GENERALIZED NEUTROSPHIC PYTHAGOREAN PRE - OPEN SETS IN GENERALIZED NEUTROSPHIC PYTHAGOREAN TOPOLOGICAL SPACES

1. P. DEVADHARSANA, 2. Dr. A. STANIS ARUL MARY
1. PG STUDENT, 2. ASSISTANT PROFESSOR

1. DEPARTMENT OF MATHEMATICS,
2. NIRMA COLLEGE FOR WOMEN

ABSTRACT: In this paper a Generalized Neutrosophic Pythagorean Generalized pre – open sets and a Generalized Neutrosophic Pythagorean Generalized pre - closed sets are introduced. Some of its properties are also studied. Also we have provided some applications of Generalized Neutrosophic Pythagorean in generalized pre - closed sets namely Generalized Neutrosophic Pythagorean p $T_{1/2}$ space and Generalized Neutrosophic Pythagorean gp $T_{1/2}$ space.

KEYWORDS: Generalized Neutrosophic Topology, Generalized Neutrosophic Pythagorean Generalized pre-closed sets , Generalized Neutrosophic Pythagorean Generalized pre-open sets, Generalized Neutrosophic Pythagorean p $T_{1/2}$ space and Generalized Neutrosophic Pythagorean gp $T_{1/2}$ space.

I. INTRODUCTION:


II. PRELIMINARIES:

Definition 2.1 A Generalized Neutrosophic Pythagorean set $A$ of $X$ is an object having the form $A = \{ <a, T_A, I_A, F_A>/a \in X \}$ where the function $(T_A)^2 + (F_A)^2 \leq 1$ and $(T_A)^2 + (F_A)^2 + (I_A)^2 \leq 2$ and $T_A \land F_A \land I_A \leq 0.5$. Here $T_A$ (a) is the truth membership, $I_A$ (a) is the indeterminacy membership and $F_A$ (a) is the false membership.

Definition 2.2 Let $A$ and $B$ be Generalized Neutrosophic Pythagorean set of the form $A = \{ <a, T_A, I_A, F_A>/a \in X \}$, $B = \{ <a, T_B, I_B, F_B>/a \in X \}$. Then
1. \( A \subseteq B \) if and only if \( T_A \subseteq T_B \) and \( I_A \geq I_B \) and \( F_A \geq F_B \) for all \( a \in X \).
2. \( A = B \) if and only if \( A \subseteq B \) and \( B \subseteq A \).
3. \( \bar{A} = \{<a, F_A(a), 1 - I_A(a), T_A(a) > / a \in X\} \)
4. \( A \cup B = \{<a, T_A(a) \lor T_B(a), I_A(a) \land I_B(a), F_A(a) \lor F_B(a) > / a \in X\} \)
5. \( A \cap B = \{<a, T_A(a) \land T_B(a), I_A(a) \lor I_B(a), F_A(a) \lor F_B(a) > / a \in X\} \)

For the sake of simplicity, we shall use the notation \( A = \{<a, T_A, I_A, F_A > / a \in X\} \) instead of \( A = \{<a, T_A(a), I_A(a), F_A(a) > / a \in X\} \). The GNP sets

\[ 0 = \{<a, 0, 1, 1> / a \in X\} \] and \( 1 = \{<a, 1, 0, 0> / a \in X\} \) are respectively the empty set and the universe of \( X \).

**Definition 2.3** A Generalized neutrosophic Pythagorean topology (GNPT in short) by subset of a non-empty \( X \) is a family of Generalized neutrosophic Pythagorean sets satisfying the following axioms.

1. \( 0,1 \in \tau \)
2. \( P_1 \cap P_2 \in \tau \), for every \( P_1, P_2 \in \tau \)
3. \( \cup P_i \in \tau \) for any arbitrary family \( \{P_i / i \in \tau \} \)

In this case the pair \((X, \tau)\) is called a (GNPTS) and any GNPS \( P \) in \( \tau \) is called GNPOS in \( X \). The complement \( \bar{A} \) of a Generalized Neutrosophic Pythagorean open set \( A \) in a GNPTS \((X, \tau)\) is called a Generalized Neutrosophic Pythagorean closed set (GNPCS in short).

