{s}{k}}x\right)\right\}$
$\mathcal{L}^{-1}\left\{\mathcal{L}(g(t))\mathcal{L}\left[\frac{x}{2\sqrt{k\pi t^3}}\exp\left(-\frac{x^2}{4kt}\right)\right]\right\}$, $\because \mathcal{L}^{-1}\left\{\exp\sqrt{\frac{s}{k}}x = \frac{x}{2\sqrt{k\pi t^3}}\exp\left(-\frac{x^2}{4kt}\right)\right\}$
Upon using convolution theorem, we arrive at the result

$$u(x,t) = \int_0^t \frac{x\exp[-x^2/4kt(t-u)]}{2\sqrt{\pi k}(t-u)^{3\backslash 2}}\bar{g}(u)du$$

Problem: 5.7

Using the Laplace transform method, solve $\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$ Subject to the initial and boundary conditions $u(x,0) = 30\cos 5x, u\left(\frac{\pi}{2},0\right) = 0, \frac{\partial u}{\partial x}(0,t) = 0$.

Solution:

Taking the Laplace transform of partial differential equation, we have
$s\bar{u}(x,s) - u(x,0) = 3\frac{d^2\bar{u}}{dx^2}$. Using the initial condition, we get $\frac{d^2\bar{u}}{dx^2} - \frac{s}{3}\bar{u} = -10\cos 5x$
Its general solution is given by $\bar{u}(x,s) = Ae^{\sqrt{\frac{s}{3}}x} + Be^{-\sqrt{\frac{s}{3}}x} + \frac{30\cos 5x}{75+s}$. Taking Laplace transform of boundary condition and using in the solution, we obtain $A = B = 0$. Thus $\bar{u}(x,s) = \frac{30\cos 5x}{75+s}$. Taking inverse Laplace transform, we get
$u(x,t) = \mathcal{L}^{-1}\left(\frac{30\cos 5x}{75+s}\right) = 30\,e^{-75t}\cos 5x$

VI. Conclusion:

In this paper, We have checked the solution of wave equation and $\mathcal{PDE}$ using Laplace transform. I have learned more ideas from this concept. In real life, we have seen more ideas can be solving with Laplace transform. Laplace transform can be playing major role inPhysics and Computer science. From this paper, I will plan to show my future research idea in this platform with the help of Laplace transform This paper work is very useful to my upcoming research work.

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