The Number Of \(i^{th}\) Smallest Parts Of \(R-M_i\) Partitions Of \(N\)

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Abstract:

In this paper, we derive the generating functions for the number of \(i^{th}\) smallest parts of \(R-M_i\) partitions of \(n\). We obtained results for even and odd partitions.

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Keywords: \(M_1\) partition; \(M_2\) partition; \(r\)-partition; over partitions; generating function.

1. Introduction

Ahlgren, Bnringmann and lovejoy [1] defined \(M_2\text{spt}(n)\) to be the number of smallest parts in the partitions of \(n\) without repeating the odd numbers and smallest part even. Hanumareddy defined [3] \(i^{th}\) smallest part and derived the relation between \(i^{th}\) smallest parts and \(i^{th}\) greatest parts of partitions and over partitions of \(n\) in general form. In this paper derivation of generating functions for the number of \(i^{th}\) smallest parts of \(r-M_i\) partitions of \(n\).

For more details on partitions see [2,4,5,6,7].

We first obtain a formula for the generating function of \(r\)-partitions of \(n\) whose \(i^{th}\) smallest part is the first part.
Definition and notations:

- A $r$-th partition over of $n$ whose $i$-th smallest parts are of the form $a^{k-1}$ is denoted by $r$-j-th partition.
- A M1 partition of $n$ is a partition with unrepeated even number and smallest part odd number.
- A M2 partition of $n$ is a partition with unrepeated odd number and with even smallest parts.
- A M1$_i$ partition of $n$ is a partition with unrepeated odd numbers and $i$-th smallest part even.
- A M2$_i$ partition of $n$ is one with unrepeated odd numbers and $i$-th smallest part even. As usual $M2_i\xi(n)$ stands for the set of such partitions and $M2_i\omega(n)$, for the cardinality of this set.
- A M1$_i\xi(n)$ stand for the set of M1$_i$ partitions and M1$_i\omega(n)$ for the cardinality of this set.

For $r$-partitions, the corresponding $M_Ji\xi_r(n)$, $M_Ji\omega_r(n)$ can be defined similarly for $J = 1$ and $J = 2$.

If $J \in \{1,2\}$, $M_Ji\omega_r(n)$ denotes the number of $j$-th smallest parts including repetitions in all $M_Ji$ partitions of $n$ and sum $M_Ji\omega_r(n)$ denotes the sum of the $j$-th smallest parts.

If $1 \leq r \leq n$ and $I \in \{1,2\}$ write $f_{r,s}(I,N)$ for the number of $r - MI$ partitions of $n$ with least part $s$.

Let us write $r - \xi_e(n)$ for the set of all $r$-partitions of $n$ with unrepeated even numbers as parts and $p_{r,e}(n)$ for the cardinality of this set. Similarly let $r - \xi_o(n)$ stand for the set of all $r$-partitions of $n$ with unrepeated odd numbers as parts and $p_{r,o}(n)$ for the cardinality of this set. Also, write $r - \omega_e(n)$ for the set of all $r$-partitions of $n$ with smallest part even and $r - \omega_o(n)$ for that of all $r$-partitions of $n$ with smallest part odd.

Then

$$r - M2\xi(n) = r - M2\omega_o(n) \cap r - \omega_e(n)$$

and

$$r - M1\xi(n) = r - M2\omega_e(n) \cap r - \omega_o(n)$$

Examples

(i) $(7,5,4,4,3,2,2)$ is $7$-M2 partition of 27

this is $7 - M2_3$, partition but not $7 - M2_4$ partition nor $7 - M2_5$ partition.

(ii) $(6,5,5,4,3,3)$ is $6$-M1 partition, which is also $6 - M1_i$, partition for $i = 1$ and $3$ but not for $i = 2$ and $4$.

The sets $r - M1\xi(n)$, $I=1,2$: 
2. Main Results

**Theorem 2.1** If \( k \in \mathbb{N} \) and \( 1 \leq k \leq \left\lfloor \frac{n}{2r} \right\rfloor \) then the number \( f_{r,2}(2k,n) \) of \( r - M2 \) partitions of \( n \) with least part \( 2k \) is

\[
f_{r,2}(2k,n) = p_{r-1,0}(n - 2kr).
\]

**Proof.** Let \( n = (\lambda_1, \lambda_2, ..., \lambda_r) \) be a \( r - M2 \) partition of \( n \) with least part \( 2k \)
i.e., \( n = (\lambda_1, \lambda_2, ..., \lambda_{r-1}, \lambda_{2k}) \)