**Definition 2.4** Let \((X, \tau)\) be a GNPTS and \( A = \{<a, T_A, I_A, F_A > / a \in X\} \) be a GNPS in \( X \). Then the interior of \( A \) is denoted by \( \text{GNP int}(A) \) and \( \text{GNP cl}(A) \) are defined as follows

1. \( \text{GNP int}(A) = \cup \{P | P \text{ is a GNPOS in } X \text{ and } P \subseteq A\} \)
2. \( \text{GNP cl}(A) = \cap \{K | K \text{ is a GNPCS in } X \text{ and } A \subseteq K\} \)

Note that for any GNPS \( A \) in \((X, \tau)\), we have \( \text{GNP cl}(A^C) = (\text{GNP int}(A))^C \text{ and } \text{GNP int}(A^C) = (\text{GNP cl}(A))^C \)

**Definition 2.5** Let \((X, \tau)\) be a GNPS subset of \( X \). Then the following holds

1. \( \text{GNP cl}(\emptyset) = \emptyset \) and \( \text{GNP cl}(X) = X \)
2. \( A \) is a GNPOS if and only if \( A = \text{GNP cl}(A) \)
3. \( \text{GNP cl}(\text{GNP cl}(A)) = \text{GNP cl}(A) \)
4. \( A \subseteq B \) implies that \( \text{GNP cl}(A) \subseteq \text{GNP cl}(B) \)
5. \( \text{GNP cl}(A \cap B) \subseteq \text{GNP cl}(A) \cap \text{GNP cl}(B) \)
6. \( \text{GNP cl}(A \cup B) = \text{GNP cl}(A) \cup \text{GNP cl}(B) \)

**Definition 2.6** Let \((X, \tau)\) be a GNPTS and let \( A \) and \( B \) be Generalized Neutrosophic Pythagorean sets. Then the following holds

1. \( \text{GNP int}(\emptyset) = \emptyset \) and \( \text{GNP int}(X) = X \)
2. \( A \) is a GNPOS if and only if \( A = \text{GNP int}(A) \)
3. \( \text{GNP int}(\text{GNP int}(A)) = \text{GNP int}(A) \)
4. \( A \subseteq B \) implies that \( \text{GNP int}(A) \subseteq \text{GNP int}(B) \)
5. \( \text{GNP int}(A \cap B) = \text{GNP int}(A) \cap \text{GNP int}(B) \)
6. \( \text{GNP int}(A \cup B) \supseteq \text{GNP int}(A) \cup \text{GNP int}(B) \)

**Definition 2.7** Let \((X, \tau)\) be a GNPTS and let \( A \) and \( B \) be two GNPS in \((X, \tau)\). If \( B \) is GNPOS, then \( \text{GNP cl}(A) \cap B \subseteq \text{GNP cl}(A \cap B) \). So if \( G \) is a GNPCS and \( H \) is any GNPS then \( \text{GNP int}(H \cup G) \subseteq \text{GNP int}(H) \cup G \)

**Definition 2.8** A GNPS \( A = \{<a, T_A, I_A, F_A>\} \) in GNPTS \((X, \tau)\) is said to be an

1. Generalized Neutrosophic Pythagorean semi closed set (GNPSCS in short) if \( \text{GNP int}(\text{GNP cl}(A)) \subseteq A \),
2. Generalized Neutrosophic Pythagorean semi open set (GNPOS in short) if \( A \subseteq \text{GNP cl}(\text{GNP int}(A)) \),
3. Generalized Neutrosophic Pythagorean pre-closed set (GNPPCS in short) if \( \text{GNP cl}(\text{GNP int}(A)) \subseteq A \),
4. Generalized Neutrosophic Pythagorean pre-open set (GNPPOS in short) if \( A \subseteq \text{GNP cl}(\text{GNP int}(A)) \),
5. Generalized Neutrosophic Pythagorean \( \alpha \)-closed set (GNP\(\alpha\)CS in short) if \( \text{GNP cl}(\text{GNP int}(\text{GNP cl}(A))) \subseteq A \).
6. Generalized Neutrosophic Pythagorean $\alpha$-open set (GNP$\alpha$OS in short) if $A \subseteq \text{GNPint}(\text{GNPcl}(\text{GNPint}(A)))$

**Definition 2.9** Let $A$ be a GNPS of a GNPTS $(X, \tau)$. Then the GNPS$\text{int}(A)$ and the GNPS$\text{cl}(A)$ are defined as:
1. $\text{GNPS int}(A) = \bigcup \{K | K$ is a GNPOS in $X$ and $K \subseteq A\}$
2. $\text{GNPS cl}(A) = \bigcap \{K | K$ is a GNPCS in $X$ and $A \subseteq K\}$

