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Thermal Stress Analyses Of Laminated Beams Under Plane Stress Condition Of Elasticity

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Abstract

Thermal stress analysis of laminates under plane stress conditions of elasticity have been performed with mixed semi-analytical model. The displacements and transverse stresses that occur naturally at an interface of laminae are considered as fundamental dependent variables and thus continuity of transverse stresses and displacements are implicitly maintained at the laminae interfaces. The mathematical model consists of defining a two-point boundary value problem (BVP) governed by a set of coupled first-order ordinary differential equations (ODEs). The accuracy and the effectiveness of the proposed model are assessed by comparing numerical results from the present investigation with the available elasticity solutions under plane stress conditions.

Keywords: semi-analytical method; laminate; plane stress; thermal load.

Introduction

Thermally induced deformations and stresses in layered composite and sandwich laminates represent a major concern in design of critical structures. These materials are also getting established in relatively new markets such as biomedical and electronic devices and also in civil structures. Due to increase in applications of composites in recent years, determination of thermally induced response is of great interest. Thermal stresses are present in laminates due to different thermal properties of the adjacent layers and due to change in temperature during the manufacturing processes and/or during service life.

Three dimensional (3D) elasticity solutions based on the solution of partial differential equations (PDEs) with appropriate boundary conditions are valuable because they represent a more realistic and closer approximation to the actual bahaviour of the structures (Tungikar and Rao 1994, Bhaskar and Varadan 1996) but 3D modelling of laminates with a large number of layers becomes intractable due to its complexity. Therefore, researchers have focused their attention on two dimensional (2D) analytical models, viz., classical lamination theory (CLT) (Timoshenko and Woinowsky-Kreiger 1959, Boley and Weiner 1960), first order shear deformation theory (FOST) (Reddy and Chao 1980, Rolfes et al. 1998) and higher-order shear deformation theories (HOSTs) (Khdeir and Reddy 1991, Kant and Khare 1994, Kapuria et al. 2003) for thermal analysis of laminates.

In this paper, a simple and efficient semi-analytical mathematical model is presented for stress analysis of laminated beam under thermal loads. A laminate under plane stress of elasticity is formulated as a two-point BVP governed by a set of coupled first-order ODEs,

$$\frac{d}{dz}y(z) = A(z)y(z) + P(z) \tag{1}$$

in the interval $-h/2 \leq z \leq h/2$ with any half of the dependent variables prescribed at the edges $z = \pm\, h/2$ under thermal loading. Here, $y(z)$ is an n-dimensional vector of fundamental variables whose number (n) equals the order of PDE, $A(z)$ is a n×n coefficient matrix (which is function of material properties in thickness direction) and $P(z)$ is n-dimensional vector of non-homogenous (loading) terms. It is clearly seen that mixed and/or non-homogeneous boundary conditions are easily admitted in this formulation.

Formulation

A layered, narrow beam composed of homogeneous isotropic or orthotropic laminae of uniform thickness subjected to thermal loading is considered. The plan dimension of a beam is $a \times b$ and thickness is h. Under such conditions, the beam domain is in a 2D plane stress condition. A simple support is assumed on the longitudinal edges, $x = 0$, a (Figure 1)

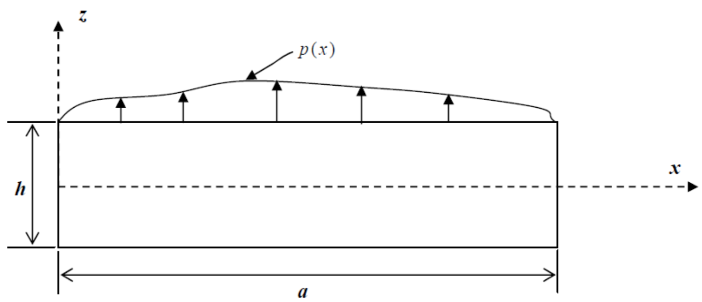


Figure 1: Beam in plane strain condition

The material constitute relations for each layer can be written as,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha_{t1}T \\ \varepsilon_z - \alpha_{t2}T \\ \gamma_{xz} \end{Bmatrix} \tag{2}$$

