IJCRT.ORG ISSN : 2320-2882
INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)
An International Open Access, Peer-reviewed, Refereed Journal

Einstein-Cartan And General Relativity's Fluid Spheres

Dr. Gajendra Prasad Gadkar

Associate Professor

Department of Physics

College of Commerce, arts &science, Patliputra University, patna

Abstract
This paper shows that the Einstein-Cartan theory-based static fluid sphere problem is looked at, and a new method for using quadrature to find an analytical solution to the Einstein-Cartan Field Equations is developed. In most cases, the application of the method yields precise and straightforward solutions.
Keywords: Static, Quadrature, Exact Solution

Introduction

Because the effect of many particles' spins cancels out, whereas the effect of mass is additive, the role of spin in the Einstein theory of gravitation is less significant. Thus, only mass-induced curvature is taken into account. Einstein-Cartan hypothesis is the speculation of Einstein's hypothesis. In this both curve and twist are consolidated. Torsion is caused by spin. One can consider Einstein-Cartan hypothesis as the hypothesis of two tensor fields, the measurement field g and the twist field Q.

Since the forecast of E-C hypothesis varies from those of general relativity just for issue filled locales, in this manner, other than cosmology a significant application field for E-C hypothesis is relativistic astronomy managing the inside of heavenly items like neutron begins with a few arrangement of twists of the constituent particles and under conditions when twist might deliver a few discernible results. As such it appears to be alluring to comprehend the full ramifications of the E-C hypothesis for limited appropriations like liquid circles with non-zero tension. Numerous researchers have considered the issue of static-fluid spheres in the E-C theory from this perspective (Prasanna 1975, Kerlick 1975, Kuchowicz 1975, and Skinner and Webb 1977).

In this paper, the Einstein-Cartan problem of static fluid sphere is looked at and a new method for using quadrature to get the solution in an analytical form is developed. In most cases, the application of the method yields precise and straightforward solutions.

Let M be a C four-dimensional, oriental, connected Hausdorff differential manifold with a Lorentz metric g defined on it for the Einstein–Cartan Field Equations. The components of a field of coframes i (in the contingent space of M) that are linearly independent at each point of M define all geometric objects other than the forms. Since we are interested in spinor fields, we assume that i is generally non-holonomic and that the associated tetrad is orthonormal. Since the complex is paracompact, there exists an association on it which we expect to be metric straight association. The metric components gij and the set of one form i, which define the covariant derivative, describe the metric g and the connection w with regard to the selected co-frame i.

Hence we have

(2.1) $g = ds^2 = g_{ij}\, \theta^i\, \theta^j$, ij themselves are completely determined by 64 functions Γ^i_{kj} such that

(2.2) $\omega^i_j = \Gamma^i_{kj}\ \theta^k$

The Einstein – Cartan field equations are

(2.3) $R^j_i - \frac{1}{2} R\, \delta^j_i = -\chi t^j_i$, where R^i_j is Ricci curvature tensor, R is curvature scalar, δ^i_j metric tensor and T^i_j is stress energy momentum tensor. $\chi = 8\pi$

(2.4) $Q^i_{jk} - \delta^i_j\ Q^i_{ik} - \delta^i_k\ Q^i_{ji} = -\chi\, S^i_{jk}$, where Q^i_{jk} is torsion tensor and S^i_{jk} is spin tensor and t^j_i and S^i_{jk} are defined through the relations

(2.5) $t_i = \eta_j\, t^j_i$, $S_{jk} = \eta_i\, S^i_{jk}$.

A static spherically symmetric matter distribution is considered which is represented by the space –time metric

(2.6) $ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$, ν and λ being functions of r. If θ^i represents an orthonormal co-frame we have from (2.1) and (2.6)

(2.7) $\theta^1 = e^{\lambda/2}\, dr$, $\theta^2 = r\, d\theta$, $\theta^3 = r \sin\theta\, d\phi$, $\theta^4 = e^{\nu/2}\, dt$,

so that $g_{ij} = \mathrm{diag}\,(1, -1, -1, -1)$.

Assuming that the spins of the individual particles composing the fluid are all aligned in the radial direction, we get for the spin tensor S_{ij} the only independent non-zero components to be S_{23} = k (say). Since the fluid is supposed to be static, has the velocity four-vector $u^i = \delta^i_4$.

Thus the non-zero components of S^i_{jk} are

(2.8) $S^4_{23} = - S^4_{32} = \mathrm{k}.$

Hence from the Cartan equations (2.4), the non-zero components of Q^i_{jk} are obtained.

