AN UPDATED SURVEY OF PENDANT DOMINATION PARAMETERS IN GRAPHS

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Abstract: Let G be any graph. A dominating set S in G is called pendant dominating set if \( \langle S \rangle \) contains at least one pendant vertex. The least cardinality of a pendant dominating set in G is called pendant domination number of G, denoted by \( \gamma_{pe}(G) \). In this survey, we present recent results on pendant dominating sets of graphs.

Keywords: Dominating set, Pendant dominating set.

1 Introduction

Let G be any graph. The concept of paired domination is an interesting concept introduced by Teresa W. Haynes in with the following application in mind. If we think of each vertex \( v \) as the possible location for a guard capable of protecting each vertex in its closed neighborhood, then domination requires every vertex to be protected. For total domination, each guard must, in turn, be protected by other guard. But for paired-domination, each guard is assigned another adjacent one, and they are designated as backups for each other. The authors in [9] introduce pendant domination for which at least one guard is assigned a backup.

2 Basic Definitions

Let G = (V, E) be any graph with |V(G)| = n and |E(G)| = m edges. Then n, m are respectively called the order and the size of the graph G. For each vertex \( v \in V \), the open neighborhood of \( v \) is the set \( N(v) \) containing all the vertices adjacent to \( v \) and the closed neighborhood of \( v \) is the set \( N[v] \) containing \( v \) and all the vertices adjacent to \( v \). Let S be any subset of V, then the open neighborhood of S is \( N(S) = \bigcup_{v \in S} N(v) \) and the closed neighborhood of S is \( N[S] = N(S) \cup S \).

The minimum and maximum of the degree among the vertices of G is denoted by \( \delta(G) \) and \( \Delta(G) \) respectively. A graph G is said to be regular if \( \delta(G) = \Delta(G) \). A vertex \( v \) of a graph G is called a cut vertex if its removal increases the number of components. A bridge or cut edge of a graph is an edge whose removal increases the number of components. A vertex of degree zero is called an isolated vertex and a vertex of a degree one is called a pendant vertex. An edge incident to a pendant vertex is called a pendant edge. The graph containing no cycle is called a tree. A complete bipartite graph \( K_{1,3} \) is a tree called as claw. Any graph containing no subgraph isomorphic to \( K_{1,3} \) is called a claw-free graph.

A set \( M \subseteq E(G) \) is called a matching of G if no two edges in M are incident in G. The two ends of an edge are said to be matched under M. If every vertex of G is matched under M, then M is called a perfect matching. The cardinality of the maximum matching is called the matching number of G, denoted by \( m(G) \).

A subset S of V(G) is a dominating set of G if each vertex \( u \in V - S \) is adjacent to a vertex in S. The least cardinality of a dominating set in G is called the domination number of G and is usually denoted by \( \gamma(G) \).

A dominating set S of a graph G is said to be paired dominating set of G if the induced subgraph \( \langle S \rangle \) contains at least one perfect matching. Any paired dominating set with minimum cardinality is called a minimum paired dominating set. The cardinality of the minimum paired dominating is called the paired
The independent domination number, denoted by \(i(G)\) and the total domination number of \(G\) is either a pendant dominating set in \(G\). A dominating set \(S\) is called a total dominating set if \(S\) contains no isolated vertex. The cardinality of the minimum total dominating set is called the total domination number of \(G\) and is denoted by \(\gamma_t(G)\). A total dominating set with cardinality \(\gamma_t(G)\) is called as \(\gamma_t\)-set.

The set \(S\) of vertices in a graph \(G\) is called an independent set if no two vertices in \(S\) are adjacent. A dominating set \(S\) of a graph \(G\) is an independent dominating set if \(S\) has no edges. The minimum cardinality of an independent dominating set is called the independent domination number, denoted by \(i(G)\) and the independence number \(\beta_0(G)\) is the maximum cardinality of an independent set of \(G\).

The corona of two disjoint graphs \(G_1\) and \(G_2\) is defined to be the graph \(G = G_1 \circ G_2\) formed from one copy of \(G_1\) and \([V(G_1)]\) copies of \(G_2\) where the \(i\)th vertex of \(G_1\) is adjacent to every vertex in the \(i\)th copy of \(G_2\). If \(G\) and \(H\) are disjoint graphs, then the join of \(G\) and \(H\) denoted by \(G + H\) is the graph such that \(V(G+H) = V(G) \cup V(H)\) and \(E(G + H) = E(G) \cup E(H) \cup uv : u \in V(G), v \in V(H)\). The line graph \(L(G)\) of a graph \(G\) is the graph whose vertex set corresponds to the edges of \(G\) such that two vertices of \(L(G)\) are adjacent if and only if the corresponding edges of \(G\) are adjacent.