The reduced material coefficients, C_{ij} for plane stress condition are,

$$C_{11} = \frac{E_1}{1-\upsilon_{13}\upsilon_{31}},\ C_{12} = C_{21} = \frac{\upsilon_{13}E_3}{1-\upsilon_{13}\upsilon_{31}},\ C_{22} = \frac{E_3}{1-\upsilon_{13}\upsilon_{31}},\ C_{33} = G_{13} \tag{3}$$

The 2D equations of equilibrium are,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0,\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \sigma_z}{\partial z} + B_z = 0 \tag{4}$$

where, B_x and B_z are the body forces per unit volume in x and z directions, respectively and from the linear theory of elasticity, the general linear strain-displacement relations in 2D are,

$$\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_z = \frac{\partial w}{\partial z} \text{ and } \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{5}$$

The above Equations (2), (4) and (5) have a total of eight unknowns in eight equations. After a simple algebraic manipulation of the above sets of equations, a set of PDEs involving only four dependent variables are obtained as follows,

$$\frac{\partial u}{\partial z} = \frac{1}{C_{33}}\tau_{xzm} - \frac{\partial w}{\partial x} \tag{6}$$

$$\frac{\partial w}{\partial z}=\frac{1}{C_{22}}\left[\sigma_{zm}C21\frac{\partial u}{\partial x}+(C_{21}\alpha_{t1}+C_{22}\alpha_{t3})T\right]$$
$$\frac{\partial \tau_{xz}}{\partial z}=\left[-C_{11}+\left(\frac{C_{12}C_{21}}{C_{22}}\right)\right]\frac{\partial^2 u}{\partial x^2}-\frac{C_{12}}{C_{22}}\frac{\partial \sigma_z}{\partial x}-\left[\left(\frac{C_{12}C_{21}}{C_{22}}-C_{11}\right)\alpha_{t1}\right]\frac{\partial T}{\partial x}-B_x$$
$$\frac{\partial \sigma_z}{\partial z}=-\frac{\partial \tau_{xz}}{\partial x}-B_z$$

This set of dependent variables is called as ‘primary variables set’. A secondary dependent variable, x _ can be expressed as a function of the primary set of variables as follows,

$$\sigma_x=C_{11}\frac{\partial u}{\partial x}+C_{12}\frac{\partial w}{\partial z}-(C_{11}\alpha_{t1}+C_{12}\alpha_{t3})T \tag{7}$$

The above PDEs defined by Equation (6) can be reduced to a coupled first-order ODEs by using Fourier trigonometric series expansion for primary variables satisfying the simple support end conditions at $x = 0$, a as follows;

$$u(x,z)=\sum u_m(z)\cos\frac{m\pi x}{a},\ w(x,z)=\sum w_m(z)\sin\frac{m\pi x}{a} \tag{8}$$

and from the basic relations of theory of elasticity, it can be shown that,

$$\tau_{xz}(x,z)=\sum \tau_{xzm}(z)\cos\frac{m\pi x}{a},\ \sigma_z(x,z)=\sum \sigma_{zm}(z)\sin\frac{m\pi x}{a} \tag{9}$$

Further, temperature variation along x direction is also expressed in sinusoidal form as,

$$T(x,z)=\sum T_m(z)\sin\frac{m\pi x}{a} \tag{10}$$

Substituting Equations (8)-(10) and its derivatives into Equation (6), the following ODEs are obtained,

$$\begin{aligned}
\frac{du_m}{dz}&=-w_m\frac{m\pi}{a}+\frac{1}{C_{33}}\tau_{xzm}\\
\frac{dw_m}{dz}&=-\frac{C_{21}}{C_{22}}u_m\frac{m\pi}{a}+\frac{1}{C_{22}}\sigma_{zm}+\left[\frac{1}{C_{22}}(C_{21}\alpha_{t1}+C_{22}\alpha_{t3})\right]T\\
\frac{d\tau_{xzm}}{dz}&=\left[-C_{11}+\left(\frac{C_{12}C_{21}}{C_{22}}\right)\right]\frac{m^2\pi^2}{a^2}u_m-\frac{C_{12}}{C_{22}}\frac{m\pi}{a}\sigma_{zm}-\left[\left(-C_{11}+\frac{C_{12}C_{21}}{C_{22}}\right)\alpha_{t1}\right]\frac{m\pi}{a}T-B_x\\
\frac{d\sigma_{zm}}{dz}&=\frac{m\pi}{a}\tau_{xzm}-B_z
\end{aligned} \tag{11}$$

