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# ON INVARIANT TENSORS OF β −CHANGES OF FINSLER METRIC BY AN $h$ −VECTOR

RAJ KUMAR SRIVASTAVA

DEPARTMENT OF MATHEMATICS

SRI JAI NARAIN P.G. COLLEGE, LUCKNOW

ABSTRACT

Let $M^n$ be an n-dimensional differentiable manifold and $F^n = (M^n, L)$ be a Finsler space with a metric L$(x, y)$. We consider a change of this metric by $\bar{L} = f(L, \beta)$, where $f$ is a positively homogeneous function of degree one in L and $\beta$, $\beta(x, y) = v_i(x, y)y^i$, $v_i(x, y)$ is an $h$ −vector in $F^n$. The purpose of the present paper is to determine the conditions under which C-reducible, quasi C-reducible, semi C reducible and S3 like Finsler spaces remains a Finsler space of the same kind under a transformed Finsler metric. We have also determined the relations between the $v$ −curvature tensor, $v$-Ricci tensor and $v$-sclar curvature with respect to the Cartan connection of Finsler spaces $F^n = (M^n, L)$ $and$ $\bar{F}^n = (M^n, \bar{L})$.

**Key words** :- Finsler space, $(\alpha, \beta)$ metric, Cartan connection, $\beta$-change , $h$-vector

INTRODUCTION

Let $M^n$ be an n-dimensional differentiable manifold and $F^n = (M^n, L)$ be a Finsler space equipped with a fundamental function L$(x, y)$ $(y^i = \dot{x}^i)$ on $M^n$. Shibata [20] has considered a change *L$(x, y)$ = $f$(L, $\beta$) which he called a $\beta$-change where $\beta(x, y) = v_i(x)y^i$, $f$ is a positively homogeneous function of degree one in L and $\beta$ and established the relation between the properties of Finsler spaces $F^n = (M^n, L)$ and $*F^n = (M^n, *$L). There are various examples of $\beta$-changes, e.g.

$$'L(x, y) = L(x, y) + \beta(x, y) \qquad (1.1)$$

$$''L(x, y) = L^2(x, y)|\beta(x, y) \qquad (1.2)$$

Matsumoto ([10]), Hashiguchi & Ichijiyo ([4]) called (1.1) as a Rander's change and established a theorem which shows a relation between Rander's change and a projective change.

The change (1.2) is called a Kropina change. If L is a Riemannian metric $\alpha(x,y) = [a_{ij}(x)y^i y^j]^{1/2}$, then the metric $^*\mathrm{L}(x,y) = f(L,\beta)$ is called an $(\alpha,\beta)$-metric ([2][18]) $'L = \alpha + \beta$ is called a Rander's metric ([10], [16]) and $''L = \alpha^2/\beta$ a Kropina metric ([18]). The properties of Finsler spaces equipped with $(\alpha,\beta)$ metric have been studied by various authors ([2], [16], [17], [18], [19]) from various standpoints in the Mathematical & Physical aspects.

During the study of conformal transformation of Finsler spaces, Izumi ([6]) introduced the concept of an $h$-vector $v_i(x,y)$ defined by $v_i|_j = 0$, where $|_j$ denotes the $v$-covariant derivative with respect to the Cartan connection $C\Gamma$, $LC_{ij}^h v_h = Kh_{ij}, C_{ij}^h = g^{hl}C_{ijl}$ is Cartan's C-tensor, $h_{ij}$ is the angular metric tensor, K = $\mathrm{L}C^i v_i|(n-1)$ and $C^i = C_{jk}^i g^{jk}$ is the torsion vector. Hence the $h$-vector $v_i(x,y)$ is a function of positional coordinates and directional arguments both satisfying $L\dot{\partial}_j v = Kh_{ij}, \dot{\partial}_j = \partial|\partial y^j$.

Prasad ([15]) has obtained the relation between the Cartan's connection of Finsler spaces $F^n = (M^n, L)$ and $'''F^n = (M^n, '''L)$, where $'''L(x,y) = L(x,y) + v_i(x,y)y^i$ and $v_i(x,y)$ is an $h$-vector in $F^n$. Singh and Srivastava ([20]) has also studied the properties of Finsler space with this metric. Singh and Srivastava ([21]) and the present author ([22]) has also studied the properties of Finsler space with the metric $\bar{L} = f(L,\beta)$, where $\beta(x,y) = v_i(x,y)y^i$ is a differentiable one form and $v_i(x,y)$ is an $h$-vector in $F^n = (M^n, L)$.