Any graph \(G\) with at least one bridge is called a bridged graph. The \(n\)-Barbell graph is the simple graph obtained by connecting two copies of a complete graph \(K_n\) by a bridge. The \(n\) Pan graph is the graph obtained by joining a cycle graph \(C_n\) to a singleton graph \(K_1\) with a bridge. The ladder graph is a Cartesian product of \(P_2\) and \(P_n\) where \(P_n\) is a path graph.

**Theorem 2.1.** [9] Let \(P_n\) be a path with \(n \geq 2\) vertices and \(C_n\) be a cycle with \(n \geq 3\) vertices. Then

\[
\gamma_{pe}(P_n \cup C_n) = \begin{cases} 
\frac{n+1}{3}, & \text{if } n \equiv 0 \pmod{3}; \\
\frac{n}{3}, & \text{if } n \equiv 1 \pmod{3}; \\
\frac{n}{3} + 1, & \text{if } n \equiv 2 \pmod{3}; 
\end{cases}
\]

**Theorem 2.2.** [9] A dominating set \(S\) is a minimal pendant dominating set if and only if for each vertex \(u \in S\) one of the following condition holds.

1. \(u\) is either an isolate or a pendant vertex of \(S\).
2. each vertex of \(S - \{u\}\) lies in a cycle.
3. there exists a vertex \(v \in V - S\) for which \(N(v) \cap S = \{u\}\).

**Theorem 2.3.** [9] Let \(T\) be any Tree. Then \(\gamma_{pe}(T) = \gamma(T)\) if and only if there is a \(\gamma\)-set which is not independent in \(T\).

**Theorem 2.4.** [9] Let \(G\) be any graph. Then \(\gamma_{pe}(G) = \gamma(G)\) if and only if \(G\) contains a \(\gamma\) set which is either an independent set in \(G\) or each vertex of \(S\) belongs to some cycle in \(S\).

**Proposition 2.1.** [9] Let \(G\) be any graph with \(n \geq 3\) vertices. Then \(n-m \leq \gamma_{pe}(G) \leq n - 1\).

Let \(G\) be the collection of graphs of following types. A cycle, path, star, wheel and a complete graph each of order 4 and a path, cycle of order 5.

**Theorem 2.5.** [9] Let \(G\) be a connected graph of order \(n\). Then \(\gamma_{pe}(G) = n - 2\) if and only if \(G \in \mathcal{G}\).

**Theorem 2.6.** [9] Let \(G\) be any graph. Then

\[
\left\lceil \frac{n}{1 + \Delta(G)} \right\rceil \leq \gamma_{pe}(G) \leq n - \Delta(G) + 1. \quad \text{Further if } G \text{ is a tree, then } \gamma_{pe}(G) = n - \Delta(G) + 1 \text{ if and only if } G \text{ is a wounded spider obtained by subdividing even number of edges of a star.}
\]

**Proposition 2.2.** [9] Let \(G\) be an acyclic graph. Then \(\gamma_{pe}(G) \leq \gamma_t(G) \leq \gamma_{pe}(G)\). Equality holds if \(G\) is either a cycle or a path of order \(4k\).

**Theorem 2.7.** [9] For any graph \(G\), \(\gamma_{pe}(G) \leq i(G) + 1\). Equality holds if \(G\) is a claw-free graph. Further, for any positive integer \(k\), there exists a graph \(H\) such that \(i(H) - \gamma_{pe}(H) = k\).

**Theorem 2.8.** [9] Let \(G\) be a graph connected with \(n\) vertices and \(H\) be any graph. Then

\[
\gamma_{pe}(G \circ H) = \begin{cases} 
n + 1, & \text{if } G \text{ is a cycle and } \gamma(H) \geq 2; \\
n, & \text{otherwise.}
\end{cases}
\]

**Proof.** For any connected graph \(G\) and any graph \(H\), we have \(\gamma(G \circ H) = n\) and hence \(\gamma_{pe}(G \circ H) \leq n + 1\). First, suppose \(G\) is not a cycle, then clearly \(V(G)\) itself a pendant dominating set in \(G\). Assume \(G\) is a cycle. If \(\gamma(H) = 1\) then for any vertex \(v \in G\), the set \(S = (V - \{v\}) \cup \{u\}\) is a pendant dominating set in \(G \circ H\), where \(\{u\}\) is a \(\gamma\)-set of \(H\). Therefore, \(\gamma_{pe}(G \circ H) = n\). Suppose \(\gamma(H) \geq 2\), since \(V(G)\) contains no pendant vertex, we must...
have that \( \gamma_{pe}(G \circ H) \geq n + 1 \). On the other hand, for any vertex \( v \) of \( H \), the set \( V(G) \cup \{v\} \) is a pendant dominating set of size \( n + 1 \). Therefore \( \gamma_{pe}(G) = n + 1 \).