Equation (11) represents the governing two-point BVP in ODEs in the domain $-h/2 \le z \le h/2$ with stress components known at the top and bottom surfaces of a beam. The basic approach to the numerical integration of the BVP defined in Equation (11) and the associated boundary condition, is to transform the given BVP into a set of IVPs- one non-homogeneous and n/2 homogeneous (Kant and Ramesh 1981). Availability of efficient, accurate and robust ODE numerical integrators for IVPs helps in computing reliable values of the primary variables through the thickness. Changes in material properties are incorporated by changing the coefficients of material matrix appropriately for each lamina. Numerical Investigation Numerical studies have been performed in various symmetric/unsymmetric composite laminates for validation of the presented analytical approach. In this paper, results of symmetric, three-layered, square sandwich ((0°/core/0°) beam with simple support end conditions and subjected to thermal load has been presented for sake of brevity. Material properties are presented in Table (1).

Table 1: Material Properties

Layer	Material Properties
Face sheet	$E_1 = 131.1$ (GPa), $E_2 = 6.90$, $E_3 = 6.90$ $G_{12} = 3.588$ (GPa), $G_{13} = 3.588$, $G_{23} = 2.332$ $\nu_{12} = 0.32$, $\nu_{13} = 0.32$, $\nu_{23} = 0.32$ $\alpha_1 = 0.023\times10^{-6}$ (k^{-1}), $\alpha_2 = 22.5\times10^{-6}$, $\alpha_3 = 22.5\times10^{-6}$ $\lambda_1 = 1.5$, $\lambda_2 = 0.5$, $\lambda_3 = 0.5$
Core sheet	$E_1 = 0.2208$ (GPa), $E_2 = 0.2001$, $E_3 = 2760$ $G_{12} = 16.56$ (GPa), $G_{13} = 545.1$, $G_{23} = 455.4$ $\nu_{12} = 0.99$, $\nu_{13} = 3\times10^{-5}$, $\nu_{23} = 3\times10^{-5}$ $\alpha_1 = 30.6\times10^{-6}$ (k^{-1}), $\alpha_2 = 30.6\times10^{-6}$, $\alpha_3 = 30.6\times10^{-6}$ $\lambda_1 = 3.0$, $\lambda_2 = 3.0$, $\lambda_3 = 3.0$

Thickness of each face sheets is one tenth (h/10) of the total thickness of the laminate. The exact 2D thermo-elasticity solutions presented by Kapuria et al. (2003) have been used for proper comparison of the obtained results.

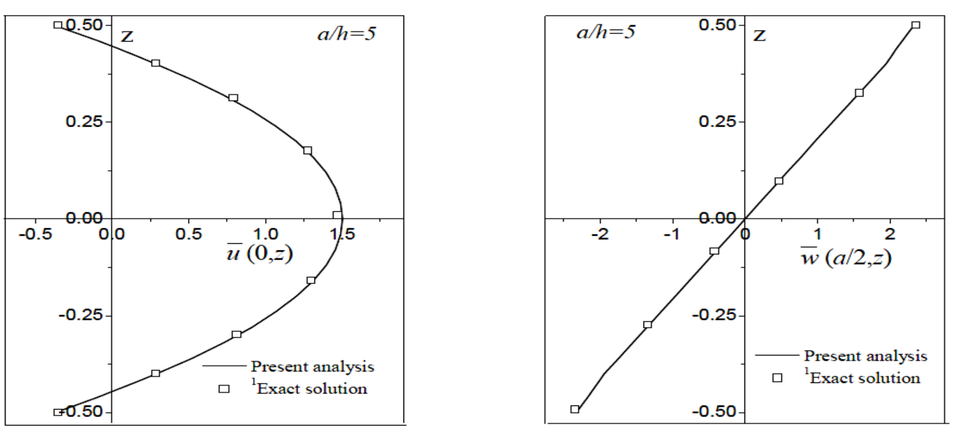
Two thermal load cases are considered here for numerical studies;

1. Equal temperature rise of the bottom and the top surface of the plate with sinusoidal inplane variations: $\Delta T(x,\pm h/2) = T_0 \sin\dfrac{\pi x}{a}$ (Case A),
2. Equal rise and fall of temperature of the top and bottom surface of the plate with sinusoidal in-plane variations: $\Delta T(x,h/2) = -\Delta T(x,-h/2) = T_0 \sin\dfrac{\pi x}{a}$ (Case B).

