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EXISTENCE AND APPROXIMATE CONTROLLABILITY OF NONLOCAL AND IMPULSIVE NEUTRAL INTEGRO-DIFFERENTIAL EVOLUTION EQUATIONS OF FINITE DELAY

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ABSTRACT

In this paper, we discuss with the existence of mild solutions as well as approximate controllability of a class of fractional order delay differential control systems with impulse is investigated in this study under the natural premise that the linear system is approximately controllable. The existence of the mild solution to the above mentioned system were determined by using the resolvent operator theory Schauder's fixed point theorem. We present an example.

Keywords: Approximate Controllability; Fractional Differential Equations; Semi-linear Control Systems; Delay Differential Equations; Nonlocal Conditions.

MSC 2010 Classifications : 26A33, 93B05, 93E99, 34K45, 35R11.

1. INTRODUCTION

In the past two decades, fractional calculus provided great challenging interest for mathematicians and physicists in fractional theory. Fractional differential equations are considered as valuable models of many phenomena in various fields, such as electrochemistry, physics, porous media, control theory, etc. Controllability plays an important role both in mathematical and control theory. The concept of exact controllability is usually too strong for the infinite dimensional space. Then, approximate controllability governed by fractional derivatives has been studied extensively. We refer the readers to the recent papers [1-14].

It is well-known that the concept of controllability is a valuable property of a control system, and controllability property plays a very important role in several control problems in both finite and infinite dimensional spaces[15-22]. In controllability of a system, we show existence of a control function that steers the solution of the system from its initial state to the desired final state, where the initial and final states may vary over the entire space. The approximate controllability means that we can steer the system to an arbitrarily small neighborhood of a final state. There are various works on approximate controllability of systems represented by differential equations, integro-differential equations,differential inclusions, neutral functional integro-differential equations, and impulsive differential equations of integer order in Banach spaces. In 1983,Zhou[24] obtained sufficient condition for the approximate controllability to a class of control systems governed by the semilinear abstract equation, which is suitable not only to the infinite-dimensional case but also to the finite-dimensional case. Latter, Mahmudov [25] investigated the approximate controllability for abstract semilinear deterministic and stochastic control systems under the natural assumption that the associated linear control system is approximately controllable by using new properties of symmetric operators, compact semigroups, the schauder fixed-point theorem, and the contraction mapping principle in 2003. In [2008], Mahmudov, [26], studied the approximate controllability for the abstract evolution equations with nonlocal conditions in Hilbert spaces. The author obtained sufficient conditions for the approximate controllability of the semilinear evolution equation by assuming the approximate controllability of the corresponding linearized equation. Jeet [27] derived the approximate controllability of non-impulsive neutral integro-differential equations of finite delay with nonlocal intial conditions using the resolvent operator theory. The authors [28,29] applied the resolvent operator theory to derive the approximate controllability of semilinear nonlocal impulsive integro-differential system in Hilbert spaces without using the delay.

Especially, the approximate controllability of fractional evolution equations is also studied in recent years. In 2015, Liu and Li [30] investigated the control systems governed by fractional evolution differential equations involving Riemann-Liouville fractional derivatives in Banach spaces under some suitable assumptions. In 2016, Fan,Dong and Li [31] studied the approximate controllability of a control system governed by a semilinear composite fractional relaxation equation in Hilbert space under the assumption that the corresponding linear system is approximately controllable. Thus, in this paper, we aim to apply the resolvent operator theory to derive sufficient conditions for the approximate controllability of nonlocal and impulsive integro-differential system of finite delay in a Hilbert space. For this, we first convert the controllability problem into a fixed point problem to show the existence of a mild solution of system and then establish the approximate controllability of the system. The resolvent operator theory, semigroup theory, fractional power theory, Krasnoselskii's fixed point theory, and Schauder's fixed point theorem are also used to prove our main results.

Let Y and W be Hilbert spaces. Also let q and τ be two positive numbers, and $0 = t_0 < t_1 < \cdots < t_r < t_{r+1} = q$. Define $\mathfrak{P}([-\tau, q], Y) = \{z: [-\tau, q] \to Y |$ z(.) is continuous at $, t \neq t_i$, $z(t_i^+)$ *and* $z(t_i^-)$ both exist and $z\ (t_i^+) = z(t_i^-)\}$, and $\mathfrak{Y} = \{$ z : $[-\tau, 0] \to Y |$ z(.) is continuous at all points except at a finite number of points s_i at which $z(s_i^+)$ *and* $z(s_i^-)$ exist and $z(s_i^+) = z(s_i^-)\}$. Consider the following nonlocal and impulsive neutral integro-differential system of finite delay:

