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CONTACT TORIC MANIFOLDS

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Abstract: Through the review of some ideas and the use of illustrative examples, we shall lay a complete basis for the definition of contact toric manifolds in this work. We are motivated by Toth and Zelditch's hypotheses on the singularity of toric integrable actions on punctured cotangent bundles of the n-torus $T^n$and of two spheres S^2.

Keywords: Symplectic and Contact manifolds , Toric integrable , Contact -Moment map .

1. Introduction

The dynamics of the geodesic flows on a compact Riemannian manifold (Q, g) and the growth rate $L^\alpha$of the norms of the square-integrable Eigen functions (L^2- normalized Eigen functions) of the Laplace operator ∇_gon the manifold Q were both studied by Toth and Zelditch recently.This took place about 2002, therefore no matter how recently we say it occurred, it must be at least that old. They demonstrated that if the metric g is flat and the square root of the Laplace operator $\sqrt{\nabla_g}$ is "Quantum completely integrable" and possesses uniformly bounded Eigen functions, then Q is finitely covered by a torus (by the Bieberbach theorems).

This is more specific in the case where the geodesic flow is toric integrable. In this sense, a torusT^n, n= dim Q, has an effective influence on the punctured cotangent bundle of T^*Q/Q of Q.

i. Commutes accompanied with dilations

   i.e $\rho_t: T^*Q/Q \to T^*Q/Q$, ρ(q, p) =(q, e^tp)

ii. Preserves the standard (typical) symplectic form on T^*Q and

iii. Preserves the energy function, $h(q,p) = g_q * (p,p)$

Where g^*denotes the metric on T^*Q dual to g. (Geodesic flow is the Hamiltonian flow of h).

Remark: Any symplectic group action that commutes with dilations on the punctured cotangent bundle retains the Liouville 1-form and is hence Hamiltonian. Therefore, the pullback metric on a finite cover of Q is toric integrable if a metric on a manifold Q is toric integrable. We must determine whether toric integrability on its own or the boundedness of eigen functions is required for the metric to be flat where Q is a torus.

Theorem 1.1: If g is a toric integrable metric on the torus T^n:=R^n/Z^n. Consequently, g is flat.

Proof: Assume that g is a toric integrable on T^n, which means that any (effective) action of the torus T^non the punctured cotangent bundle $T*T^n/_{T^n}$, which maintains the symplectic form commutes with dilation are free. Any action of a group G on a punctured cotangent bundle $\frac{T*Q}{Q}$ which commutes with dilation induces an action of Gon the orbit space $S(T*Q) = (T*Q/Q)/R$, which is the cosphere bundle of Q. Additionally, any such symplectic action results in a contact action on $S(T*Q)$.We know that dim $S(T*T^n) = 2dimT^n -$

1. This contributes to the definition of a contact toric manifold as a triple with the form(M, ξ, T), where M is a $2n-1$ dimensional manifold.
A torus $T^n$is effectively preserved when there is a contact structure on N and $\tau: T^n \times N \to N$.Additionally, a topological classification of compact connected toric manifolds with non free torus actions and evidence that none of these manifolds could possess the homotopy type of $S(T*T^n) \cong T^n \times S^{n-1}$are provided. These manifolds are categorized by rational polyhedral cones on simple polytopes, with exceptional series occurring in dimensions of 3 and 5. And this theorem implies that some classes of matrices on tori are flat thanks to the work of Toth and Zeldich, who named these classes "toric integrable" (see (2)).

A word about notation: The vector space dual of a Lie group G is denoted by g^*, and its Lie algebra is represented by G throughout this work. When a Lie group G acts on a manifold N we designate the action by an element $g \in G$on a point $x \in N$by $g.x$; $G.x$represents the G-orbit of x and the vector field induced on m by an element X of the Lie algebra g of G is denoted by $X_N = \frac{d}{dt}|0(expt\, tX)$.
A contact form and $\xi = ker\alpha$ co-oriented contact structure are always indicated by the symbols x and α.

