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# A BRIEF STUDY OF PROJECTIVE BOUNDED SETS, STRONG PROJECTIVE CONVERGENCE, STRONG PROJECTIVE CONTINUITY

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## 1.ABSTRACT:

In this chapter projective-bounded set has been defined. A necessary and sufficient condition has been given when a set $X$ is $\sigma_p(f) - \text{bounded}$. We also define strong projective convergence and strong projective continuity. Some theorems have been established concerning these notions. We have also considered a case of family of functions when projective convergence and strong projective convergence coincide.

Keyword: Norm, projective bounded set, strong projective convergence and strong projective continuity.

## 2. Definitions:

We use the idea of 'Norm' for the study of the subject-matter suggested by the heading of this chapter, so we begin with the definition of 'Norm'.

A linear space $L$ is said to be normed[1] if to each element $f \in L$ there corresponds a non-negative number $\|f\|$ which is called the **norm** of $f$, and is such that

(i) $\|f\| \geq 0$ for each $f \in L$

[1]Cooke, R.G. "Linear operator" Macmillan, London 1953, p-35

(ii) $\|f\| = 0$ if and only if $f = 0$

(iii) $\|af\| = |a|\,\|f\|$, for every scalar $a$, and

(iv) $\|f+g\| \leq \|f\| + \|g\|$ for every $f, g \in L$

In $\sigma_p(f), 1 \leq p < \infty$, the norm is given by

$$\|f\|_p = \left(\int_0^\infty |f(x)|^p\,dx\right)^{\frac{1}{p}}$$

In $\sigma_\infty(f)$, the norm is given by

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in[0,\infty)} |f(x)|$$

**3.<u>Projective bounded sets:</u>**

If $\alpha^*(f) \geq \beta(f)$, and if $\left|\int_0^\infty f(x)g(x)dx\right| \leq K(g)$

for every $f$ in a set $X$ in $\alpha(f)$ and every $g$ in $\beta(f)$, where $K(g)$ is a positive constant depending on $g$, then we say that the set $X$ is **projective-bounded** (p-bd) relative to $\beta(f)$, or $\alpha(f)\beta(f)-bd$. When $\beta(f) = \alpha^*(f)$, we say that $X$ is $p-bd$ in $\alpha(f)$, or $\alpha(f)\beta(f)-bd$. [cf. Infinite Matrices and Sequence Spaces, 293]

if $\alpha^*(f) \geq \beta(f)$, and we take a set $X$ in $\alpha(f)$ to be the family $f_\lambda(x)$ with $\lambda \in [0,\infty)$, then we say that $f_\lambda(x)$ is $\alpha(f)\beta(f)-bd$ if

$$\left|\int_0^\infty f_\lambda(x)g(x)dx\right| \leq K(g) \qquad \ldots\ldots \quad (4.30)$$

for all $\lambda \in [0,\infty)$ and every $g$ in $\beta(f)$.

It is obvious from the definition of $\alpha(f)\beta(f)-\text{convergence}$ given in 3.2, and that of $\alpha(f)\beta(f)-\text{boundedness}$ given above, that $\alpha(f)\beta(f)-\text{convergence}$ does not necessarily imply $\alpha(f)\beta(f)-\text{boundedness}$, since even if

$$\int_0^\infty f_\lambda(x)g(x)dx$$

tends to finite limit as $\lambda \to \infty$,if may not be necessarily bounded as $\lambda$ runs through $[0,\infty)$. But, unlike this, in the case of sequence spaces (see Infinite Matrices and Sequence Space, 293)

$\alpha\beta -$ convergence always implies $\alpha\beta -$ boundedness.

**Theorem (3.I):** A set $X$ is $\sigma_p(f) - bd$ if and only if, $\|f\|_p \leq M$ for every $f$ in $X$, where $1 \leq p \leq \infty$.

**Proof:** Let $f \in X$ in $\sigma_p(f)$, and $g$ be any function in $\sigma_p^*(f) = \sigma_q(f)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

That the condition is sufficient follows at once from Holder's inequality

$$\left|\int_0^\infty f(x)g(x)dx\right| \leq \|f\|_p \cdot \|g\|_q$$

We now prove the necessity of the condition.