(2.9) $Q^4_{23} = - Q^4_{32} = - \mathrm{x\ k}.$

Thus for a perfect fluid distribution with pressure p and density ρ the field equation (2.3) finally reduce to

(2.10) $8\pi p = 16\pi^2 k^2 - \frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right)$,

(2.11) $8\pi\rho = 16\pi^2 k^2 + \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)$,

(2.12) $\frac{e^{\lambda}}{r^2} = \frac{1}{r^2} - \frac{\nu'^2}{4} - \frac{\nu''}{2} + \frac{\nu'\lambda'}{4} + \frac{\nu'+\lambda'}{2r}$, where dashes denote differentiation with respect to r.

The conservation law gives us the relations

(2.13) $\nabla_i\,[(\rho + p)\ u^i] = 0$ (matter conservation)

(2.14) $\nabla_i\,(k\,u^i) = 0$ (spin conservation) and

(2.15) $p' + \frac{1}{2}(\rho + p)\ \nu' + \lambda\,k\,(k' + \frac{k\,\nu'}{2}) = 0.$ If the equation of hydrostatic equilibrium is used

(2.16) $p' + \frac{1}{2}(\rho + p)\ \nu' = 0$,

The following equation is obtained.

(2.17) $k' + \frac{k\,\nu'}{2} = 0.$

From (2.17) we have

(2.18) $k = A_1\, e^{-\nu/2}$, where A_1 is a constant of integration.

In principle we have a completely determined system if an equation of state is specified. However, as is well known that in practice this set of equations is formidable to solve using a pre-assigned equation of state , except perhaps for the case ρ = p, which may not be physically meaningful. Secondly, we have the equation of boundary conditions to be chosen for fitting the solutions in the interior and the exterior of the star. A very interesting aspect of the Einstein- Cartan theory is that outside the fluid distribution the equations reduce to Einstein's equations for empty space viz. $Rij = 0$, since there is no spin density.

Now, if we define

(2.19) $\bar{\rho} = \rho - 2\pi\ k^2$, $\bar{p} = p - 2\pi\ k^2$, then the equations (2.10) and (2.11) take the usual general relativistic form for a static fluid sphere as given by

(2.20) $8\pi\ \bar{p} = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right)$,

(2.21) $8\pi\bar{\rho} = \frac{1}{r^2} + e^{-\lambda}\left(-\frac{1}{r^2} + \frac{\lambda'}{r}\right)$, along with (2.12). The equation of continuity (2.15) now becomes

(2.22) $\frac{d\bar{p}}{dr} + \frac{1}{2}\ (\ \bar{\rho}\ +\ \bar{p}\ )\ \ \nu' = 0.$

In $\bar{p}$ and $\bar{\rho}$ the square term of spin behaves as the effective repulsive force. The repulsion can be important if the expression $2\pi\, k2$ is of the same order as the energy density ρ. It is clear from these equations that it is the $\bar{p}$ and not p which is continuous across the boundary r = $r0$ of the fluid sphere. The continuity of $\bar{p}$ across the boundary ensures that of ($\nu'\, e\nu$). Further with $\bar{p}$ and $\bar{\rho}$ replacing p and ρ respectively, we are assured that the metric coefficients are continuous across the boundary. Hence we shall apply the usual boundary conditions to the solutions of equations (2.12), (2.20) and (2.21).

The boundary conditions are

$$(2.23) \qquad [e^{-\lambda}]_{r=r_0} = [e^{\nu}]_{r=r_0} = (1 - \frac{2m}{r_0}),$$

$\bar{p}$ = 0 at r=$r0$, where $r0$ is the radius and m is the mass of the fluid sphere. The total mass, as measured by an external observer, inside the fluid sphere of radius $r0$ is given by

$$(2.24) \qquad \mathrm{m} = 4\,\pi \int_0^{r_0} \bar{\rho}\, r^2\, \mathrm{d\,r} = 4\,\pi \int_0^{r_0} \rho\, r^2\, \mathrm{d\,r} - 8\,\pi^2 \int_0^{r_0} k^2(r)\; r^2\, \mathrm{d\,r}.$$

Thus the total mass of the fluid sphere is modified by the correction,

$$8 \quad \pi^2 \int_0^{r_0} k^2(r)\; r^2\, \mathrm{dr}.$$

Solution of the Field Equations

We have to solve equation (2.12) for ν and λ. This may be fulfilled by quadrature in a number of ways e.g. Tolman specifies various conditions on the functions ν and λ that simplify, the equation and allow immediate integration while Adler in 1974 and Whitman in 1977 find λ by judicious choice of ν (r). We note that λ may be obtained if ν is given and vice-versa. Once ν and λ are obtained, p and ρ follow directly from equations (2.10) and (2.11).