2. Group Actions On Contact Manifolds

Definition 2.1: A 1-form α on a manifold N is contact if $d\alpha/\xi$ is non-degenerate and $\xi = ker\alpha$ is a co-dimensional -1 distribution if $\alpha_x \neq 0$ for all $x \in N$. Such that the manifold N is invariably odd dimensional and the vector bundle $\xi \to N$ is required to have fibers of even dimensions.

Definition 2.2: A co-dimensional distribution of -1. If the oriented line bundle that serves as its annihilator ξ^0 is a global section that is nowhere vanishing, then ξ on N is co-orientable. It is co-oriented if one component $\xi^0{}_+$ of $\xi^0/\{0\} \subseteq t*N$ (ξ minus the zero section) is chosen.

Definition 2.3: A co-oriented contact structure ξ on N is a co-oriented codimension-1 distribution thus $\xi^0/\{0\}$ is a symplectic sub manifold of the cotangent bundle $T*N$(the cotangent bundle is given the canonical symplectic form).We use the symbol $\xi^0{}_+$ to represent the component of $\xi^0/\{0\}$ and then think of it as its symplectization of (N, ξ).We are aware that N is odd-dimensional and that $\xi^0{}_+$ is a line bundle on N, hence the line bundle is dimensional.

Definition 2.4: Suppose a lie group G acting on a manifold M while maintaining a 1-form α, the corresponding α- moment map $\psi_\alpha: N \to g^*$for this group is defined by $< \psi_\alpha(x), X >= \alpha_x(X_N(x))$ for all $x \in N$ and X $\in g$ of G, Where $X_N(x) = \frac{d}{dt}|0(\ expt\, tX).x$is the vector field produced by X.

Remark: If a Lie group G acts on a contact manifold $(N, \xi = ker\alpha)$ through a contact action. Define $\psi: T*N \to$ $\psi(p,q)(X) = q(X(p))$ then the contact moment map μ: $\xi^0{}_+ \to g^*$ is the restriction $\mu = \psi|_{\xi^0+}$.

3. Contact Toric Manifolds With Examples

If the element that fixes every point on m is the identity, then the group G's action on the manifold N is said to be effective.
Definition 3.1: A co-oriented contact manifold $(N,\ \xi = ker\alpha)$with a totally integrable structure that preserves the contact structure ξ is known as a contact toric G- manifold (i.e an effective contact action of a torus G such that $2dimG = dimN + 1$). As a result, component $\xi^0{}_+$ of $\xi^0_+/\{0\}$ has an action of G that is a fully integrable Hamiltonian action, and also $\xi^0{}_+$ is a symplectic toric manifold.

Remark: We shall treat a contact toric G-manifold as a triple $(N,\ \xi = ker\alpha\,, \mu$: $\xi^0{}_+ \to g^*)$equivalent to the symplectic situation.

Definition 3.2: Let $(N,\ \xi = ker\alpha)$ be a co-oriented contact manifold with an action of a Lie group G that preserves the co-oriented contact structure $\xi = ker\alpha$.The **moment cone**$C(\psi)$for the map ψ: $\xi^0{}_+ \to g^*$is the set $\psi\,(\xi^0{}_+) \cup \{0\}$.Assuming that α is a G-invariant contact form, we can write$C(\psi) = \{(tf|f\ \in \psi_\alpha(N)\ t \in [0,\infty\,)\}$, where $\psi_\alpha: N \to g^*$ stands for the α-moment map.

Example 3.3: Let $S^3 = \{(x_0, y_0, x_1, y_1) \in R^4 : ({x_0}^2 + {y_0}^2) + ({x_1}^2 + {y_1}^2) = 1$ be the standard 3-sphere with the contact form $\alpha = (x_0 dy_0 - y_0 dx_0) + (x_1 dy_1 - y_1 dx_1)$ the 2- torusT^2=$S^1 \times S^1$={(θ_0, θ_1)} acts on $S^3$by rotation of (x_0, y_0) $and(x_1, y_1)$ $planesby\theta_0$ and θ_1 respectively. (i.e T^2 action on S^3 is generated by the vector fields $H_i = \left(x_i \partial_{y_i} - y_i \partial_{x_i}\right)$ for i=0,1 on S^3) this shows that action preserves the contact stucture and is effective. Therefore S^3 is a contact toric T^2 -manifold. Consider the α - moment map $\psi_\alpha: S^3 \to g^*$, where we identify the lie algebra of the torus as $g^* \cong (R)^{2^*}$= span$\{{e_0}^*, {e_1}^*\}$ with $e_i \in g$ generating H_i.

