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Approximation Of Scale Parameters Of Weibull Distribution Under General Entropy Loss Function With Failure Censored Data

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Abstract

This paper describes the Approximate Bayes Estimate of the unknown Scale parameter of Weibull distribution-based on failure censored data. The scale parameter of the Weibull distribution is considered with a Natural Conjugate Gamma Prior in order to obtain the Bayes Estimate. The Weibull parameters are derived on the Asymmetric Loss Function known as General Entropy loss Function (GELF) function. Lindley's approximation is used to obtain Approximate Bayes estimators. The result from Bayesian method is used to compare with Bayes and Maximum likelihood estimate (MLE) methods. The simulation shows that the results from Bayes is better with Approximate Bayesian method than MLE.

Keywords: General Entropy loss function, Maximum likelihood estimation, Bayesian Estimation, Lindley Approach, Weibull distribution. Failure Censoring.

1. Introduction

The Weibull distribution was introduced by the Swedish physicist Weibull [1959], it has been used in many different fields like material science, engineering, physics, chemistry, meteorology, medicine, pharmacy, economics and business, quality control, biology, geology and geography. The estimation of its parameters has been discussed by a number of authors.[Zakerzadeh and Jafari [2014], Doostparast [2006], Modarress, Kaminskiy and Krivtsov [2006], Sun and Berger[1998] and Kundu and Joarder [2006] and Kundu [2007]]. In reliability analysis, the most useful form is the two-parameter formula for the probability density function, where the time to failure is calculated using the two parameters shape and scale. This is a continuous distribution, theoretically with time going out to infinity. The shape parameter describes how the failure rate changes over time. The scale parameter simply adjusts the distribution to fit over the correct range of time, stretching the distribution wider or narrower. Although it was first identified by Fréchet (1927) and first applied by Rosin & Rammler (1933) to describe the size distribution of particles in connection with his studies on strength of material. Weibull (1939,1951) showed that the distribution is also useful in describing the wear out of fatigue failures. Estimation and properties of the Weibull distribution is studied by many author's like see Kao (1959), Johnson;Kotz; Balakrishnan; (1994), Lieblein and Zelen, (1956),Mann(1968).

The probability density function of weibull distribution are given respectively as

$$f(x) = \frac{p}{\theta} x^{(p-1)} \exp\left(-\frac{x^P}{\theta}\right) \quad ; \qquad x, \theta, p > 0 \quad , \qquad (1)$$

Where $'\theta'$ is the scale and 'p' is shape parameters.

The most widely used loss function in estimation problems is quadratic loss function given as $L(\hat{\theta}, \theta) = k(\hat{\theta} - \theta)^2$, where $\hat{\theta}$ is the estimate of $\theta,$ the loss function is called quadratic weighed loss function. If k=1, we have

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \quad , \qquad (2)$$

known as squared error loss function. This loss function is symmetrical because it associates the equal importance to the losses due to overestimation and under estimation with equal magnitudes however in some estimation problems such an assumption may be inappropriate. Overestimation may be more serious than underestimation or Vice-versa. Ferguson (1985), Canfield (1970), Basu and Ebrabimi (1991). Zellner (1986) Soliman (2000) derived and discussed the properties of varian's (1975) asymmetric loss function for a number of distributions.

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\frac{\hat{\theta}}{\theta}$. In this case Calabria and Pulcini (1994) points out that a useful asymmetric loss function is the Entropy loss

$$L(\delta) \alpha [\delta^P - p \, log_e(\delta) - 1 \quad ; \text{Where } \delta = \frac{\hat{\theta}}{\theta},$$

and whose minimum occurs at $(\hat{\theta} = \theta)$,where p>0, a positive error $(\hat{\theta} > \theta)$ causes more serious consequences that a negative error and vice-versa. For small |p| value the function is almost symmetric, when both $\hat{\theta}$ and θ are measured in a logarithmic scale and is approximately.

$$L(\delta) = b[\delta - log_e(\delta) - 1]; \quad b > 0; \qquad \text{where } \delta = \frac{\hat{\theta}}{\theta}. \qquad (3)$$

In a Bayesian setup, the unknown parameter is viewed as random variable. The uncertainty about the true value of parameter is expressed by a prior distribution. The parametric inference is made using the posterior distribution which is obtained by incorporating the observed data into the prior distribution using the Bayes theorem, the first theorem of inference. Hence we update the prior distribution in the light of observed data. Thus the uncertainty about the parameter prior to the experiment is represented by the prior distribution and the same after the experiment is represented by the posterior distribution.