**Definition 2.10** Let $A$ be a GNPS in $(X, \tau)$, then:
1. $\text{GNPS cl}(A) = A \cup \text{GNP cl}(\text{GNP cl}(A))$
2. $\text{GNPS int}(A) = A \cap \text{GNP cl}(\text{GNP int}(A))$

**Definition 2.11** A GNPS $A = \{<a, T_A, I_A, F_A>\}$ in a GNPTS $(X, \tau)$ is said to be an
1. Generalized Neutrosophic Pythagorean regular open set (GNPROS) if $A = \text{GNPint}(\text{GNP cl}(A))$
2. Generalized Neutrosophic Pythagorean regular closed set (GNPRCS) if $A = \text{GNP cl}(\text{GNP int}(A))$

**Definition 2.12** A GNPS $A$ of a GNPTS $(X, \tau)$ is a Generalized Neutrosophic Pythagorean generalized closed set (GNP$G$CS) if $\text{GNP cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a GNPOS in $X$.

**Definition 2.13** Let $A$ be a GNPS $A$ of a GNPTS $(X, \tau)$, then:
1. $\text{GNP cl}(A) = A \cup \text{GNP cl}(\text{GNP cl}(A))$
2. $\text{GNP int}(A) = A \cap \text{GNP cl}(\text{GNP int}(A))$

**Definition 2.14** Let $A$ be a GNPS in $(X, \tau)$, then:
1. $\text{GNP cl}(A) = A \cup \text{GNP cl}(\text{GNP cl}(A))$
2. $\text{GNP int}(A) = A \cap \text{GNP cl}(\text{GNP int}(A))$

**Definition 2.15** A GNPS $A$ of a GNPTS $(X, \tau)$ is said to be a Generalized Neutrosophic Pythagorean $\alpha$ generalized closed sets (GNP$\alpha$GCS in short) if $\text{GNP cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a GNPOS in $X$.

**Definition 2.16** Let $(X, \tau)$ be a GNPTS and $A = \{<a, T_A, I_A, F_A>\}$ be a GNPS in $X$. The pre interior of $A$ is denoted by $\text{GNP pre int}(A)$ and is defined by the union of all GNPPS$\text{pos}$ in $X$ which are contained in $A$. The intersection of all GNPPS$\text{pos}$ $\text{pos}$ containing $A$ is called the pre closure of $A$ and is denoted by $\text{GNP pcl}(A)$.
1. $\text{GNP pre int}(A) = \bigcup \{G | G$ is a GNPPS in $X$ and $G \subseteq A\}$
2. $\text{GNP pcl}(A) = \bigcap \{K | K$ is a GNPPCS in $X$ and $A \subseteq K\}$

**Definition 2.17** If $A$ is a GNPS in $X$, then $\text{GNP pcl}(A) = A \cup \text{GNP cl}(\text{GNP int}(A))$.

**Definition 2.18** For any GNPS $A$ in $(X, \tau)$, we have:
1. $X \setminus \text{GNP int}(A) = \text{GNP cl}(X \setminus A)$
2. $X \setminus \text{GNP cl}(A) = \text{GNP int}(X \setminus A)$

**III. GENERALIZED NEUTROSOPHIC PYTHAGOREAN GENERALIZED PRECLOSED SETS**

**Definition 3.1** A GNPS $A$ is said to be Generalized Neutrosophic Pythagorean generalized pre-closed set (GNPGPCS) in $(X, \tau)$ if $\text{GNP pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a GNPOS in $X$. The family of all GNPGPCSs of a GNPTS $(X, \tau)$ is denoted by GNPGPCS$(X)$.

**Example 3.2** Let $X = \{a, b\}$ and $\tau = \{0, U, 1\}$ be a GNPT on $X$, where $U = \{<a, 0.2, 0.6, 0.6>, <b, 0.3, 0.6, 0.7>\}$. Then the GNPS $A = \{<a, 0.1, 0.6, 0.7>, <b, 0.2, 0.6, 0.8>\}$ is a GNPPCS in $X$.