By reducing each part by \( 2k - 2 \) we get

\[
n - (2k - 2)r = (\lambda_1 - (2k - 2), \lambda_2 - (2k - 2), ..., \lambda_{r-1} - (2k - 2), 2) \quad \text{and} \quad n - (2k - 2)r - 2 = (\lambda_1 - (2k - 2), \lambda_2 - (2k - 2), ..., \lambda_{r-1} - (2k - 2))
\]
is \( (r - 1) - M2 \) partition of \( n - (2k - 2)r - 2 \) with unrepeated odd parts hence, \( \in (r - 1)\xi_0(n) \)

In this way we get a \( (r - 1) - M2 \) partition of \( n - (2k - 2)r - 2 \) from a \( (r - 1) - M2 \) partition of \( n \).

Coversely let \( (\mu_1, \mu_2, ..., \mu_{r-1}) \in (r - 1)\xi_0(n - 2kr) \)

Then, \( (\mu_1 + 2k, \mu_2 + 2k, ..., \mu_{r-1} + 2k, 2k) \in r - M2\xi(n) \)

The correspondence

\[
n = (\lambda_1, \lambda_2, ..., \lambda_{r-1}, \lambda_{2k}) \iff (\lambda_1 - 2k, ..., \lambda_{r-1} - 2k)
\]
is one - one and onto from \( r - M2\xi(n) \) to \( (r - 1)\xi_0(n - 2kr) \)

Hence \( f_{r,2}(2k,n) = p_{r-1,0}(n - 2kr) \).

In a similar way we can prove Theorem 2.2.

**Theorem 2.2.** If \( k \in \mathbb{N} \) and \( 1 \leq k \leq \left\lfloor \frac{n}{2r-1} \right\rfloor \) then the number \( f_{r,1}(2k - 1,n) \) of \( n \) with least part \( 2k - 1 \) is

\[
f_{r,1}(2k - 1,n) = p_{r,e}(n - (2k - 1)r).
\]
Proof. The correspondence

\[(\lambda_1, \ldots, \lambda_{r-1}, \lambda_r = 2k - 1) \leftrightarrow (\lambda_1 - (2k - 1), \ldots, \lambda_{r-1} - (2k - 1))\]

can easily be verified as above, to be one-one and onto between \(r - M1\xi(n)\) and \((r - 1) - \xi_e(n - (2k - 1)r)\).

The following theorem is well known. However, we present the proof for completeness.

**Theorem 2.3.** The generating function for the number of divisors of \(n\) is \(\sum_{r=1}^{\infty} \frac{q^r}{1-q^r}\).

Proof. Since \(\frac{n}{r} = t \leftrightarrow n = tr\), \(d(n)\) is the number of partition of \(n\) with equal parts, so the generating function is (ref: mathword.wolfram.com)

\[
\sum_{r=1}^{\infty} \frac{q^r}{1-q^r} = \sum_{r=1}^{\infty} \left[ q^r + q^{2r} + q^{3r} + \cdots \right] = \sum_{r=1}^{\infty} \left[ q^r(1 + q^r + q^{2r} + q^{3r} + \cdots) \right] = \sum_{r=1}^{\infty} \frac{q^r}{1-q^r}
\]

**Corollary 2.4.** If \(k \in \mathbb{N}, 1 \leq k \leq n\) and \(\frac{n-a}{r} = 1\), then \(\sum q^n = \sum q^{a+tr}\)

Therefore \(\sum q^n = \sum q^{a+tr}\).

**Proposition 2.5.** There is a one one correspondence between \(r\)-partition of \(n\) and \(r\)-partition of \(n+r\) with smallest part \(\geq 2\). Under this correspondence \(r - M2\), partition of \(n\) correspond to \(r - M1\), partition of \(n+r\).

Proof. Associate with each \(r\)-partition \((\lambda_1, \ldots, \lambda_r)\) of \(n\) the \(r\)-partition \((\mu_1, \ldots, \mu_r)\) where \(\mu_i = \lambda_i + 1 \forall i\). This correspondence is one one and onto between the sets mentioned. Since \(\lambda_i\) is even(odd) iff \(\lambda_i + 1\) is odd(even) this stands \(r - M2\) parts onto \(r - M1\) partitions and \(r - M2\) partitions onto \(r - M1\) partitions and vice-versa.
Theorem 2.6: Given $n$, $r \leq n$ and $\alpha_1, \alpha_2, \ldots, \alpha_l$ there is a one-one correspondence between decreasing $l$-tuples $(\mu_1, \ldots, \mu_l)$ such that $\alpha_1 \mu_1 + \ldots + \alpha_l \mu_l = n$ and $l$-tuples

$$(a_1, \ldots, a_l)$$

such that $\mu_{j-1} = \mu_j + a_{j-1}$ and $(\mu_1^{(a_1)}, \mu_2^{(a_2)}, \ldots, \mu_l^{(a_l)})$ can be reduced to $(\alpha_i^{(a_i)})$ by successive subtraction method.