The paper deals with the methods to obtain the approximate Bayes estimators of the weibull distribution by using Lindley approximation technique(Lindley(1980)) for type-II censored samples. A bivariate prior density for the parameters and Entropy loss function (ELF) are used to obtain the approximate Bayes Estimators. A statistical software R is used for numerical calculations for different approximate Bayes estimators and their relative mean squared errors by preparing programs to present the statistical properties of the estimators

2. The Estimator

Let $x_1, x_2, \dots \dots \dots x_n$ be the life times of 'n' items that are put on test for their lives, follow a weibull distribution with density given in equation (1) . The failure times are recorded as they occur until a fixed number 'r' of times failed. Let $= (x_{(1)}, x_{(2)}, \dots \dots \dots \dots \dots, x_{(n)})$, where $x_{(i)}$ is the life time of the ith item . Since remaining (n-r) items yet not failed thus have life times greater than $x_{(r)}$.

The likelihood function can be written as

$$L(x|\theta, p) = \frac{n!}{(n-r)!}\left(\frac{p}{\theta}\right)^r \prod_{i=1}^{r} x_i^{(p-1)} \exp\left(-\frac{\delta}{\theta}\right) \quad ; \tag{4}$$

where $\delta = \sum_{i=1}^{r} x_i^P + (n-r)x_r^P$.

The logarithm of the likelihood function is

$$\log L(x|\theta, p) \ \propto \ r \, log \, p - r \log \, \theta + (p-1)\sum_{i=1}^{r} \log x_i - \frac{\delta}{\theta} \ ; \tag{5}$$

assuming that 'p' is known, the maximum likelihood estimator $\hat{\theta}_{ML}$of θ can be obtain by using equation(5) as

$$\hat{\theta}_{ML} = r/\delta \tag{6}$$

If both the parameters p and θ are unknown their MLE's $\hat{p}_{ML}$and $\hat{\theta}_{ML}$ can be obtained by solving the following equation

$$\frac{\delta \log L}{\delta \theta} \ = \frac{r}{\theta} - \delta = 0 \, , \tag{7}$$

$$\frac{\delta \log L}{\delta P} = \frac{r}{P} + \sum_{i=1}^{r} \log x_i - \theta \delta_1 = 0, \tag{8}$$

where

$\delta_1 = \sum_{i=1}^{r} x_i^P \log x_i + (n-r)x_r^P \log x_r$, eliminating θ between the two equations of (7-8) and simplifying we get

$$\hat{\theta}_{ML} = \frac{\delta}{r} \ \ , \tag{9}$$

equation (7-8) may be solved for Newton-Raphson or any suitable iterative Method and this value is substituted in equation (8) by replacing θ with $\hat{\theta}_{ML}$, we get $\hat{p}_{ML}$ as

$$\hat{p}_{ML} = \frac{r\delta}{\delta^*} \tag{10}$$

where $\delta^* = [r\delta_1 - \delta \sum_{i=1}^{r} log x_i]$

3. Bayes Estimator of θ when shape Parameter p is known

If p is known, assume gamma prior $\gamma(\alpha, \beta)$ as natural conjugate prior for θ as

$$g(\theta|\underline{x}) = \frac{\beta^\alpha}{\Gamma\alpha}\left(\frac{1}{\theta}\right)^{(\alpha+1)} \exp\left(-\frac{\beta}{\theta}\right); (\alpha, \beta) > 0, \theta > 0 \, . \tag{11}$$

The posterior distribution of θ using equation (1) and (11) we get

$$h(\theta|\underline{x}) = \frac{(\delta+\beta)^{r+\alpha}}{\Gamma(r+\alpha)}\left(\frac{1}{\theta}\right)^{(r+\alpha+1)} \exp\left(-\frac{(\delta+\beta)}{\theta}\right) . \quad (12)$$

