IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

SOLUTION OF THE EXPONENTIAL DIOPHANTINE EQUATION $25^x + 483^y = z^2$

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ABSTRACT: The Diophantine equation $25^x + 483^y = z^2$, where x, y, z are non-negative integers, was examined by the authors in this paper and showed that $(x, y, z) = (0, 1, 22)$ is the unique non-negative integer solution of this Diophantine equation.

KEY WORDS: Exponential Diophantine equation; Congruence Modulo.

MATHEMATICS SUBJECT CLASSIFICATION: 11D61, 11D72.

1. INTRODUCTION

Diophantine equation is a key component of number theory, which is an intriguing area of mathematics. A vast area of mathematics known as "Theory of numbers" is connected to several other mathematical disciplines.

The field of computer science has many applications for number theory. Among many other challenges, number theorists use computers to test conjectures, factor big integers, and determine primes. The goal of any Diophantine equation is to solve for all of the unknowns. Aggarwal et al. [1-3] find the non-negative integer solution of the Diophantine equations, $143^x + 45^y = z^2$, $143^x + 85^y = z^2$ and $143^x + 485^y = z^2$.

In [4], a conjecture was formulated by Catalan that the Diophantine equation $a^x - b^y = 1$, where $a, b, x, y \in Z$ has unique solution $\{a = 3, b = 2, x = 2, y = 3\}$ under condition $\min\{a, b, x, y\} > 1$. In [5], Gupta and Kumar discussed the exponential Diophantine equation $n^x + (n + 3m)^y = z^{2k}$, where n is a number of the form $6r + 1$ and $x, y, z, m, k, r \in W$, here W is the set of whole numbers. In [6], Gupta and Kumar discussed the exponential Diophantine equation $a^u + (a + 5b)^v = c^{2w}$, where a is a number of the form $5r + 1$ and $u, v, w, b, c, r \in W$.

In [7], Gupta and Kumar discussed the exponential Diophantine equation $u^{\alpha} + (u + 11v)^{\beta} = w^{2\gamma}$ and concluded that this equation has no solution in W under some conditions. In [8], Gupta and Kumar studied the Diophantine equation $k^{a} + 37^{b} = c^{2}$, where $k \equiv 2(mod\ 111)$. Gupta et al. [9] examined non-linear exponential Diophantine equation $x^{\alpha} + (1 + my)^{\beta} = z^{2}$. Gupta et al. [10] studied the non-linear exponential Diophantine equation $(x^{a} + 1)^{m} + (y^{b} + 1)^{n} = z^{2}$. Hoque and Kalita [11] determined the solution of the Diophantine equation $(p^{q} - 1)^{x} + p^{qy} = z^{2}$. In [12], Kumar et al. discussed on the equations $31^{x} + 41^{y} = z^{2}$and $61^{x} + 71^{y} = z^{2}$, where x, y, z are natural numbers.

In [13], Kumar et al. discussed on the equation $p^{x} + (p + 12)^{y} = z^{2}$ and show that under some conditions this equation has no solution in W. Sroysang [14-17] examined the Diophantine equations,$3^{x} + 85^{y} = z^{2}$, $4^{x} + 10^{y} = z^{2}$,$3^{x} + 45^{y} = z^{2}$ and $143^{x} + 145^{y} = z^{2}$ for non-negative integer solution.In [18], Viriyapong and Viriyapong discussed the Diophantine equation $n^{x} + 19^{y} = z^{2}$, where $n \equiv 2(mod\ 57)$ and prove that it has a unique solution $\{n = 2, x = 3, y = 0, z = 3\}$.

The primary goal of this paper is to find the solution of the Diophantine equation $25^{x} + 483^{y} = z^{2}$, where x, y, z are non-negative integers.

2. PRELIMINARIES

PROPOSITION 2.1 Catalan's Conjecture [4]: The Diophantine equation $a^{x} - b^{y} = 1$, where a, b, x and y are integers such that $\min\{a, b, x, y\} > 1$, has a unique solution $(a, b, x, y) = (3, 2, 2, 3)$.

LEMMA 2.2 The Diophantine equation $25^{x} + 1 = z^{2}$, where x, z are non-negative integers, has no non-negative integer solution.

PROOF: Suppose that x, z are non-negative integers such that $25^{x} + 1 = z^{2}$. If $x = 0$, then $z^{2} = 2$ which is impossible. Thus,$x \geq 1$. Now $z^{2} = 25^{x} + 1 \geq 25^{1} + 1 = 26$. Thus $z \geq 6$. Now, we consider the equation $z^{2} - 25^{x} = 1$. By Proposition 2.1, we have $x = 1$. It follows that $z^{2} = 26$. This is a contradiction. Hence, the Diophantine equation $25^{x} + 1 = z^{2}$, where x, z are non-negative integers, has no non-negative integer solution.

LEMMA 2.3 The Diophantine equation $483^{y} + 1 = z^{2}$, where y, z are non-negative integers, has a unique solution $(y, z) = (1,22)$.