The purpose of the present paper is to determined the conditions under which C-reducible, quasi C-reducible, semi C-reducible and S3-like Finsler spaces remains a Finsler space of the same kind under a transformed Finsler metric.

$$\bar{L} = f(L,\beta)$$

We have also determined the relations between the $v$-curvature tensor, $v$-Ricci tensor and $v$-sclar curvature with respect to the Cartan connection of Finsler spaces $F^n = (M^n, L)\ and\ \bar{F}^n = (M^n, \bar{L})$

The terminology and notations are referred to well known Matsumoto's book ([14]) unless otherwise stated.

## THE FINSLER SPACE $\bar{F}^n = (M^n, \bar{L})$

Let $F^n = (M^n, L)$ be an $n-$dimensional Finsler space with a fundamental function $L(x,y)$. We consider a change of the metric defined by

$$\bar{L} = f\{L(x,y), \beta(x,y)\} \tag{2.1}$$

and have another Finsler space $\bar{F}^n = (M^n, \bar{L})$, where $\beta(x,y) = v_i(x,y)y^i$, $v_i$ is an $h$-vector in $F^n = (M^n, L)$ and $f(L,\beta)$ is a positively homogeneous function of degree one in L and $\beta$. We shall call the Finsler space $\bar{F}^n = (M^n, \bar{L})$ as a generalized Finsler space. Throughout the paper the quantities of the Finsler space $\bar{F}^n$ will be denoted by putting bar $(-)$ on the top of the corresponding quantities of the Finsler space $F^n$. We shall use the following notations

$$f_1 = \partial f|\partial L, \quad f_2 = \partial f|\partial \beta, \quad f_{11} = \partial^2 f|\partial L\partial L, \quad f_{12} = \partial^2 f|\partial L\partial \beta \text{ etc.}$$

Since $\bar{L}$ is a positively homogeneous function of degree one in L and $\beta$, hence we have

$$f = f_1 L + f_2 \beta, \; Lf_{12} + \beta f_{22} = 0, \; Lf_{11} + \beta f_{12} = 0 \tag{2.2}$$

If $l_i$, $h_{ij}$, $g_{ij}$ denote the element of support, angular metric tensor and metric tensor of $F^n$ respectively, then the corresponding tensors of $\bar{F}^n = (M^n, \bar{L})$ are given by ([21])

$$\bar{l}_i = f_1 l_i + f_2 v_i \tag{2.3}$$

$$\bar{h}_{ij} = r' h_{ij} + s_0 m_i m_j \tag{2.4}$$

$$\bar{g}_{ij} = r' g_{ij} + r_0 v_i v_j + r_{-1}(v_i y_j + v_j y_i) + r'_{-2} y_i y_j \tag{2.5}$$

Where we put

$$r = ff_1/L, \; s = ff_2, \; s_0 = ff_{22}, \; r' = f(f_1 + Kf_2)/L,$$

$$m_i = v_i - \beta y_i/L^2, \; r_0 = s_0 + f_2^2, \; s_{-1} = ff_{12}/L, \; r_{-1} = s_{-1} + rf_2/f,$$

$$s_{-2} = f(f_{11} - f_1/L)/L^2, \; r_{-2} = s_{-2} + r^2/f^2, \; r'_{-2} = r_{-2} - Ks/L^3 \tag{2.6}$$

The reciprocal tensor $\bar{g}^{ij}$ of $\bar{g}_{ij}$ can be written as ([21])

$$\bar{g}^{ij} = (1/r')g^{ij} - u'_0 v^i v^j - u'_{-1}(v^i y^j + v^j y^i) - u'_{-2} y^i y^j \quad , \tag{2.7}$$