**Theorem 3.3** Every PFCS is a PFPGPCS but not conversely.
Proof: Let $A$ be a GNPCS in $X$ and let $A \subseteq U$ and $U$ is a GNPOS in $(X, \tau)$. Since $\text{GNP pcl}(A) \subseteq \text{GNP cl}(A)$ and $A$ is a GNPCS in $X$, $\text{GNP pcl}(A) \subseteq \text{GNP cl}(A) = A \subseteq U$. Therefore $A$ is a GNPGPCS in $X$.

**Example 3.4** Let $X = \{a, b\}$ and $\tau = 0, U, 1\}$ be a PFT on $X$, where $U = \{<a, 0.2, 0.6, 0.6>, <b, 0.3, 0.6, 0.7>\}$. Then the GNS $A = \{<a, 0.1, 0.6, 0.7>, <b, 0.2, 0.6, 0.8>\}$ is a GNPGPCS in $X$ but not a GNPCS in $X$. 
Theorem 3.5: Every GNPαCS is a GNPGPCS but not conversely.
Proof: Let A be a GNPαCS in X and let A ⊆ U and U is a GNPOS in (X, τ). By hypothesis, GNPcl(GNPInt(GNPcl(A))) ⊆ A. Since A ✈ GNPcl(A), GNPcl(GNPInt(A)) ⊆ GNPcl(GNPInt(GNPcl(A))) ⊆ A. Hence GNPpcl(A) ⊆ A ✈ U. Therefore A is a GNPGPCS in X.

Example 3.6: Let X = {a, b} and let τ = {0, U, 1} be a GNP on X, where U = {< a, 0.6, 0.5, 0.4 >, < b, 0.3, 0.5, 0.7 >}. Then the GNP A = {< a, 0.3, 0.5, 0.7 >, < b, 0.1, 0.5, 0.9 >} is a GNPGPCS in X but not a GNPαCS in X since GNPcl(GNPInt(GNPcl(A))) = {< a, 0.4, 0.5, 0.6 >, < b, 0.7, 0.5, 0.3 >} ✈ A.

Theorem 3.7: Every GNPGPCS is a GNPGPCS but not conversely.
Proof: Let A be a GNPGPCS in X and let A ⊆ U and U is a GNPOS in (X, τ). Since GNPpcl(A) ⊆ GNPcl(A) and by hypothesis, GNPcl(A) ⊆ U. Therefore A is a GNPGPCS in X.

Example 3.8: Let X = {a, b} and let τ = {0, U, 1} be a GNP on X, where U = {< a, 0.6, 0.5, 0.4 >, < b, 0.3, 0.5, 0.7 >}. Then the GNP A = {< a, 0.3, 0.5, 0.7 >, < b, 0.1, 0.5, 0.9 >} is a GNPGPCS in X but not a GNPCS in X since A ✈ U but GNPcl(A) = {< a, 0.4, 0.5, 0.6 >, < b, 0.7, 0.5, 0.3 >} ✈ U.

Theorem 3.9: Every GNPPCS is a GNPGPCS but not conversely.
Proof: Let A be a GNPPCS in X. By Definition 2.11, A = GNPcl(GNPInt(A)). This implies GNPcl(A) = GNPcl(GNPInt(A)). Therefore GNPcl(A) = A. That is A is a GNPCS in X. By Theorem 3.3, A is a GNPGPCS in X.

Example 3.10: Let X = {a, b} and let τ = {0, U, 1} be a PFT on X, where U = {< a, 0.4, 0.4, 0.6 >, < b, 0.7, 0.4, 0.3 >}. Then the GNP A = {< a, 0.3, 0.5, 0.7 >, < b, 0.2, 0.5, 0.8 >} is a GNPGPCS but not a GNPPCS in X since GNPcl(GNPInt(A)) = 0 ✈ A.

Theorem 3.11: Every GNPPCS is a GNPGPCS but not conversely.
Proof: Let A be a GNPPCS in X and let A ⊆ U and U is a GNPOS in (X, τ). By Definition 2.8, GNPcl(GNPInt(A)) ⊆ A. This implies that GNPpcl(A) = A ✈ GNPcl(GNPInt(A)) ⊆ A. Therefore GNPpcl(A) ⊆ U. Hence A is a GNPGPCS in X.

Example 3.12: Let X = {a, b} and let τ = {0, U, 1} be a GNP on X, where U = {< a, 0.2, 0.3, 0.8 >, < b, 0.3, 0.5, 0.7 >}. Then the GNP A = {< a, 0.4, 0.3, 0.7 >, < b, 0.4, 0.6, 0.6 >} is a GNPGPCS but not a GNPPCS in X since GNPcl(GNPInt(A)) = 1 ✈ A.