Under General Entropy Loss Function, the Bayes estimator $\hat{\theta}_{BE}$ of θ using equations (3) and (12) given by

$$\hat{\theta}_{BE} = \frac{(\delta+\beta)}{(\alpha+r)}, \quad (13)$$

4. The Bayes estimators with θ and p unknown

The joint prior density of θ and p is given by

$$G(\theta|p) = g_1(\theta|p).\, g_2(p)$$

$$G(\theta|p) = \frac{1}{\eta\Gamma v} p^{-v}\left(\frac{1}{\theta}\right)^{(v+1)} .\exp\left[\left\{-\frac{1}{\theta p}+\frac{p}{\eta}\right\}\right] \ ; (\theta, p, \eta, v) > 0, \quad (14)$$

where $g_1(\theta|p) = \frac{1}{\Gamma v} p^{-v}\left(\frac{1}{\theta}\right)^{(v+1)} .\exp\left[-\frac{1}{\theta p}\right]$, (15)

And $g_2(p) = \frac{1}{\eta}\exp\left(-\frac{p}{\eta}\right)$, (16)

The joint posterior density of θ and p is

$$h^*(\theta, p|\underline{x}) = \frac{\frac{1}{\eta\Gamma v}p^{-v}\left(\frac{1}{\theta}\right)^{(v+1)}\exp\left[-\left\{\frac{1}{\theta p}+\frac{p}{\eta}\right\}\right]\left(\frac{p}{\theta}\right)^r \prod_{i=1}^{r} x_i^{(p-1)} e^{-p/\theta}}{\iint \frac{1}{\eta\Gamma v}p^{(r-v)}\left(\frac{1}{\theta}\right)^{(r+v+1)} \prod_{i=1}^{r} x_i^{(p-1)} .\exp\left[-\left\{\frac{1}{\theta p}+\frac{p}{\eta}+\frac{p}{\theta}\right\}\right] d\theta dp}, \quad (17)$$

Approximate Bayes Estimators

The Bayes estimators of a function $\mu = \mu(\theta, p)$ of the unknown parameter θ and p under squared error loss is the posterior mean

$$\hat{\mu}_{BS} = E(\mu|\underline{x}) = \frac{\iint \mu(\theta,p) G(\theta,p|\underline{x}) d\theta dp}{\iint G(\theta, p|\underline{x}) .d\theta .dp}, \quad (18)$$

To evaluate (18), consider the method of Lindley approximation

$$E(\mu(\theta,p)|\underline{x}) = \frac{\int \mu(\theta).e^{(l(\theta)+\rho(\theta))} d\theta}{\int e^{(l(\theta)+\rho(\theta))} .d\theta}, \quad (19)$$

where $(\theta) = \log g(\theta)$, and $g(\theta)$ is an arbitrary function of θ and $l(\theta)$ is the logarithm likelihood function.

The Lindley approximation for two parameters is given by

$$E(\hat{\mu}(\theta,p)|\underline{x}) = \mu(\theta,p) + \frac{A}{2} + \rho_1 A_{12} + \rho_2 A_{21} + \frac{1}{2}[l_{30}B_{12} + l_{21}C_{12} + l_{12}C_{21} + l_{03}B_{21}]$$
(20)

where

$$A = \sum_{1}^{2}\sum_{1}^{2} \mu_{ij}\sigma_{ij} ; \quad l_{\eta\epsilon} = (\delta^{(\eta+\epsilon)} l|\delta\theta_1^{\eta}\delta\theta_2^{\epsilon});$$

where$(\eta+\epsilon)=3$ for i,j= 1,2 ; $\rho_1=(\delta\rho|\delta\theta_i)$;

$$\mu_i=\frac{\delta\mu}{\delta\theta_i}\ ;\ \ \mu_{ij}=\frac{\delta^2\mu}{\delta\theta_i\delta\theta_j}\ ;\forall i\neq j\ ;$$

$$A_{ij}=U_i\sigma_{ij}\ ;\ \ B_{ij}=\left(U_i\sigma_{ij}+U_j\sigma_{ij}\right)\sigma_{ii}\ ;$$

$$C_{ij}=3U_i\sigma_{ii}\sigma_{ij}+U_j\left(\sigma_{ii}\sigma_{jj}+2\sigma_{ij}^2\right);$$