Where $v^i = g^{ij} v_j, \; v^2 = g^{ij} v_i v_j, \; \nu = v^2 - \beta^2/L^2, u'_0 = f^2 s_0/L^2 \tau' r',$

$$u'_{-1} = (f^2/r'\tau' L^2)(r_{-1} + Kf_2^2/L), \; \tau' = (f^2/L^2)(r' + \nu s_0),$$

$$u'_{-2} = r'_{-2}/rr' - (u'_{-1}/r)(\nu r_{-1} - Ks\beta/L^3) \tag{2.8}$$

From the homogeneity, it follows that .

$$s_0\beta + s_{-1}L^2 = 0, \qquad s_{-1}\beta + s_{-2}L^2 = -r, \qquad r_0\beta + r_{-1}L^2 = s,$$

$$s\beta + rL^2 = f^2, \qquad r_{-1}\beta + r_{-2}L^2 = 0 \tag{2.9}$$

From the definition of $m_i$, it is evident that

(a) $m_i l^i = 0$ (b) $m_i v^i = m_i m^i = v^2 - \beta^2|L^2 = \nu$ where $m^i = g^{ij} m_j$,

(c) $h_{ij} m^i = h_{ij} v^i = m_j$ (d) $C_{ij}^h m_h = \frac{K}{L} h_{ij}$ (2.10)

Differentiating (2.5) with respect to $y^k$, the torsion tensor $\bar{C}_{ijk}$ of $\bar{F}^n$ is given by

$$\bar{C}_{ijk} = r' C_{ijk} + \frac{1}{2} r'_{-1}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{r_{02}}{2} m_i m_j m_k \tag{2.11}$$

where $r'_{-1} = r_{-1} + (K/L) r_0, \; r_{02} = \frac{\partial r_0}{\partial \beta}$ (2.12)

or $\bar{C}_{ijk} = r' C_{ijk} + V_{ijk},$ (2.13)

where $V_{ijk} = \frac{r'_{-1}}{2}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{r_{02}}{2} m_i m_j m_k$ (2.14)

Contracting (2.13) by $\bar{g}^{kl}$ and using (2.10), we have

$$\bar{C}_{ij}^l = C_{ij}^l + V_{ij}^l, \tag{2.15}$$

where

$$V_{ij}^{l} = -Q^l\left(r'C_{imj}v^m + r'_{-1}m_im_j\right) + (r'_{-1}/2r')\left(h_i^lm_j + h_j^lm_i\right) + (m^l/r' - \nu Q^l)\ \left(r_{02}m_im_j + r'_{-1}h_{ij}\right)/2 \tag{2.16}$$

$$Q^l = u'_0v^l + u'_{-1}y^l, \quad h_i^l = g^{lk}h_{ik}, \quad m^l = g^{kl}m_k$$

Putting $l = j$ in (2.16) and using (2.10) we have,

$$V_{ij}^{j} = -\left(u'_0v^j + u'_{-1}y^j\right)\left(r'C_{imj}v^m + r'_{-1}m_im_j\right) + (r'_{-1}/2r')\left(h_i^jm_j + h_j^jm_i\right) + \left\{\frac{m^j}{r'} - \nu\left(u'_0v^j + u'_{-1}y^j\right)\right\}\left\{r_{02}m_im_j + r'_{-1}h_{ij}\right\}/2$$

or $V_{ij}^{j} = \frac{1}{2}\frac{r_{02}}{r'}m_i\nu - \frac{\nu^2}{2}u'_0\,r_{02}m_i + \frac{r'_{-1}}{2r'}[m_i + (n-1)m_i] - \frac{\nu}{2}u'_0r'_{-1}m_i - r'u'_0C_{i\beta\beta} - u'_0r'_{-1}\nu\,m_i$

or $V_{ij}^{j} = \left[\frac{(n+1)r'_{-1}}{2r'} - \frac{3}{2}u'_0r'_{-1}\nu + \frac{r_{02}\,\nu}{2(r'+\nu s_0)}\right]m_i - r'u'_0C_{i\beta\beta}$ (2.17)

Here and in the following the subscript $\beta$ denotes contraction with respect to an $h$-vector $v^i$.