Theorem 3.13: Every GNPαGCS is a GNPGPCS but not conversely.
Proof: Let A be a GNPαGCS in X and let A ⊆ U and U is a GNPOS in (X, τ). By Definition 2.14, A ✈ GNPcl(GNPInt(GNPcl(A))) ⊆ U. This implies GNPcl(GNPInt(GNPcl(A))) ⊆ U and GNPcl(GNPInt(A)) ⊆ U. Therefore GNPpcl(A) = A ✈ GNPcl(GNPInt(A)) ⊆ U. Hence A is a GNPGPCS in X.

Example 3.14: Let X = {a, b} and let τ = {0, U, 1} be a PFT on X, where U = {< a, 0.4, 0.4, 0.4 >, < b, 0.6, 0.3, 0.4 >}. Then the GNP A = {< a, 0.4, 0.5, 0.4 >, < b, 0.7, 0.3, 0.3 >} is a GNPGPCS but not a GNPαGCS in X since GNPcl(A) = 1 ✈ U.

Theorem 3.15: GNPCS and GNPGPCS are independent to each other.

Example 3.16: Let X = {a, b} and let τ = {0, U, 1} be a GNP on X, where U = {< a, 0.4, 0.5, 0.6 >, < b, 0.2, 0.5, 0.8 >}. Then the GNP A = U is a GNPCS but not a GNPGPCS in X since A ✈ U but GNPpcl(A) = {< a, 0.6, 0.5, 0.4 >, < b, 0.8, 0.5, 0.2 >} ✈ U.

Example 3.17: Let X = {a, b} and let τ = {0, U, 1} be a GNP on X, where U = {< a, 0.7, 0.4, 0.3 >, < b, 0.7, 0.4, 0.3 >}. Then the GNP A = {< a, 0.6, 0.3, 0.4 >, < b, 0.5, 0.3, 0.2 >} is a GNPGPCS but not a GNPCS in X since GNPInt(GNPcl(A)) ✈ A.

Theorem 3.18: GNUPCS and GNPGPCS are independent to each other.

Example 3.19: Let X = {a, b} and let τ = {0, U, 1} be a GNP on X, where U = {< a, 0.4, 0.5, 0.6 >, < b, 0.1, 0.5, 0.8 >}. Then the GNP A = U is a GNPCS but not a GNPGPCS in X since A ✈ U but GNPpcl(A) = {< a, 0.6, 0.5, 0.4 >, < b, 0.8, 0.5, 0.1 >} ✈ U.
Theorem 4. Let $A \subseteq X$. Replacing $\text{GNP}$ by $\text{GPO}(X)$ then $\text{V} \subseteq \text{GPint}(\text{GNPcl}(A))$ whenever $V \subseteq A$ and $V$ is $\text{GNPCs}$ in $X$.

Proof: Let $A \in \text{GNPGO}(X)$. Then $\bar{A}$ is a $\text{GNPGCS}$ in $X$. Therefore $\text{GNPcl}(\bar{A}) \subseteq U$ whenever $\bar{A} \subseteq U$ and $U$ is a $\text{GPOS}$ in $X$. That is $\text{GNPcl}(\text{GNPcl}(\bar{A})) \subseteq U$. This implies $U \subseteq \text{GPint}(\text{GNPcl}(A))$ whenever $U \subseteq A$ and $U$ is $\text{GNPCs}$ in $X$. Replacing $U$ by $V$, we get $V \subseteq \text{GPint}(\text{GNPcl}(A))$ whenever $V \subseteq A$ and $V$ is $\text{GNPCs}$ in $X$.

The converse of the above statements need not be true which can be seen from the following examples.

Remark 3.21: The union of any two $\text{GNPGPCS}$s is not a $\text{GNPGPC}$ in general as seen in the following example.

Example 3.22: Let $X = \{a, b\}$ be a $\text{GNPT}$ and let $\tau = \{0, U, 1\}$ be a $\text{GNPT}$ on $X$, where $U = \{< a, 0.2, 0.4, 0.7 >, < b, 0.2, 0.4, 0.6 >\}$. Then the $\text{GNPS}$ $A = \{< a, 0.2, 0.4, 0.7 >, < b, 0.2, 0.4, 0.7 >\}$ is a $\text{GNPGPCS}$ but not a $\text{GNPGSCS}$ in $X$ since $A \subseteq U$ but $\text{GNPcl}(A) = 1 \not\subseteq U$.