The variations of temperature across the thickness, for case A and for case B, are presented by Kapuria et al. (2003) by solving exactly 2D thermal problem of heat conduction equation for all layers. Same variations are used here for proper comparison of the obtained results. Following normalizations have been used in all examples considered here for the comparison of the results;

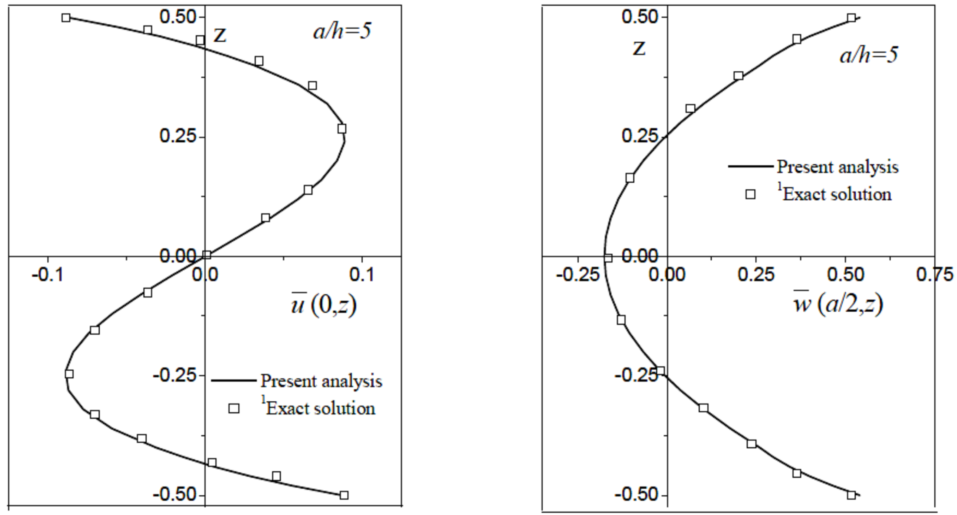
$$s = \frac{a}{h},\ \bar{u} = \frac{100u}{h\alpha_2 T_0 s},\ \bar{w} = \frac{100w}{h\alpha_2 T_0 s^2},\ \bar{\sigma}_x = \frac{\sigma_x}{E_2\alpha_2 T_0},\ \bar{\tau}_{xz} = \frac{s\tau_{xz}}{E_2\alpha_2 T_0},\ \bar{\sigma}_z = \frac{\sigma_z}{E_2 T_0} \tag{12}$$

Figures 2 and 3 show the through thickness variations of in-plane displacement ($\bar{u}$) and transverse displacement ($\bar{w}$) for an aspect ratio of 5 for case A and case B, respectively.



(a) (b)

Figure 2: Through thickness variation (Case A) of normalized
(a) in-plane displacement $\bar{u}$ (b) transverse displacement $\bar{w}$



(a) (b)

Figure 3: Through thickness variation (Case B) of normalized
(a) in-plane displacement $\bar{u}$ (b) transverse displacement $\bar{w}$

Excellent agreement between exact and present results is observed. Through thickness variation of in-plane displacement ($\bar{u}$) is found to be in cubic in nature and transverse displacement ($\bar{w}$) are almost linear with small change in slope at the interface between core and face sheets is observed in case A. On the other hand, in case B, smooth non-linear variation is observed for in-plane displacement without mid-surface stretching. Symmetric behavior of all quantities about the mid-surface is also observed due to symmetric configuration of laminate with respect to geometry and material properties.

Concluding remarks

A simple semi-analytical methodology for thermal analysis of laminates under plane stress condition of elasticity is described in this paper. The proposed mathematical model is very simple, efficient and highly accurate. A two-point BVP governed by a set of linear coupled first order ODEs is formed by assuming all primary variables in the form of trigonometric functions along the inplane directions and methodology is free from any simplified assumptions. Another important feature of this approach is that both the displacements and stresses are computed simultaneously with the same degree of accuracy.

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