From equations (2.15) and (2.17), we have

$\therefore \bar{C}_i = C_i - r'u'_0C_{i\beta\beta} + \sigma m_i$ (2.18)

where $\sigma = \frac{(n+1)r'_{-1}}{2r'} - \frac{3}{2}u'_0r'_{-1}\nu + \frac{r_{02}\,\nu}{2(r'+\nu s_0)}$ (2.19)

From equations (2.7) and (2.18), we have

$$\bar{C}^i = g^{-ij}\bar{C}_j = \frac{1}{r'}C^i + \frac{\sigma}{r'}m^i - u'_0C^i_{\beta\beta} - \left(u'_0v^i + u'_{-1}y^i\right)\left(C_\beta - r'u'_0C_{\beta\beta\beta} + \sigma\nu\right)$$

or $\bar{C}^i = \frac{1}{r'}C^i + N^i$ (2.20)

where $N^i = \frac{\sigma}{r'}m^i - u'_0C^i_{\beta\beta} - \left(u'_0v^i + u'_{-1}y^i\right)\left(C_\beta - r'u'_0C_{\beta\beta\beta} + \sigma\nu\right)$ (2.21)

$$\bar{C}^2 = \bar{C}^i\bar{C}_i = \frac{1}{r'}C^2 + \phi \tag{2.22}$$

where $\phi = \sigma^2\nu\left(\frac{1}{r'} - u'_0\nu\right) + C_\beta\left\{\frac{2\sigma}{r'} - u'_0(1 + 2\sigma\nu)\right\} + u'_0C_{i\beta\beta}\left(r'u_0'^2C_{\beta\beta\beta}v' - 2\sigma u'_0\nu r'v^i - 2C^i\right)$
$+u'_0C_{\beta\beta\beta}\left(r'u'_0C_\beta - 2\sigma\right)$ (2.23)

From equations (2.11), (2.15) and (2.16), the $v$-curvature tensor of $\bar{F}^n$ with respect to Cartan connection is written as

$$\bar{S}_{ijkl} = \bar{C}_{ilp}\bar{C}^p_{jk} - \bar{C}_{ikp}\bar{C}^p_{jl}$$

or $\bar{S}_{ijkl} = r'S_{ijkl} + A_{(kl)}\{h_{il}K_{jk} + h_{jk}K_{il}\}$ (2.24)

where $K_{jk} = \lambda_1m_jm_k + \lambda_2h_{jk}$ (2.25)

and $A_{kl}(\ldots.)$ denotes the interchange of indices k, $l$ and subtraction.

$\lambda_1 = \frac{r_{-1}^2}{4r'}(1 - 2u'_0\nu r') + \frac{\nu r_{02}r'_{-1}}{4(r'+\nu s_0)} + \frac{K}{L}\left\{\frac{r'r_{02}}{2(r'+\nu s_0)} - r'r'_{-1}u'_0\right\}$ (2.26)

$\lambda_2 = \frac{r'_{-1}\nu}{8(r'+\nu s_0)} + \frac{Kr'_{-1}}{2L}\{(1 - u'_0r'\nu)\} - \frac{K^2}{2L^2}r'^2u'_0$ (2.27)

The tensor $K_{jk}$ defined above is symmetric and indicatory.

From equations (2.7), (2.24), (2.25), (2.26) and (2.27), we have

$\bar{S}_{jl} = \bar{g}^{ik}\bar{S}_{ijkl} = S_{jl} - r'u'_0 S_{ijkl}v^i v^k + K_1 h_{jl} + K_2 m_j m_l$ (2.28)

where $K_1 = (3-n)\lambda_1|r' - u'_0(2\lambda_2 + \lambda_1 \nu)$, (2.29)

$K_2 = \{(4-2n)\lambda_2 - \lambda_1 \nu\}|r' + u'_0 \nu(2\lambda_2 + \lambda_1 \nu)$ (2.30)

$S_{jl} = g^{ik}S_{ijkl}$ (2.31)

From equations (2.7) and (2.8), we have

$\bar{S} = \bar{g}^{jl}\bar{S}_{jl} = \frac{1}{r'}S - 2u'_0 S_{jl}v^j v^l + r'^2 u'^2_0 S_{ijkl}v^i v^j v^k v^l + \{(n-1)K_1 + K_2\nu\}|r' - u'_0\nu(K_1 + K_2\nu)$ (2.32)

$S = g^{jl}S_{jl}$ (2.33)

**Definition (2.1)**:- A non Riemannian Finsler space $F^n = (M^n, L)$ with dimension $n \geq 3$ is said to be a quasi-C-reducible if the $(h)hv$-torsion tensor $C_{ijk}$ is written as ([14])

$$C_{ijk} = B_{ij}C_k + B_{jk}C_i + B_{ki}C_j,$$

where $B_{ij}$ is a symmetric and indicatory tensor and $C_i$ is the torsion vector.