If $X$ is $\sigma_p(f) - bd$, then

$$|F_f(g)| = \left|\int_0^\infty f(x)g(x)dx\right| \leq K(g) \qquad \ldots\ldots (3.1)$$

for every $f$ in $X$ and every $g$ in $\sigma_q(f)$. $F_f(g)$ is a number for every $f \in X$ in $\sigma_p(f)$ and every $g$ in $\sigma_q(f)$,

therefore, $\|F_f(g)\| = |F_f(g)|$, and so, by equation (3.1)

$$\|F_f(g)\| \leq K(g) \qquad \ldots. (3.2)$$

for each $g$ in $\sigma_q(f)$, as $f$ runs through the set $X$.

Hence it follows from equation (3.2), by the Banach-Steinhaus Theorem on uniform boundedness. [see Cooke, (1), 319-20, Zaanen, (1), 135, or Banach and Steinhaus, (1)] that

$$\|F_f(g)\| \leq M \cdot \|g\|_q \qquad \ldots. (3.3)$$

for every $f$ in $X$ and every $g$ in $\sigma_q(f)$.

Now, for each $f$ in $\sigma_q(f)$,

$$F_f(g) = \int_0^\infty f(x)g(x)dx$$

is a bounded linear functional on $\sigma_q(f)$.

So, it follows from equation (3.3), by definition of the norm of a bounded linear functional [see Cooke, (2), 350, or Zaanen, (1), 138], that

$$\|F_f\| \leq M \qquad \ldots. (3.4)$$

for every $f$ in $X$ ; and also, for each $f$ in $X$ we have

$$\|F_f\| = \sup|F_f(g)|$$

for all $g$ in $\sigma_q(f)$ with $\|g\|_q \leq 1$ ; i.e.,

$$\|F_f\| = \sup\left|\int_0^\infty f(x)g(x)dx\right| \quad .... \quad (3.5)$$

for all given $\sigma_q(f)$ with $\|g\|_q \leq 1$.

But,

$$\|f\|_p = \sup\left|\int_0^\infty f(x)g(x)dx\right| \quad .... \quad (3.6)$$

for all $g$ in $\sigma_q(f)$ with $\|g\|_q \leq 1$.

Hence, by equations (3.5) and (3.6), we have

$$\|F_f\| = \|f\|_p \quad .... (3.7)$$

Therefore, by equations (3.4) and (3.7),

$$\|f\|_p \leq M$$

for every $f$ in $X$, and thus the condition is necessary.

In $\sigma_q(f)$ , $1 \leq p < \infty$

$$\|f\|_p = \left(\int_0^\infty |f(x)|^p dx\right)^{\frac{1}{p}}$$

and in $\sigma_q(f)$,

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in[0,\infty)}|f(x)|$$

Hence, we have

**Corollary 1:** If $1 \leq p < \infty$, a set $X$ is $\sigma_q(f) - bd$ if and only if,

$$\int_0^\infty |f(x)|^p dx \leq M^p$$

for every f in X. [cf. Infinite Matrices and Sequence Space, 298(10.4.IV), and 299, (10.4.V)]

**corollary 2:** A set X is $p-bd$ in $\sigma_q(f)$ if and only if, $|f(x)| \le M$ for almost all $x \ge 0$ and every f in X. (M is the same for all the functions $f(x)$ in the set X). [cf. Infinite Matrices and Sequence Space, 298, (10.4.III)].

## 4. Strong projective convergence and strong projective continuity

If $\alpha^*(f) \ge \beta(f)$, and $f_\lambda(x)$ in $\alpha(f)$ satisfies the condition that to every $\varepsilon > 0$ and every $p-bd$ set $\mho$ in $\beta(f)$ corresponds a positive number $N(\varepsilon, \mho)$ such that

$$\left|\int_0^\infty g(x)\{f_\lambda(x) - f_{\lambda'}(x)\}dx\right| \le \varepsilon \qquad \dots.(4.1)$$

for every g in $\mho$ and all $\lambda, \lambda' \ge N(\varepsilon, \mho)$, then $f_\lambda(x)$ is said to be **strong projective convergent** ($\underline{p}-cgt$) relative to $\beta(f)$, or $\underline{\alpha(f)\beta(f)}-cgt$ when $\beta(f) = \alpha^*(f)$, we say that $f_\lambda(x)$ is $\underline{p}-cgt$ in $\alpha(f)$, or $\underline{\alpha(f)}-cgt$. [cf. Infinite Matrices and Sequence Space, 302].