We define

(3.1) ν = 2 log Y

Then using equation (2.12), we get the differential equation

$$(3.2) \qquad Y'' - (\frac{1}{r} + \frac{\lambda'}{r})\, Y' + (\frac{e^{\lambda}}{r^2} - \frac{\lambda'}{2r} - \frac{1}{r^2})\, \mathrm{Y} = 0$$

It is not always possible to get a traceable solution from the analytic specification of the equation of state. In these cases numerical and graphic technique are easy to apply. Exact solution in terms of known functions are most easily obtained by requiring one of the field variable to satisfy some subsidiary condition which simplify the full set of equations. Once the field equations are solved in this manner, an equation of state then can be extracted. Such solutions may be useful in understanding a system in the extreme, relativistic limit where we cannot specify a priori what the equation of state might be.

As stated above, the set of equations (2.10) to (2.12) cannot be solved without either choosing an equation of state or making a specific assumption on one of the functions p, ρ, ν and λ. For this we assume

$$(3.3) \qquad e^{\lambda(\mathrm{r})} = \mathrm{A}\; r^n, \text{ where A and n are constants.}$$

Substitution of equation (3.3) in (3.2) provides

$$(3.4) \qquad Y'' - (\frac{1}{r} + \frac{n}{2r})\, Y' + (\mathrm{A}\, r^{n-2} - \frac{n}{2r^2} - \frac{1}{r^2})\, \mathrm{Y} = 0.$$

This is a second order differential equation in Y for the general value of n and A.

We solve it for n = - 2

Equation (3.4) reduces to

$$(3.5) \qquad Y'' - \frac{A}{r^4}\, \mathrm{Y} = 0$$

which has the solution

$$(3.6) \qquad \mathrm{Y} = \frac{A}{6r^2} + B_4\, \mathrm{r} + C_4.$$

Thus

(3.7) $e^{\nu} = (\frac{A}{6r^2} + B_4 r + C_4)^2$, $e^{\lambda} = \frac{A}{r^2}$, where B_4 and C_4 are constants.

In this case pressure and density are

(3.8) $$8\pi r^2 \rho(r) = 1 - \frac{2r^2}{A} + 16\pi^2 A_1^2 (\frac{A}{6r^2} + B_4 r + C_4)^{-2}$$

(3.9) $$8\pi r^2 p(r) = -1 + \frac{2}{A}\left[\frac{-\frac{A}{r^3} + B_4 r^3}{\frac{A}{6r^2} + B_4 r + C_4}\right]$$

$$+ 16\pi^2 (\frac{A}{6r^2} + B_4 r + C_4)^{-2} .$$

Spin density K is given by

(3.10) $$K = A_1 [\frac{A}{6r^2} + B_4 r + C_4]^{-1} .$$

The constants A, $B4$, $C4$ and $A1$ are given by

(3.11) $$A = \frac{1}{R_b}(1 - \frac{2M}{R_b})^{-1}$$

(3.12) $$B_4 = \frac{1}{3R_b^5(1-\frac{2M}{R_b})} + \frac{M}{R_b^2(1-\frac{2M}{R_b})^{1/2}} ,$$

(3.13) $$C_4 = (1 - \frac{2M}{R_b})^{1/2} - \frac{1}{2R_b^4(1-\frac{2M}{R_b})} + \frac{M}{R_b(1-\frac{2M}{R_b})^{1/2}}$$

(3.14) $$8\pi R_b^2 \rho(R_b) = 1 - \frac{2R_b^2}{A} + 16\pi A_1^2 (\frac{A}{6R_b^2} + B_4 R_b + C_4)^{-2}$$

Conclusion

The Einstein –Cartan Field Equations are written for a perfect fluid distribution with pressure p and density ρ. For solving this, the method of quadrature is used. Since exact solution in terms of known functions can be obtained by requiring one of the field variable to satisfy some subsidiary condition which simplify the full set of equations, we define ν = 2 log Y. Also it is assumed that $e\lambda$(r) = A rn , where A and n are constants. The equation is solved for n = -2. The equations for pressure p, density ρ and spin density K are obtained.

REFERENCES

[1] Adler, R.J., Journal of Mathematical Physics, 15, 729, 1974.
[2] Hehl F.W., Gen. Relativity Gravitation,5,491,1974
[3] Kerlick, G.D., Astrophysics Journal, 185, 631, 1973.
[4] Kerlick, G.D., Ph.D. Thesis, Princeton University (U.S.A), 1975.
[5] Kuchowicz B., Acta Phys.Pol.B6, 555, 1975
[6] Prasanna, Phys. Rev.D11, 2076, 1975
[6] Singh T. and Yadav R.B.S., Acta Phys. Pol. B9, 837-847, 1978
[7] Skinner r., Webb I., Acta Phys. Pol. B8, 81, 1977
[8] Trautman, A.1st Naz., Alta Mat. Symp. Mat 139, 1973
[9] Tolman, R.C., "Relativity, Thermodynamics and cosmology", Oxford Univ. Press, Oxford, 1966.
[10] Whitman, P.G., Journal of Mathematical Physics, 18, 686, 1977.