The Cartesian product of two graphs \( G_1 \) and \( G_2 \) is the graph, denoted by \( G_1 \times G_2 \), with \( G_1 \times G_2 = V(G_1) \times V(G_2) \) (where \( \times \) denotes the Cartesian product of sets) and two vertices \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) in \( V(G_1 \times G_2) \) whenever \([u_1 = v_1 \text{ and } (u_2, v_2) \in E(G_2)] \) or \([u_2 = v_2 \text{ and } (u_1, v_1) \in E(G_1)]\). If each \( G_1 \) and \( G_2 \) is a path \( P_m \) and \( P_n \) (respectively), then we will call \( P_m \times P_n \), a \( m \times n \) grid graph. For notational convenience, we denote \( P_m \times P_n \) by \( P_{m,n} \).

**Theorem 2.9.** [7] For all \( n \geq 2 \), \( \gamma_{pe}(P_{2,n}) = \left\lceil \frac{2n}{3} \right\rceil \).

**Theorem 2.10** [7] For all \( n \geq 4 \), \( \gamma_{pe}(P_{3,n}) = n \).

**Theorem 2.11.** [7] For all \( n \geq 5 \), \( \gamma_{pe}(P_{4,n}) = \left\lceil \frac{4n}{3} \right\rceil \).

**Theorem 2.12.** [7] For all \( n \geq 6 \),

\[
\gamma_{pe}(P_{5,n}) = \begin{cases} 
\frac{5n}{3}, & \text{if } n \equiv 0, 3 \pmod{6}; \\
\frac{5n + 1}{3}, & \text{otherwise}.
\end{cases}
\]

**Theorem 2.13.** [8] Let \( G \) be a path with \( n \) vertices. Then

\[
\gamma_{pe}(\mathcal{Z}(G)) = \begin{cases} 
\frac{3n}{2}, & \text{if } n = 5; \\
\frac{n}{3}, & \text{otherwise}.
\end{cases}
\]

**Theorem 2.14.** [8] Let \( G \cong S_n \) be a crown graph with \( 2n \) vertices. Then \( \gamma_{pe}(\mathcal{Z}(G)) = \gamma(\mathcal{Z}(G)) \).

**Theorem 2.15** [8] Let \( G \cong H_n \) be a helm graph. Then \( \gamma_{pe}(\mathcal{Z}(G)) = 2n - 2 \).

**Definition 2.1.** [8] For \( m \geq 3 \), Jahangir graph \( J_{n,m} \) is a graph of order \( nm + 1 \), consisting of a cycle of order \( nm \) with one vertex adjacent to exactly \( m \) vertices of \( C_{nm} \) at a distance \( n \) to each other. Jahangir graph \( J_{2,8} \) is shown in Fig.2.

**Theorem 2.16.** [8] Let \( G \cong J_{n,m} \) be a Jahangir graph with \( m \geq 3 \) and \( n \equiv 0 \text{ or } 1 \pmod{3} \), then

\[
\gamma_{pe}(\mathcal{Z}(J_{n,m})) = \begin{cases} 
\frac{mn}{3}, & \text{if } m = 3; \\
\frac{mn}{3} - 2, & \text{otherwise}.
\end{cases}
\]

**Theorem 2.17.** [8] Let \( G \cong J_{n,m} \) be a Jahangir graph with \( m \geq 3 \) and \( n \equiv 2 \pmod{3} \), then

\[
\gamma_{pe}(\mathcal{Z}(G)) = \begin{cases} 
\frac{mn}{3}, & \text{if } m \equiv 0 \text{ or } 1 \pmod{3}; \\
\frac{mn}{3}, & \text{if } m \equiv 2 \pmod{3}.
\end{cases}
\]

**Definition 2.2.** [8] The gear graph is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle. The gear graph \( G \) has \( 2n + 1 \) vertices and \( 3n \) edges. In Fig.3. we display \( G_8 \).

**Theorem 2.18.** [8] Let \( G_n \) be a gear graph with \( n \geq 3 \). Then
\[
\gamma(\mathcal{S}(G_n)) = \begin{cases} 
\frac{4n}{3}, & \text{if } n \equiv 0 \text{ or } 1 \text{ (mod 3)} \\
\frac{4n}{3} + 1, & \text{if } n \equiv 2 \text{ (mod 3)} 
\end{cases}
\]

**Proposition 2.3.** [8] Let G be any connected graph of order n. Then \(1 \leq \gamma(G) \leq \gamma_{pe}(G) \leq \gamma(\mathcal{S}(G)) \leq \gamma_{pe}(\mathcal{S}(G)) \leq 2n\). Further, \(\gamma_{pe}(\mathcal{S}(G)) = 2\) if and only if G contains an edge of degree at least \(n - 2\).