Where σ_{ij} is the (i,j)th element of the inverse of matrix $\{-l_{jj}\}; i,j=1,2$ s.t. $l_{ij}=\frac{\delta^2 l}{\delta\theta_i\delta\theta_j}$. All the function in above equations are evaluated at MLE of (θ_1,θ_2).In our case $(\theta_1\theta_2)=(\theta,p); So\ \mu(\theta)=\mu(\theta,p)$

To apply Lindley approximation (19), we first obtain σ_{ij} , elements of the inverse of $\{-l_{jj}\}; i,j=1,2,$ which can be shown to be

$$\sigma_{11}=\frac{Y}{D},\sigma_{12}=\sigma_{21}=\frac{Q}{D},\sigma_{22}=\frac{Z}{D}; \tag{21}$$

where $Y=\left(\frac{r}{p^2}+\frac{\delta_{11}}{\theta}\right); Q=-\frac{\delta_{11}}{\theta^2}; Z=\frac{1}{\theta^2}\left(-r+\frac{2\delta}{\theta}\right);$

$$\delta_{11}=\sum_{\mathrm{i=1}}^{\mathrm{r}}x_i^p\,(logx_i)^2+(n-r)x_r^p.(logx_r)^2;$$

and $D=\left[\frac{1}{\theta^2}\left(-r+\frac{2\delta}{\theta}\right)\left(\frac{r}{p^2}+\frac{\delta_{11}}{\theta}\right)-\frac{\delta_{11}^2}{\theta^4}\right];$

To evaluate ρ_i , we take the partial derivatives of the logarithm of joint prior $G(\theta|p)$ as,

$$G(\theta|p)=\frac{1}{\eta\Gamma v}p^{-v}\left(\frac{1}{\theta}\right)^{(v+1)}.\exp\left[\left\{-\frac{1}{\theta p}+\frac{p}{\eta}\right\}\right]\ ;(\theta,p,\eta,v)>0,$$

$$\Rightarrow \log[G(\theta|p)]=constant-vlogp-(v+1)log\theta-\frac{1}{\theta p}-\frac{p}{\eta}$$

Therefore

$$\rho_1=\frac{\partial\rho}{\partial\theta}=\frac{1-(v+1)\theta p}{\theta^2 p}\ ; \tag{22}$$

and

$$\rho_2=\frac{1-v\theta p}{p^2\theta}-\frac{1}{\eta}\ ; \tag{22a}$$

Further more

$$L_{21}=-\frac{2}{Q^3}\delta_{11}\,;L_{12}=\frac{\delta_{12}}{\theta^2}\,;L_{03}=-\left(\frac{2r}{p^3}+\frac{\delta_{13}}{Q}\right); \tag{22b}$$

$$and\ \ L_{30}=\frac{2}{Q^3}\left(\frac{3\delta}{Q}-r\right); \tag{22c}$$

where $\mu=\mu(\theta,p)\,;(i\neq j)=1,2$

$$A_{12}=\mu_1\sigma_{11}+\mu_2\sigma_{21};\ \ A_{21}=\mu_2\sigma_{22}+\mu_1\sigma_{12}; \tag{22d}$$

$B_{12} = (\mu_1\sigma_{11} + \mu_2\sigma_{12})\sigma_{11}\ ;\ \ B_{21} = (\mu_2\sigma_{22} + \mu_1\sigma_{21})\sigma_{22}\ ;$ (22e)

$C_{12} = 3\mu_2\sigma_{11}\sigma_{12} + \mu_2(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2);$ (22f)

$C_{21} = 3\mu_2\sigma_{22}\sigma_{21} + \mu_1(\sigma_{11}\sigma_{22} + 2\sigma_{21}^2);$ (22g)