Given $\mu_1 > \mu_2 > \ldots > \mu_l$ write $\mu_j = \mu_{j+1} + a_{j+1}$ for $j < l$ and $a_l = \mu_l$

apply successive subtraction method to $(\mu_1^{(a_1)}, \mu_2^{(a_2)}, \ldots, \mu_l^{(a_l)})$. Subtract $\mu_i$ from each part. Since zeros cannot be a part we get $r_i$ partition $(\mu_1^{(a_1)}, \ldots, \mu_{i-1}^{(a_{i-1})})$ of $n_i$ where

$$r_i = r - \alpha_i, n_i = n - \alpha_i \mu_i$$

and $\mu_i = \mu_j - \mu_i$. Subtract $\mu_i$ from each part of this partition and get $r_2$ partition

$$n_2 = (\mu^{(a_1)}, \ldots, \mu^{(a_i-1)})$$

where

$$\mu_j = \mu_j^{(a_1)} - \mu^{(a_i-1)} \forall j,$$

$$r_2 = r - \alpha_2 = r - (\alpha_1 + \alpha_2)$$

and $n_2 = n_i - \alpha_{i-1} \mu_{i-1}^{(a_i)} = n_i - \alpha_{i-1} (\mu_{i-1} - \mu_i)$.

Repeating this process we get a finite sequence of $(l-k)$ partitions

$$(\mu_1^{(a_1)}, \ldots, \mu_{l-k}^{(a_{l-k})})$$

where

$$r_k = r_{k-1} - \alpha_k = r - (\alpha_1 + \ldots + \alpha_k)$$

and $n_k = n_{k-1} - \alpha_{k-1} \mu_{k-1}^{(a_{k-1})}$ and $\mu_j^{(k)} = \mu_j^{(k-1)} - \mu_{j-k+1}^{(k-1)}$ when $k = l-1$ we get $r_{l-1} = (\alpha_i)$ partition of $n$. $n_{l-1} = (\mu^{(a_i)})$ where

$$n_{l-1} = n_{l-2} - \alpha_{l-2} \mu_{l-2}^{(l-2)} = \mu_l - \mu_2.$$ Thus the given $r$-partition of $n$ reduces to the partition $$(a_{l-k}^{(a_{l-k})}),$$ write $a_1 = \mu_1 - \mu_2, a_2 = \mu_2 - \mu_3, \ldots, a_{l-1} = \mu_{l-1} - \mu_l$ and $a_l = \mu_l$

$$a_1 + a_2 + \ldots + a_l = \mu_l.$$ Conversely assume that

$$\mu_i = a_i, \mu_{l-j} = \mu_{l-j+1} + a_{l-j} = a_{l-j} + a_{l-(j-1)} + \ldots + a_l, write r_0 = r = \sum a_i$$

$$n_0 = n, \mu_j = \mu_i, \mu_k = \mu_j^{(k-1)} - \mu_{j-k+1}^{(n-1)}, 1 \leq j \leq k - 1, n_j = n_{j-1} - \alpha_{j-1} \mu_{j-1}^{(j-1)}.$$

We apply the successive subtraction method for $(\mu_1^{(a_1)}, \mu_2^{(a_2)}, \ldots, \mu_l^{(a_l)})$ using the above notation and finally get the $\alpha_i$ partition $(\mu_i - \mu_2)^{a_i} = (\alpha_i^{a_i}).$

2.7. Corollary: The number of $r-M2$ partitions of $n$ with $i^{th}$ smallest part coinciding with the first is equal to

the number of $\alpha$-partitions of $m$ with equal parts $\alpha$ being the frequency of the smallest part of the $r$-partition of $n$. 
2.8. Example:

List of $6I_3$ : partitions of $20$ : $(34)$

$(\alpha_1 + \alpha_2 + \alpha_3 = 6)$

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List of $6 - M_{13}$ partitions : (12)

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List of $6 - M_{23}$ partitions : (3)

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number of $6$ partitions that are not $6 - M_{I3}$ partitions ; $34 - 15 = 19$
The remaining 19 partitions are listed below:

List of remaining $6I_3$ : partitions of 20 :

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References