If $\alpha^*(f) \ge \beta(f)$, and $f_\lambda(x)$ in $\alpha(f)$ satisfies the condition that to every $\varepsilon > 0$ and every $p-bd$ set $\mho$ in $\beta(f)$ corresponds a positive number $\delta(\varepsilon, \mho)$ such that equation (4.40) holds for every g in $\mho$ and all non-negative $\lambda$ and $\lambda'$ such that $|\lambda - \lambda'| < \delta(\varepsilon, \mho)$, then $f_\lambda(x)$ is said to be **strong projective continuous** ($\underline{p}-continuous$) relative to $\beta(f)$, or $\underline{\alpha(f)\beta(f)}-continuous$ when $\beta(f) = \alpha^*(f)$, we say that $f_\lambda(x)$ is $\underline{p}-continuous$ in $\alpha(f)$, or $\underline{\alpha(f)}-continuous$.

By taking $\mho$ to consist of one function only, we see from the definitions that $\underline{\alpha(f)\beta(f)}-convergence$ implies $\alpha(f)\beta(f)-convergence$, and that $\underline{\alpha(f)\beta(f)}-continuity$ implies $\alpha(f)\beta(f)-continuity$.

**Theorem (4.I):** Every parametric convergent family in $\sigma_\infty(f)$ is $\underline{p}-cgt$ in $\sigma_\infty(f)$.

**Proof:** We have $\alpha^*_\infty(f) = \sigma_1(f)$. Let $\mho$ be a $p-bd$ set in $\sigma_1(f)$; then by theorem (3.I), cor.1,

$$\int_0^\infty |g(x)|dx \le M \qquad \dots.\ (4.2)$$

for every g in $\mho$.

Let $f_\lambda(x)$ belong to $\sigma_\infty(f)$ and be parametric convergent $(\lambda - cgt)$, then corresponding to every $\varepsilon > 0$, we can choose a positive number $N(\varepsilon)$ such that, for almost all $x \ge 0$

$$|f_\lambda(x) - f_{\lambda'}(x)| \leq \frac{\varepsilon}{M} \qquad \ldots.(4.3)$$

for all $\lambda, \lambda' \geq N$.

Hence by equations (4.2) and (4.3),

$$\left|\int_0^\infty g(x)\{f_\lambda(x) - f_{\lambda'}(x)\}dx\right| \leq \frac{\varepsilon}{M} \cdot \int_0^\infty |g(x)|dx \leq \varepsilon$$

for every $g$ in $\mho$ and all $\lambda, \lambda' \geq T(\varepsilon)$, for every $\varepsilon > 0$

therefore $f_\lambda(x)$ is $\underline{p} - cgt$ in $\sigma_\infty(f)$.

**Theorem (4.II):** Every family $f_\lambda(x)$ in $\sigma_\infty(f)$, such that $f_\lambda(x) \in \zeta_1(f)_\lambda$, is $\underline{\sigma_\infty(f)} - \text{continuous}$.

Then proof is similar to that of (4.4.I).

**Theorem (4.III):** For every parametric convergent family in $\sigma_1(f), p - \text{convergence}$ and $\underline{p} - \text{convergrence}$ coincide. [cf., Infinite Matrices and Sequence Space, 304, (10.5, II)].

**Proof:** We have $\alpha_1^*(f) = \sigma_\infty(f)$. If $\mho$ is a $p - bd$ set in $\sigma_\infty(f)$, then, by theorem (3.I), cor 2,

$$|g(x)| \leq M \qquad \ldots. \ (4.4)$$

for every $g$ in $\mho$ and almost all $x \geq 0$.

Let $f_\lambda(x)$ belong to $\sigma_1(f)$ and be $\lambda - cgt$, and suppose that it is $\sigma_1(f) - cgt$. Then, since $\sigma_\infty(f)$ is normal and $f_\lambda(x)$ is $\lambda - cgt$, we can takingg$(x) = 1$ for all $x \geq 0$, for every $\varepsilon > 0$ determine a positive number $N(\varepsilon)$ such that

$$\int_0^\infty |f_\lambda(x) - f_{\lambda'}(x)|dx \leq \frac{\varepsilon}{M} \qquad \ldots. \ (4.5)$$

for all $\lambda, \lambda' \geq N$

Hence by equations (4.4) and (4.5),

$$\left|\int_0^\infty g(x)\{f_\lambda(x) - f_{\lambda'}(x)\}dx\right| \leq M \int_0^\infty |f_\lambda(x) - f_{\lambda'}(x)|dx \leq \varepsilon$$

for every $g$ in $\mho$ and all $\lambda, \lambda' \geq T$.