Then we have

$$\begin{aligned}\psi_\alpha(x_0, y_0, x_1, y_1)(e_i) &= \alpha_{(x_0,y_0,x_1,y_1)}((e_i)^{\#}((x_0, y_0, x_1, y_1)) \\ &=\alpha_{(x_0,y_0,x_1,y_1)}(\left(x_i \partial_{y_i} - y_i \partial_{x_i}\right)) \\ &= ({x_i}^2 + {y_i}^2)\end{aligned}$$

From this, we have $\psi_\alpha(x_0, y_0, x_1, y_1) = ({x_0}^2 + {y_0}^2){e_0}^* + ({x_1}^2 + {y_1}^2){e_1}^*$.

As $({x_0}^2 + {y_0}^2) + ({x_1}^2 + {y_1}^2) = 1$

This shows that the image of the α-moment map is$\psi_\alpha(S^3)$=$\{t_0{e_0}^* + t_1{e_1}^*$: $t_0 + t_1$=1, $t_i \geq 0\}$

That is, $\psi_\alpha(S^3)$ is the standard 1-simplex in g^* .Then the moment cone is C(ψ) = $\{ s_0{e_0}^* + s_1{e_1}^*: s_i \geq 0\}$ this is the first quadrant in $g^* \cong (R)^{2^*}$.

Remark: If we expand the aforementioned example to higher dimensions, the torus T^{n+1} interacts with the sphere S^{2n+1} via rotations caused by the vector fields $H_i = \left(x_i \partial_{y_i} - y_i \partial_{x_i}\right)$ for $i = 0,1,2, \dots \dots \dots, n$ on S^{2n+1}. As a result, the action S^{2n+1} is a contact toric T^{n+1} manifold and the image of the α-moment map is $\psi_\alpha(S^{2n+1}) = \{\sum_{i=0}^{n} t_i {e_i}^* : \sum_{i=0}^{n} t_i$=1, $t_i \geq 0\}$. To put it another way, $\psi_\alpha(S^{2n+1})$ is $g^* \cong (R)^{2^*}$ is default n-simplex. As a result, C(ψ)= $((R)^{n+1} \geq 0)^*$is the moment cone.

Example 3.4: Consider the manifold$N = S^1 \times T^2 = \{(x, \theta_0, \theta_1)\}$ with the contact form $\alpha = cosx d\theta_0 + sinx d\theta_1$, the 2-torus $T^2 = \{(\varphi_0, \varphi_1)\}$ acts on N $= S^1 \times T^2 = \{(x, \theta_0, \theta_1)$ by term wise addition on the second factor ,this action is free and preserves the contact structure. Thus $S^1 \times T^2$ is a contact toric manifold. Consider the α-moment map is $\psi_\alpha: S^1 \times T^2 \to g^* \cong span\{\left(d\varphi_0|0\right., (d\varphi_1|0)\}$

Then we have $\psi_\alpha(x, \theta_0, \theta_1)\left(\partial_{\varphi_0}|0\right) = \alpha_{(x,\theta_0,\theta_1)}((\partial_{\varphi_0}|0)^{\#}(x, \theta_0, \theta_1)$

$$\begin{aligned}&= \alpha_{(x,\theta_0,\theta_1)}((\frac{d}{dt}\Big|t = 0\ expt\left(\partial_{\varphi_0}|0\right).(x, \theta_0, \theta_1)\Big) \\ &= \alpha_{(x,\theta_0,\theta_1)}((\frac{d}{dt}\Big|t = 0\,.(x, \theta_0 + 1, \theta_1)\Big) \\ &= \alpha_{(x,\theta_0,\theta_1)}((\partial_{\varphi_0}\big|(x, \theta_0, \theta_1)) \\ &= cosx\end{aligned}$$

Similarly $\psi_\alpha(x, \theta_0, \theta_1)\left(\partial_{\varphi_1}|0\right) = sinx$.