Substituting the above values in equation (20),we get

$$E\left(\hat{\mu}(\theta,p)\middle|\underline{x}\right) = \mu(\theta,p) + \frac{1}{2}[\mu_{11}\sigma_{11} + \mu_{21}\sigma_{21} + \mu_{12}\sigma_{12} + \mu_{22}\sigma_{22}] + \rho_1(\mu_1\sigma_{11} + \mu_2\sigma_{21}) + \rho_2(\mu_2\sigma_{22} + \mu_1\sigma_{12}) + \frac{1}{2}[l_{30}\,(\mu_1\sigma_{11} + \mu_2\sigma_{13})\sigma_{11} + l_{21}(3\mu_1\sigma_{11}\sigma_{12}) + \mu_2\sigma_{11}\sigma_{22} + 2\mu_2\sigma_{12}^2) + \mu_{12}\big(3\mu_2\sigma_{22}\sigma_{21} + \mu_1\sigma_{11}\sigma_{22} + 2\mu_1\sigma_{21}^2 + l_{03}\big(\mu_2\sigma_{22}^2\,\mu_1\sigma_{21}\sigma_{22}\big)\big]\ , \quad (23)$$

$$E\left(\hat{\mu}(\theta,p)\middle|\underline{x}\right) = \mu(\theta,p) + \frac{1}{2}\left[\mu_{11}\frac{Y}{D} + \mu_{21}\frac{Q}{D} + \mu_{12}\frac{Q}{D}\right] + \mu_{22}\frac{Z}{D} + \frac{1-\theta p(v+1)}{\theta^2 p}\left(\mu_1\frac{Y}{D} + \mu_2\frac{Q}{D}\right) + \rho_2\mu_2\sigma_{22} + \rho_2\mu_1\sigma_{12} + \frac{1}{2}l_{30}\mu_1\sigma_{11}^2 + \frac{1}{2}l_{30}\mu_2\sigma_{11}\sigma_{12} + \frac{3}{2}\mu_1 l_{21}\sigma_{11}\sigma_{12} + \frac{1}{2}l_{21}\mu_2\sigma_{11}\sigma_{12} + \frac{1}{2}l_{21}2\mu_2\sigma_{12}^2 + \frac{1}{2}l_{12}3\mu_2\sigma_{22}\sigma_{21} + \frac{1}{2}l_{03}\mu_1\sigma_{21}\sigma_{22} \quad (24)$$

$$E\left(\hat{\mu}(\theta,p)\middle|\underline{x}\right) = \mu_1[\frac{Y}{D}\left\{\frac{1-\theta p(v+1)}{\theta^2 p}\right\} + \rho_2\sigma_{12} + \frac{l_{30}}{2}\sigma_{11}^2 + \frac{3}{2}l_{21}\sigma_{11}\sigma_{12} + \frac{l_{12}}{2}\sigma_{11}\sigma_{22} + \frac{l_{03}}{2}\sigma_{21}\sigma_{22}]\ + \mu_2[\frac{\theta}{D}.\frac{(1-\theta p(v+1))}{\theta^2 p} + \rho_2\sigma_{22} + \frac{l_{30}}{2}\sigma_{11}\sigma_{12} + \frac{1}{2}l_{21}\sigma_{11}\sigma_{22} + l_{21}\sigma_{12}^2 + \frac{3}{2}l_{12}\sigma_{22}\sigma_{21} + \frac{l_{03}}{2}\sigma_{22}^2] + \mu(\theta,p) + A, \quad (26)$$

$$E\left(\hat{\mu}(\theta,p)\middle|\underline{x}\right) = \mu(\theta,p) + A + \mu_1\phi_1 + \mu_2\phi_2\ , \quad (25)$$

where

$$\phi_1 = \frac{1}{\theta^2 D^2}\,[\,\frac{YD}{p}\big(1-\theta p(v+1)\big) - \delta_{11}D\left\{\frac{1-v\theta p}{\theta^2 p} - \frac{1}{\eta}\right\} + \frac{Y^2}{\theta}\left(\frac{3\delta}{\theta} - r\right) + + \frac{3}{\theta^3}\sigma_{11}^2 Y + \frac{1}{2}\delta_{12}YZ + \frac{\delta_{11}^2\delta_{12}}{\theta^4} + \frac{1}{2}\left(\frac{2r}{p^3} + \frac{\delta_{13}}{\theta}\right)\delta_{11}\,Z]\quad ; \quad (26)$$

$$\phi_2 = \frac{1}{\theta^4 D^2}\,[\frac{(\{\eta(1-vp)-\theta^2 p)\}zD\theta^2}{p\eta} - \frac{\delta_{11}D\big(1-\theta p(v+1)\big)}{p} - \frac{3}{2}\left(\frac{3\delta}{\theta} - r\right)\frac{r\delta_{11}}{\theta} - \delta_{11}\theta YZ - \frac{2\delta_{11}^3}{\theta} - \frac{3}{2}\delta_{11}\delta_{12}Z - \frac{Z^2\theta^3}{2}\left(\frac{2r\theta+\delta_{13}p^3}{p^3}\right), \quad (27)$$

All the function of right hand side are to be evaluated for $\hat{\theta}_{ML}$ and $\hat{p}_{ML}$.