Therefore $f_\lambda(x)$ is $\underline{\sigma_1(f)} - cgt$.

Also $\underline{p}-\text{convergence}$ implies $p-\text{convergence}$.

Hence the result follows.

**Theorem (4.IV):** For every family $f_\lambda(x)$ in $\sigma_1(f)$, such that $f_\lambda(x) \in \zeta_1(f)_\lambda$, $p-\text{continuity}$ and $\underline{p}-\text{continuity}$ coincide.

The proof is similar to that of (4.III)

**Theorem (4.V):** when $\beta(f)$ is normal and is such that sections of functions in it belong to it, then the necessary and sufficient condition that $f_\lambda(x)$ in $\alpha(f)$ should be $\underline{\alpha(f)\beta(f)}-\text{cgt}$ is that, to every $\varepsilon > 0$ and every $p-\text{bd}$ set $\mho$ in $\beta(f)$, these corresponds a positive number $N(\varepsilon, \mho)$ such that

$$\int_0^\infty |g(x)\{f_\lambda(x) - f_{\lambda'}(x)\}|dx \leq \varepsilon$$

for every $g$ in $\mho$ and all $\lambda, \lambda' \geq N$. [cf. Infinite Matrices and Sequence Space, 303, (10.5.I)].

**Proof:** It follows from the definition of $\underline{p}-\text{convergence}$ that the condition is sufficient.

To prove that it is necessary, we construct the sections

$$h_r(x) = \begin{cases} g(x) & \text{for } 0 \leq x \leq r \\ 0 & \text{for } x > r \end{cases}$$

of every $g$ in $\mho$, and consider the set $V$ which consists of all functions $\theta_r(x)$ such that

$$|\theta_r(x)| = |h_r(x)|, \text{for all } x \geq 0$$

By hypothesis, $\beta(f)$ is such that sections of functions in it belong to it, so $h_r(x) \in \beta(f)$; also $\beta(f)$ is normal, therefore

$\theta_r(x) \in \beta(f)$ for all $\omega(x)$ in $\beta^*(f)$,

$$\left|\int_0^\infty \theta_r(x)\omega(x)dx\right| \leq \int_0^\infty |\theta_r(x)\omega(x)|dx$$

$$= \int_0^\infty |h_r(x)\omega(x)|dx$$

$$= \int_0^\infty |g(x)\omega(x)|dx$$

$$\leq \int_0^\infty |g(x)\omega(x)|dx$$

for every $r > 0$; hence $V$ is $p - bd$ in $\beta(f)$.

Consequently, if $f_\lambda(x)$ is $\underline{\alpha(f)\beta(f)} - cgt$

$$\left|\int_0^\infty \theta_r(x)\{f_\lambda(x) - f_{\lambda'}(x)\}dx\right| \leq \varepsilon$$

for all $\lambda, \lambda' \geq N(\varepsilon, \mho)$ for every $\varepsilon > 0$.

Given any two fixed number $\lambda, \lambda' \geq N$,

We choose the signs of $\theta_r(x)$ so that $\theta_r(x)\{f_\lambda(x) - f_{\lambda'}(x)\}$ is non-negative.

Then we have, for all $\lambda, \lambda' \geq N(\varepsilon, \mho)$,

$$\int_0^r |g(x)\{f_\lambda(x) - f_{\lambda'}(x)\}|dx \leq \varepsilon$$

for every $r > 0$ and every $g$ in $\mho$; the result thus follows.

In (4.V), unlike the case of sequence spaces [see Infinite Matrices and Sequence Space, 303, (10.5, I)], we have to make the additional hypothesis that $\beta(f)$ is normal and is such that sections of functions in it belong to it, because sections of a sequence always belong to the sequence space $\phi$, where as the sections of a function do not necessarily belong to the function space $\phi(f)$, since a section of a function is not necessarily essentially bounded, but functions in $\phi(f)$ are, by definition essentially bounded.

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