IV. GENERALIZED NEUTROSOPHIC PYTHAGOREAN GENERALIZED PRE-OPEN SETS

In this section we introduce Generalized Neutrosophic Pythagorean generalized pre-open sets and studied some of its properties.

Definition 4.1: A $\text{GNPS}$ $A$ is said to be a Generalized Neutrosophic Pythagorean generalized pre-open set ($\text{GNPGPOS}$) in $(X, \tau)$ if the complement $\bar{A}$ is a $\text{GNPGPC}$ in $X$.

The family of all $\text{GNPGPOS}$s of a $\text{GNPTS}$ $(X, \tau)$ is denoted by $\text{GNPGPO}(X)$.

Example 4.2: Let $X = \{a, b\}$ and let $\tau = \{0, U, 1\}$ be a $\text{GNPT}$ on $X$, where $U = \{< a, 0.2, 0.3, 0.8 >, < b, 0.3, 0.3, 0.6 >\}$. Then the $\text{GNPS}$ $A = \{< a, 0.8, 0.3, 0.2 >, < b, 0.7, 0.3, 0.2 >\}$ is a $\text{GNPGPOS}$ in $X$.

Theorem 4.3: For any $\text{GNPTS}$ $(X, \tau)$, we have the following:

1. Every $\text{GNPOS}$ is a $\text{GNPGPOS}$
2. Every $\text{GNPSOS}$ is a $\text{GNPGPOS}$
3. Every $\text{GNPZOS}$ is a $\text{GNPGPOS}$
4. Every $\text{GNPPOS}$ is a $\text{GNPGPOS}$.

The converse of the above statements need not be true which can be seen from the following examples.

Example 4.4: Let $X = \{a, b\}$ and let $\tau = \{0, U, 1\}$ be a $\text{GNPT}$ on $X$, where $U = \{< a, 0.2, 0.4, 0.8 >, < b, 0.2, 0.4, 0.6 >\}$. Then the $\text{GNPS}$ $A = \{< a, 0.8, 0.4, 0.2 >, < b, 0.7, 0.4, 0.2 >\}$ is a $\text{GNPGPOS}$ in $(X, \tau)$ but not a $\text{GNPOS}$ in $X$.

Example 4.5: Let $X = \{a, b\}$ and let $\tau = \{0, U, 1\}$ be a $\text{GNPT}$ on $X$, where $U = \{< a, 0.4, 0.3, 0.6 >, < b, 0.2, 0.3, 0.7 >\}$. Then the $\text{GNPS}$ $A = \{< a, 0.7, 0.3, 0.3 >, < b, 0.7, 0.3, 0.2 >\}$ is a $\text{GNPGPOS}$ but not a $\text{GNPZOS}$ in $X$.

Example 4.6: Let $X = \{a, b\}$ and let $\tau = \{0, U, 1\}$ be a $\text{GNPT}$ on $X$, where $U = \{< a, 0.6, 0.5, 0.4 >, < b, 0.4, 0.4, 0.5 >\}$. Then the $\text{GNPS}$ $A = \{< a, 0.7, 0.5, 0.3 >, < b, 0.5, 0.5, 0.4 >\}$ is a $\text{GNPGPOS}$ but not a $\text{GNPPOS}$ in $X$.

Theorem 4.7: Let $(X, \tau)$ be a $\text{GNPTS}$. If $A \in \text{GNPGO}(X)$ then $V \subseteq \text{GPint}(\text{GNPcl}(A))$ whenever $V \subseteq A$ and $V$ is $\text{GNPCs}$ in $X$.

Proof: Let $A \in \text{GNPGO}(X)$. Then $\bar{A}$ is a $\text{GNPGCS}$ in $X$. Therefore $\text{GNPcl}(\bar{A}) \subseteq U$ whenever $\bar{A} \subseteq U$ and $U$ is a $\text{GPOS}$ in $X$. That is $\text{GNPcl}(\text{GNPcl}(\bar{A})) \subseteq U$. This implies $U \subseteq \text{GPint}(\text{GNPcl}(A))$ whenever $U \subseteq A$ and $U$ is $\text{GNPCs}$ in $X$. Replacing $U$ by $V$, we get $V \subseteq \text{GPint}(\text{GNPcl}(A))$ whenever $V \subseteq A$ and $V$ is $\text{GNPCs}$ in $X$.