Therefore we have $\psi_\alpha(x, \theta_0, \theta_1) = cosx.\left(\partial_{\varphi_0}|0\right) + sinx.\left(\partial_{\varphi_1}|0\right)$. This shows that the image of the α-moment map is $\psi_\alpha(S^1 \times T^2) = \{ cosx.(d\varphi_0|0) + sinx.(d\varphi_1|0): x \in S^1\}$. And the moment cone is

$$\begin{aligned}C(\psi) &= \{s\, cosx.(d\varphi_0|0) + s\, sinx.(d\varphi_1|0): x \in S^1, s \geq 0\} \\ &=\{s_0(d\varphi_0|0) + sinx.(d\varphi_0|0): s_0 s_1 \in R\} \\ &= g^* \cong (R^2)^* \cong R^2\end{aligned}$$

In light of this, we can now generalize this to the same manifold $S^1 \times T^2 = \{(x, \theta_0, \theta_1)\}$ and the contact form $\alpha_n = cosnx d\theta_0 + sinnx d\theta_1$with n positive integers followed by the moment cone $C(\psi)$=$g^* \cong (R^2)^* \cong R^2$.

4. Properties Of Contact Moment Maps

Definition 4.1: If there is a contactomorphism (diffeomorphism) $\varphi: N \to N'$(such that $\varphi^* \xi' = \xi$ and $\varphi^* {\psi'}_\alpha = \psi_\alpha$) that preserves co-orientation, two contact toric G-manifolds (N, $\xi = ker\alpha$,ψ_α: N→ g^*) and (N', $\xi' = ker\alpha'$,${\psi'}_\alpha: N' \to g^*$) are isomorphic.The term "isomorphism of contact toric manifolds" will be used to describe such a map. We designate the collection of isomorphisms of (N, $\xi = ker\alpha$,ψ_α: N→ g^*) by $Iso(N', \xi' = ker\alpha', {\psi'}_\alpha: N' \to g^*)$=Iso (N).

Definition 4.2: If g^* is the dual of a torus' Lie algebra G, then the rational polyhedron cone $C \sqsubset R^{n+1}$ is the intersection of precisely k facets whose set of normals may be completed to an integral base $Z^{n+1}$is good if there exists a minimal set of primitives $\{v_j\} \in Z^{n+1}$with $d \geq n+1$such that $C = \sum_{j=1}^{d}\{x \in R^{n+1}: l_j(x) = \langle x, v_j\rangle \geq 0\}$ and any co-dimension k face of C.

i.e Each $\{v_j\}$ is a basic component of the lattice $Z^{n+1} \sqsubset R^{n+1}$, where d$\geq n+1$ is the facet count.

Definition 4.3: If ${}^{N}\!/_{G}$ represents the orbit space of the torus's action on a manifold N, where$\pi: N \to {}^{N}\!/_{G}$ serves as the orbit map, and $\psi_\alpha: N \to S(g^*)$ serves as the moment map for the torus' action on a manifold N that maintains the contact form α, the **orbital moment map** is given by the α-moment map, a map called $\bar{\psi}_\alpha: {}^{N}\!/_{G} \to S(g^*)$ is produced.

Definition 4.4: Two contact toric G-manifolds $(N,\ \xi = ker\alpha\,, \psi_\alpha$: N→ $g^*)$ and $(N',\ \xi' = ker\alpha'\,, \psi'_\alpha: N' \to g^*)$ are **locally isomorphic** if there exists a homeomorphism $\rho: N/G \to N'/G$ and for any point $x \in N/G$ there is a neighbourhood $V \subseteq N/G$ of x and an isomorphism of contact toric manifolds $\rho_V: \pi^{-1}(V) \to (\pi')^{-1}(\rho(V))$ such that $\pi' \circ \rho_V = \bar{\rho} \circ \pi$.

In conclusion: We reviewed all the fundamental notions derived from contact geometry, such as contact form, and introduced the concept of Symplectic manifolds and different geometrical objects related to them, such as moment maps. In order to make these notions more clear, a few contact toric G-manifold instances are given. And at long last, I've begun my search for evidence to support the comprehensive classification of compact linked contact toric manifolds.

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6. References

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