**Proposition 2.4.** [8] Let G be any graph. If \(\text{diam}(G) = 2\) then \(\gamma_{pe}(\mathcal{S}(G)) \leq \delta(G) + 1\). Equality holds if G is a path.

**Proposition 2.5.** [8] Let G be a connected graph of order \(n \geq 2\). Then \(\gamma_{pe}(\mathcal{S}(G)) \leq \gamma(G) + \delta(G)\).

### 3 Vertex Removal

We observe that the pendant domination parameter value of a graph G may be increases or decreases or remains same when a point is removed from G. For an example in a complete graph \(K_m\) (\(m > 2\)) or complete bipartite graph \(K_{m,n}\) removal of any one point it does not affect the number of \(\gamma_{pe}\). In a sunlet graph the removal of a vertex of degree one it decreases the value of \(\gamma_{pe}\) by one. In barbell graph \(v_1, v_2\) are the adjacent vertices connected two copies of complete graphs. If we removal of the vertex \(v_1\) in barbell graph increases the value of \(\gamma_{pe}\) by 2. Hence we can define the point set \(V(G)\) of G into three subsets

\[
V^0_{pe} = \{(u, v) \in V : \gamma_{pe}(G - uv) = \gamma_{pe}(G)\}
\]

\[
V^-_{pe} = \{(u, v) \in V : \gamma_{pe}(G - uv) < \gamma_{pe}(G)\}
\]

\[
V^+_{pe} = \{(u, v) \in V : \gamma_{pe}(G - uv) > \gamma_{pe}(G)\}
\]

**Theorem 3.1.** [6] If \(G \cong P_n\) and \(n \geq 3\), then we have

(i) If \(n \equiv 0 \text{ (mod 3)}\) or \(n \equiv 1 \text{ (mod 3)}\) then \(v_i \in V^0_{pe}\), if \(i = 1\) or \(n\);

(ii) If \(n \equiv 2 \text{ (mod 3)}\), then \(v_i \in V^-_{pe}\), if \(i = 1\) or \(2\) or \(3\) (mod 3).

**Theorem 3.2.** [6] If \(C_n\) is a cycle with \(n \geq 4\) vertices, then

\[
V(C_n) \in \begin{cases} 
V^-_{pe}, & \text{if } n \equiv 2 \text{ (mod 3)}; \\
V^0_{pe}, & \text{Otherwise}
\end{cases}
\]

### 4 Edge Removal

In this section, we analyse the effect of edge removal in the pendant domination number \(\gamma_{pe}(G)\) of graph G. As in the case of vertex removal, we can observe that the pendant domination number \(\gamma_{pe}(G)\) of a graph G may increase or decrease remain same when an edge is removed from G. Hence we can partition the edge set \(E(G)\) of G into 3 subsets as \(E^+_pe, E^-pe\) and \(E^0 pe\) below.

\[
E^-_{pe} = \{(u, v) \in E : \gamma_{pe}(G - uv) \leq \gamma_{pe}(G)\}
\]

\[
E^0_{pe} = \{(u, v) \in E : \gamma_{pe}(G - uv) = \gamma_{pe}(G)\}
\]

\[
E^+_{pe} = \{(u, v) \in E : \gamma_{pe}(G - uv) \geq \gamma_{pe}(G)\}
\]

**Theorem 4.1.** [6] Let \(P_n\) be a path with \(n \geq 3\) vertices, then we have

(i) If \(n \equiv 0 \text{ (mod 3)}\), then
\[
\left(v_i, v_{i+1}\right) \in \begin{cases} E^-_{pe}, & i = 1 \text{ or } n - 1; \\ E^+_{pe}, & i = 0 \text{ (mod 3)}; \\ E^0_{pe}, & i = 1 \text{ (mod 3)} 
\end{cases}
\]

(ii) If \( n \equiv 1 \text{ (mod 3)} \), then

\[
\left(v_i, v_{i+1}\right) \in \begin{cases} E^0_{pe}, & i = 2 \text{ (mod 3)} \text{ or } i = 0 \text{ or } n - 1; \\ E^+_{pe}, & i = 0 \text{ (mod 3)} 
\end{cases}
\]

(iii) If \( n \equiv 2 \text{ (mod 3)} \), then

\[
\left(v_i, v_{i+1}\right) \in \begin{cases} E^-_{pe}, & i = 0 \text{ (mod 3)} \text{ or } n - 1; \\ E^0_{pe}, & \text{Otherwise} 
\end{cases}
\]

REFERENCES