From equations (2.11), (2.18) and (2.19), we have

$$\bar{C}_{ijk} = r'C_{ijk} + \frac{1}{6\sigma}A_{(ijk)}\left[\{3r'_{-1}h_{ij} + r_{02}m_i m_j\}\bar{C}_k\right] + \frac{1}{6\sigma}A_{(ijk)}\{\left(3r'_{-1}h_{ij} + r_{02}m_i m_j\right)\left(r'u'_0 C_{k\beta\beta} - C_k\right)\},$$

where $A_{(ijk)}(\ldots.)$ denotes the cyclic interchange of indices $i, j, k$ and summation.

Hence, we have

**LEMMA (2.1)** :- The Cartan tensor $\bar{C}_{ijk}$ of the generalized Finsler space $\bar{F}^n$ can be written in the form.

$\bar{C}_{ijk} = A_{(ijk)}\left(\bar{B}_{ij}\bar{C}_k\right) + Q_{ijk}$, (2.34)

where $\bar{B}_{ij} = \frac{1}{6\sigma}(3r'_{-1}h_{ij} + r_{02}m_i m_j)$ (2.35)

$Q_{ijk} = \frac{1}{6\sigma}A_{(ijk)}\{2\sigma r'C_{ijk} + (3r'_{-1}h_{ij} + r_{02}m_i m_j)(r'u'_0 C_{k\beta\beta} - C_k)\}$ (2.36)

Since the tensor $\bar{B}_{ij}$ is symmetric and indicatory, using the above lemma, we have the following.

**THEOREM (2.1)** :- Finsler space $\bar{F}^n = (M^n, \bar{L})$ is quasi C-reducible if $Q_{ijk} = 0$

**COROLLARY (2.1)** :- If $F^n = (M^n, L)$ is a Riemannian space, then $\bar{F}^n = (M^n, \bar{L})$ is transformed to a quasi C-reducible Finsler space.

**Definition (2.2)**:- A Finsler space $F^n = (M^n, L)$ of dimension $(n \geq 3)$ with $C^2 \neq 0$ called semi C-reducible if the $(h)hv$- torsion tensor $C_{ijk}$ is written as ([13])

$$C_{ijk} = \frac{p}{n+1}\left(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\right) + \frac{t}{C^2}C_i C_j C_k,$$

where $p$ and $t$ are scalar function such that $p + t = 1$

**THEOREM (2.2)** :- If $F^n = (M^n, L)$ is a Riemannian space, then $\bar{F}^n = (M^n, \bar{L})$ is transformed to a semi C-reducible Finsler space.

**Proof :-** If $F^n$ is a Riemannian space then from equation (2.4), (2.11), (2.18), (2.19) and (2.22), we have

$$\bar{C}_{ijk} = \frac{r'_{-1}}{2r'\sigma}\left(\bar{h}_{ij}\bar{C}_k + \bar{h}_{jk}\bar{C}_i + \bar{h}_{ki}\bar{C}_j\right) + \nu\frac{(r'r_{02} - 3r'_{-1}s_0)}{2r'\sigma(r'+\nu s_0)\bar{C}^2}\bar{C}_i\bar{C}_j\bar{C}_k \qquad (2.37)$$

$$= \frac{p}{n+1}\left(\bar{h}_{ij}\bar{C}_k + \bar{h}_{jk}\bar{C}_i + \bar{h}_{ki}\bar{C}_j\right) + \frac{t}{\bar{C}^2}\bar{C}_i\bar{C}_j\bar{C}_k \qquad (2.38)$$

where $p = \frac{r'_{-1}(n+1)}{2r'\sigma}$, $\quad t = \frac{\nu(r'r_{02} - 3r'_{-1}s_0)}{2r'\sigma(r'+\nu s_0)}$