Theorem 4.8: Let $(X, \tau)$ be a $\text{GNPTS}$. Then for every $A \in \text{GNPGO}(X)$ and for every $A \in \text{GNPS}(X)$, $\text{GNPint}(A) \subseteq B \subseteq A$ implies $A \in \text{GNPGPO}(X)$.
Proof: By hypothesis $\bar{A} \subseteq \bar{B} \subseteq (\text{GNPint}(A))$. Let $\bar{B} \subseteq U$ and $U$ be a GNPPOS. Since $\bar{A} \subseteq \bar{B}$, $\bar{A} \subseteq U$. But $\bar{A}$ is a GNPGPCS, $\text{GNPpcl}(\bar{A}) \subseteq U$. Also $\bar{B} \subseteq (\text{GNPint}(A)) = \text{GNPpcl}(\bar{A})$. Therefore $\text{GNPpcl}(\bar{B}) \subseteq \text{GNPpcl}(\bar{A}) \subseteq U$. Hence $\bar{B}$ is a GNPGPCS. Which implies $B$ is a GNPPOS of $X$.

Remark 4.9: The intersection of any two GNPPOSs is not a GNPPOS in general.

Theorem 4.10: A GNS $A$ of a GNPTS $(X, \tau)$ is a GNPPOS if and only if $F \subseteq \text{GNPint}(A)$ whenever $F$ is a GNPCS and $F \subseteq A$.

Proof:
Necessity: Suppose $A$ is a GNPPOS in $X$. Let $F$ be a GNPCS and $F \subseteq A$. Then $\bar{F}$ is a Pythagorean Fuzzy Open set in $X$ such that $\bar{A} \subseteq \bar{F}$. Since $\bar{A}$ is a GNPGPCS, we have $\text{GNPpcl}(\bar{A}) \subseteq \bar{F}$. Hence $(\text{GNPint}(A)) \subseteq \bar{F}$. Therefore $F \subseteq \text{GNPint}(A)$.

Sufficiency: Let $A$ be a GNPPOS of $X$ and let $F \subseteq \text{GNPint}(A)$ whenever $F$ is a GNPCS and $F \subseteq A$. Then $A \subseteq \bar{F}$, and $\bar{F}$ is a GNPPOS. By hypothesis, $\text{GNPint}(A) = \bar{F}$. Which implies $\text{GNPpcl}(\bar{A}) \subseteq \bar{F}$. Therefore $\bar{A}$ is a GNPGPCS of $X$. Hence $A$ is a GNPPOS of $X$.

Corollary 4.11: A GNS $A$ of a GNPTS $(X, \tau)$ is a GNPPOS if and only if $F \subseteq \text{GNPint}(\text{GNPcl}(A))$ whenever $F$ is a GNPCS and $F \subseteq A$.

Proof:
Necessity: Suppose $A$ is a GNPPOS in $X$. Let $F$ be a GNPCS and $F \subseteq A$. Then $\bar{F}$ is a GNPPOS in $X$ such that $\bar{A} \subseteq \bar{F}$. Since $\bar{A}$ is a GNPGPCS, we have $\text{GNPcl}(\bar{A}) \subseteq \bar{F}$. Therefore $\text{GNPcl}(\text{GNPcl}(\bar{A})) \subseteq \bar{F}$. Hence $\text{GNPint}(\text{GNPcl}(\bar{A})) \subseteq \bar{F}$. Therefore $F \subseteq \text{GNPcl}(\text{GNPcl}(\bar{A}))$.

Sufficiency: Let $A$ be a GNPPOS of $X$ and let $F \subseteq \text{GNPint}(\text{GNPcl}(A))$ whenever $F$ is a GNPCS and $F \subseteq A$. Then $\bar{A} \subseteq \bar{F}$, and $\bar{F}$ is a GNPPOS. By hypothesis, $\text{GNPint}(\text{GNPcl}(A)) = \bar{F}$. Hence $\text{GNPcl}(\text{GNPint}(\bar{A})) \subseteq \bar{F}$, which implies $\text{GNPpcl}(\bar{A}) \subseteq \bar{F}$. Hence $A$ is a GNPPOS of $X$.