Approximate Bayes Estimators Under Entropy loss function

With equations(3),(23)-(27), the Approximate Bayes estimators under Entropy Loss Function, using Lindley's approximation are given as

Special Cases

(i) substituting $\mu(\theta,p) = \frac{1}{\theta}$ in equation(20) then;

The approx. Bayes Estimator of θ, under Entropy loss function is

$$\hat{\theta}_{ABE} = \left[E_h\left(\frac{1}{\theta}\right)\right]^{-1}$$

which gives

$$\hat{\theta}_{ABE} = \theta \left[1 + \frac{y}{\theta^2 D} - \frac{\phi_1}{\theta}\right]^{-1}; \quad (28)$$

(ii) substituting $\mu(\theta, p) = \frac{1}{p}$ in equation(22) then ;

The approximate Bayes estimator of p under Entropy loss function is

$$\hat{p}_{ABE} = \left[E_h\left(\frac{1}{p}\right)\right]^{-1},$$

which gives

$$\hat{p}_{ABE} = p \left[1 + \frac{Z}{p^2 D} - \frac{\phi_2}{p}\right]^{-1}; \quad (29)$$

5. Simulations and Numerical Comparison

The simulations and numerical calculations are done by using R-Language programming and results are presented below.

1. We have taken the different sizes of samples n=25, 35, 45, 55 and 80 with failure censoring. The Approximate Bayes estimators under General Entropy loss function(GELF), the MLE's of θ , the Bayes estimators under GELF and their respective MSE's are obtained by repeating the steps 2000 times, parameters of prior distribution α =2 , β =3 and hyper parameters of joint prior distribution v= 10 and η = 35 with Weibull parameters $\theta = 2$ and p=1.5. The estimated values and their MSE's (in square parenthesis) are presented in the tables (1).

Table(1)

Mean and MSE's of θ and p

($\theta = 2, p = 1.5, v = 10, \eta = 35$)

n	r	$\hat{\theta}_{ML}$	$\hat{\theta}_{BE}$	$\hat{\theta}_{ABE}$
25	20	1.150681	1.188578	1.878511
		[0.000360671]	**[0.0003292025]**	**[7.37974e-06]**
35	30	1.804628	1.751401	2.616886
		[1.90850e-05]	**[1.104086e-05]**	**[0.0001902745]**
45	40	1.669568	1.879024	2.661863
		[5.45926e-05]	**[5.151276e-05]**	**[0.000290217]**
55	45	2.023018	1.999028	2.728531
		[2.64916e-07]	**[4.726138e-10]**	**[0.000197525]**
80	60	3.953938	0.7554244	2.796781
		[0.001908936]	**[0.000774484]**	**[0.000317430]**

Conclusions

1. Table (1) presents the MLE of parameter θ, Bayes estimates of θ under GELF (for known p) , Approximate Bayes estimate GELF (for θ and p both unknown) and their respective MSE's. It also presents the mean and MSE's of p and Approximate Bayes estimates of p (for θ and p both unknown) under GELF. The estimates of θ have minimum MSE's for sample size n=25, as the sample sizes increase it started decreasing. At sample size n≥ 80 ,it shows the tendency of increasing MSE's .We observe here that the effective range of sample size 'n' for better

estimate of parameters are as (i) For $\hat{\theta}_{ML}$ and $\hat{\theta}_{ABE}$,the effective range of n is 25-80, (ii)For the other estimates $\hat{\theta}_{BE}$,the MSE's continue to be increasing, so their effective sample size range is much larger than $\hat{\theta}_{ML}$ and $\hat{\theta}_{ABE}$. Among all the estimators $\hat{\theta}_{ABE}$ under GELF has the lowest MSE.

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