Here $p + t = 1$

Hence $\bar{F}^n$ is a semi-C-reducible Finsler space

**Definition (2.3):-** A Finsler space $F^n = (M^n, L)$ of dimension $(n \geq 3)$ with $C^2 \neq 0$ is called C-reducible if the $(h)hv$- torsion tensor $C_{ijk}$ is of the form ([9])

$$C_{ijk} = \frac{1}{n+1}\left(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\right)$$

Let $W_{ijk} = C_{ijk} - \frac{1}{(n+1)}\left(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\right) \qquad (2.39)$

Here $W_{ijk}$ is symmetric and indicatory tensor If $F^n$ is a C-reducible Finsler space then $W_{ijk} = 0$

From equations (2.4), (2.11), (2.18) and (2.19), we have

$$\bar{C}_{ijk} - \frac{1}{n+1}\left(\bar{h}_{ij}\bar{C}_k + \bar{h}_{jk}\bar{C}_i + \bar{h}_{ki}\bar{C}_j\right)$$

$$= r'\left[C_{ijk} - \frac{1}{n+1}\left(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\right)\right] + a_{ijk}$$

or $\bar{W}_{ijk} = r'W_{ijk} + a_{ijk} \qquad (2.40)$

where $a_{ijk} = \frac{1}{(n+1)}A_{(ijk)}\{(\beta_1 h_{ij} + \beta_2 m_i m_j)m_k - s_0 m_i m_j c_k + \left(u'_0 r' s_0 m_i m_j + r'^2 u'_0 h_{ij}\right)C_{k\beta\beta}\} \qquad (2.41)$

$\beta_1 = \frac{r'_{-1}}{2} - \frac{r'\sigma}{n+1}, \qquad \beta_2 = \frac{r_{02}}{6} - \frac{s_0\sigma}{n+1}$

**THEOREM (2.3) :-** The following statements are equivalent

(a) $F^n$ is a C-reducible Finsler space

(b) $\bar{F}^n$ is a C-reducible Finsler space

iff the tensor $a_{ijk}$ vanishes.

**Definition (2.4):-** A non Riemannian Finsler space $F^n = (M^n, L)$ with dimension $n > 3$ is said to be S3-like ([8]) if the v-curvature tensor $S_{ijkl}$ satisfies.

$$S_{ijkl} = \frac{S}{(n-1)(n-2)}\{h_{ik}h_{jl} - h_{il}h_{jk}\}$$

Where S is the vertical scalar curvature

We define the tensor

$$E_{ijkl} = S_{ijkl} - \frac{S}{(n-1)(n-2)}\{h_{ik}h_{jl} - h_{il}h_{jk}\} \qquad (2.43)$$

$E_{ijkl}$ vanishes iff the space $F^n$ is S3-like.

From equations (2.4), (2.24), (2.32) and (2.43) we have

$$\bar{E}_{ijkl} = \bar{S}_{ijkl} - \frac{\bar{S}}{(n-1)(n-2)}\{\bar{h}_{ik}\bar{h}_{jl} - \bar{h}_{il}\bar{h}_{jk}\}$$

or $\bar{E}_{ijkl} = r'E_{ijkl} + \tau_{ijkl}$ (2.44)

where $\tau_{ijkl} = A_{(kl)}\left[h_{il}K_{jk} + h_{jk}K_{il} - \frac{r'^2\Omega}{(n-1)(n-2)}h_{jk}h_{jl} - \frac{s_0}{(n-1)(n-2)}(S + r'\Omega)(h_{jl}m_i m_k + h_{ik}m_l m_j)\right]$ (2.45)

$\Omega = r'u_0'^2 S_{ijkl}v^i v^j v^k v^l + \{(n-1)K_1 + K_2 v\}/r' - u_0' v(K_1 + K_2 v) - 2S_{jl}u_0' v^j v^l$ (2.46)

We have the following theorem

**<u>THEOREM (2.4)</u> :-** The following statements

(a) $F^n = (M^n, L)$ is an S3-like Finsler space.

(b) $\bar{F}^n = (M^n, \bar{L})$ is an S3-like Finsler space.

are equivalent iff the tensor $\tau_{ijkl}$ vanishes.

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