Theorem 4.12: For a GNS $A$, $A$ is a GNPPOS and a GNPGPCS in $X$ if and only if $A$ is a GNPPOS in $X$.

Proof:
Necessity: Let $A$ be a GNPPOS and a GNPGPCS in $X$. Then $\text{GNPpcl}(A) \subseteq A$. This implies $\text{GNPcl}(\text{GNPint}(A)) \subseteq A$. Since $A$ is a GNPPOS, it is a GNPPOS. Hence $A \subseteq \text{GNPint}(\text{GNPcl}(A))$. Therefore $A = \text{GNPint}(\text{GNPcl}(A))$. Hence $A$ is a GNPPOS in $X$.

Sufficiency: Let $A$ be a GNPPOS in $X$. Therefore $A = \text{GNPint}(\text{GNPcl}(A))$. Let $A \subseteq U$ and $U$ be a GNPPOS in $X$. This implies $\text{GNPpcl}(A) \subseteq A$. Hence $A$ is a GNPPOS in $X$.

V. APPLICATIONS OF PYTHAGOREAN FUZZY GENERALIZED PRE-CLOSED SETS

In this section we provide some applications of Generalized Neutrosophic Pythagorean generalized pre-closed sets.

Definition 5.1: A GNPTS $(X, \tau)$ is said to be a Generalized Neutrosophic Pythagorean $pT_{1/2}$ (GN $pT_{1/2}$) space if every GNPGPCS in $X$ is a GNPCS in $X$.

Definition 5.2: A GNPTS $(X, \tau)$ is said to be a Generalized Neutrosophic Pythagorean $gp T_{1/2}$ (GNPg$pT_{1/2}$) space if every GNPGPCS in $X$ is a GNPPCS in $X$.

Theorem 5.3: Every GNP$pT_{1/2}$ space is a GNPgp$pT_{1/2}$ space. But the converse is not true in general.

Proof: Let $X$ be a GNP$p T_{1/2}$ space and let $A$ be a GNPGPCS in $X$. By hypothesis $A$ is a GNPCS in $X$. Since every GNPCS is a GNPPCS, $A$ is a GNPPCS in $X$. Hence $X$ is a GNPgp $T_{1/2}$ space. But the converse need not be true which can be seen in the following example.

Example 5.4: Let $X = \{a, b\}$ and let $\tau = \{0, 1\}$ be a GNPT on $X$, where $U = \{< a, 0.8, 0.4, 0.2>, < b, 0.8, 0.4, 0.2>\}$. Then $(X, \tau)$ is a GNPgp$p T_{1/2}$ space. But it is not a GNP$p T_{1/2}$ space since the GNS $A = \{< a, 0.2, 0.4, 0.6>, < b, 0.3, 0.4, 0.6>\}$ is GNPPOS but not a GNPCS in $X$. 


Theorem 5.5: Let \((X, \tau)\) be a GNPTS and \(X\) is a GNPP\(\tau_{1/2}\) space then

(i) Any union of GNPGPCSs is a GNPGPCS.
(ii) Any intersection of GNPGPOSs is a GNPGPOS.

Proof:
(i) Let \(\{A_i\}_{i \in I}\) is a collection of GNPGPCSs in a GNPP\(\tau_{1/2}\) space \((X, \tau)\). Therefore every GNPGPCS is a GNPCS. But the union of GNPCS is a GNPCS. Hence the union of GNPGPCS is a GNPGPCS in \(X\).
(ii) It can be proved by taking complement in (i).

Theorem 5.6: A GNPTS \(X\) is a GNPP\(\tau_{1/2}\) space if and only if GNPGPO(X) = GNPPPO(X).

Proof:
Necessity: Let \(A\) be a GNPGPOS in \(X\), then \(\bar{A}\) is a GNPGPOS in \(X\). By hypothesis \(\bar{A}\) is a GNPPCS in \(X\). Therefore \(A\) is a GNPPPOS in \(X\). Hence GNPGPO(X) = GNPPPO(X).

Sufficiency: Let \(A\) be a GNPGPCS in \(X\). Then \(\bar{A}\) is a GNPGPOS in \(X\). By hypothesis \(\bar{A}\) is a GNPPPOS in \(X\). Therefore \(A\) is a GNPPPCS in \(X\). Hence \(X\) is a GNPP\(\tau_{1/2}\) space..

REFERENCES
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