PROOF: Suppose that y, z are non-negative integers such that $483^{y} + 1 = z^{2}$. If $y = 0$, then $z^{2} = 2$ which is impossible. Thus, $y \geq 1$. Now $z^{2} = 483^{y} + 1 \geq 483^{1} + 1 = 484$. Thus $z \geq 22$. Now, we consider the equation $z^{2} - 483^{y} = 1$. By Proposition 2.1, we have $y = 1$. It follows that $z^{2} = 484 \Rightarrow z = 22$. Hence, $(y, z) = (1,22)$ is the unique non-negative integer solution of the Diophantine equation $483^{y} + 1 = z^{2}$.

3. MAIN RESULTS

THEOREM 3.1$(x, y, z) = (0, 1, 22)$ is the unique non-negative integer solution of the Diophantine equation $25^x + 483^y = z^2$, where x, y, z are non-negative integers.

PROOF: Let x, y, z be non-negative integers such that $25^x + 483^y = z^2$. By Lemma 2.2, we have $y \geq 1$. Since, 25^x and 483^y are odd, thereforez^2 is even $\Rightarrow z$ is even. Thus$z^2 \equiv 0 (mod\ 4)$.

Now $25 \equiv 1 (mod\ 4) \Rightarrow 25^x \equiv 1 (mod\ 4)$. It follows that $483^y \equiv 3 (mod\ 4) \Rightarrow y$ is odd. Now, we will divide the number x into two cases.

CASE 1. $x = 0$.

In this case by Lemma 2.3, we obtain that $y = 1$ and $z = 22$.

CASE 2. $x \geq 1$.

Since, $25 \equiv 0 (mod\ 5) \Rightarrow 25^x \equiv 0 (mod\ 5)$.

Also,$483 \equiv 3 (mod\ 5) \Rightarrow 483^y \equiv 2 (mod\ 5)$ or $483^y \equiv 3 (mod\ 5)$.

Thus$z^2 \equiv 2 (mod\ 5)$ or $z^2 \equiv 3 (mod\ 5)$. In fact, $z^2 \equiv 0 (mod\ 5)$ or $z^2 \equiv 1 (mod\ 5)$ or $z^2 \equiv 4 (mod\ 5)$. This is a contradiction.

Hence, $(x, y, z) = (0, 1, 22)$ is the unique non-negative integer solution of the Diophantine equation $25^x + 483^y = z^2$, where x, y, z are non-negative integers.

COROLLARY 3.2 The Diophantine equation $25^x + 483^y = w^4$, where x, y, w are non-negative integers, has no non-negative integer solution.

PROOF: Let x, y, w be non-negative integers such that $25^x + 483^y = w^4$. Let $z = w^2$. Then the equation $25^x + 483^y = w^4$ becomes $25^x + 483^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (0, 1, 22)$. Thus $w^2 = z = 22$. But there exists no non-negative integer w such that $w^2 = 22$. Hence, the Diophantine equation $25^x + 483^y = w^4$, where x, y, w are non-negative integers, has no non-negative integer solution.

COROLLARY 3.3 $(x, y, s) = (0, 1, 11)$ is the unique non-negative integer solution of the Diophantine equation $25^x + 483^y = 4s^2$, where x, y, s are non-negative integers.

PROOF: Let x, y, s be non-negative integers such that $25^x + 483^y = 4s^2$. Let $z = 2s$. Then the equation $25^x + 483^y = 4s^2$ becomes $25^x + 483^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (0, 1, 22)$. Thus $2s = z = 22 \Rightarrow s = 11$. Hence, $(x, y, s) = (0, 1, 11)$ is the unique non-negative integer solution of the Diophantine equation $25^x + 483^y = 4s^2$, where x, y, s are non-negative integers.

COROLLARY 3.4$(x, y, u) = (0, 1, 2)$ is the unique non-negative integer solution of the Diophantine equation $25^x + 483^y = 121u^2$, where x, y, u are non-negative integers.

PROOF: Let x, y, u be non-negative integers such that $25^x + 483^y = 121u^2$. Let $z = 11u$. Then the equation $25^x + 483^y = 121u^2$ becomes $25^x + 483^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (0, 1, 22)$. Thus$11u = z = 22 \Rightarrow u = 2$. Hence, $(x, y, u) = (0, 1, 2)$ is the unique non-negative integer solution of the Diophantine equation $25^x + 483^y = 121u^2$, where x, y, u are non-negative integers.

COROLLARY 3.5 The Diophantine equation $25^x + 483^y = 4v^4$, where x, y, v are non-negative integers, has no non-negative integer solution.

PROOF: Let x, y, v be non-negative integers such that $25^x + 483^y = 4v^4$. Let $z = 2v^2$. Then the equation $25^x + 483^y = 4v^4$ becomes $25^x + 483^y = z^2$. By Theorem 3.1, we have $(x, y, z) = (0, 1, 22)$. Thus $2v^2 = z = 22 \Rightarrow v^2 = 11$. But there exists no non-negative integer v such that $v^2 = 11$. Hence, the Diophantine equation $25^x + 483^y = 4v^4$, where x, y, v are non-negative integers, has no non-negative integer solution.

4. CONCLUSION

In this paper, authors studied the Diophantine equation $25^x + 483^y = z^2$, where x, y, z are non-negative integers.They demonstrated that $(x, y, z) = (0, 1, 22)$ is the unique non-negative integer solution of this Diophantine equation with the help of Catalan's Conjecture. The method mentioned in this work can be used in the future to solve different Diophantine equations.

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