$$\begin{cases} \frac{d}{dt} z(t) + Az(t) = \int_0^t a(t-s) z(s) ds + B\, w(t) + h(t, y_t),\ t \epsilon I = [0, q], , t \neq t_i\,, \\ \Delta z|_{t=t_i} = \mathfrak{X}_i(z(t_i)),\ i = 1,2, \dots r, \\ z(t) = \xi(t) + \phi(z)(t), t \,\epsilon\, [-\tau, 0], \end{cases} \quad (1.1)$$

where $z(.)\epsilon\, \mathfrak{P}([-\tau, q], Y)$ is a state function; $-A$ generates an compact analytic semigroup $T(t)_{t \geq 0}$, of bounded linear operator in a Hilbert space Y; $a(t)$ is a closed linear operator on $D(A)$ for each $t \geq 0$; the time history function $z_t(.)\epsilon\, \mathfrak{Y}$ is defined by $z_t(s) = z(t+s),\ s \in [-\tau, 0]$; w(.) is the control function in Hilbert space $L^2(I, W)$; the operator $B: W \to Y$ is linear and continuous; the nonlinear function $h: [0, q]$x $\mathfrak{Y} \to$ Y is continuous; $\mathfrak{X}_i : \mathfrak{Y} \to Y, i = 1,2, \dots r$ are impulsive functions, $\Delta z(t_i)$ defines the jump of a function y at t_i as $\Delta z(t_i) = z(t_i^+) - z(t_i^-)$; $\xi \in \mathfrak{Y}$ and ϕ maps continuously from the space $\mathfrak{P}([-\tau, q], Y)$ to $\mathfrak{Y}$

This paper is organized as follows: Some notations, basic definitions, assumptions and preliminary results are given in section 2. In section 3, we focus on define the continuous solution, well definedness and controllability results of the solution of (1.1). in the final section we provide an example to justify this theory.

2. PRELIMINARIES

In this section, we give some basic definitions, notations, hypotheses, and preliminary results which we shall need throughout the paper. Let $\mathcal{L}(z)$ be a Banach space of bounded linear operators from Y into itself with the operator norm. We denote by z_α the Hilbert space $D(A^\alpha), 0 < \alpha < 1$, endowed with the norm $||A^\alpha||$. We also denote $\mathfrak{P}([-\tau, q], Y)$ by $\mathfrak{P}$.

We assume the following conditions throughout the paper:

C1 The operator $A: D(A) \subseteq Y \to Y$ generates an analytic semigroup $T(t)_{t \geq 0}$, on Y and $\rho(A) \supset \Lambda_\theta = \{\Upsilon \in\ C\,\{0\}: \arg(\Upsilon) < \theta\}$ and $\|R(\Upsilon, A)\|\ \leq M_0 |\Upsilon|^{-1}$ for some constants $M_0 > 1, \theta \in\ (^{\Pi}/_{2}, \Pi)$ and for each $\Upsilon \epsilon\, \Lambda_\theta$, where $R(\Upsilon, A)$ is the resolvent of A.

C2 The operator $a(t): D(a(t)) \subseteq Y \to Y$ is linear and closed with $D(A) \subseteq D(a(t))$ for each $t \geq 0$. For any $u \in D(A)$, the function $a(.)u$ is strongly measurable on (0,∞). There is a $\xi(.)\ \epsilon\ L^1_{loc}(R^+)$ such that $\hat{\xi}(\Upsilon)$ can be obtained for $\boldsymbol{Re}(\boldsymbol{\Upsilon}) > 0$ and $\|\ a(t)\ \| \leq \xi(t)\ \|\ u\ \|\ \_$ for each $t > 0$ and $u\ \epsilon\ D(A)$, here $\hat{\xi}$ denotes the Laplace transform and $\|\ u \|_1 = \|u\| + \|\ Au \|$. In addition, the function $\hat{a}: \Lambda_{\Pi/2} \to\ \mathcal{L}(D(A), Y)$ has an analytic extension to Λ_θ such that $\|\ \hat{a}(\Upsilon) u\ \|\ \leq \|\ \hat{a}(\Upsilon)\|\ \ \|u\|_1$ for each $u\ \epsilon\ D(A)$, and $\|\ \hat{a}(\Upsilon)\|\ \to 0\ as\ |\Upsilon| \to \infty$.

C3 There is a subspace $X \subseteq D(A)$ that is dense in $D(A)$ and a constant $C_1 > 0$ such that $\hat{a}(\Upsilon)(X) \subseteq D(A)$, $\| A\hat{a}(\Upsilon)u \| \leq C_1 \|u\|$ for each $u \in X$ and $\Upsilon \in \Lambda_\theta$.

In the continuation, for $\kappa > 0$ and $\vartheta \in \left(\frac{\pi}{2}, \theta\right)$, $\Lambda_{\kappa,\vartheta} = \{\Upsilon \in C\backslash\{0\}: |\Upsilon| > \kappa, |\arg(\Upsilon)| < \vartheta\}, \Gamma_{\kappa,\vartheta}, \Gamma_{\kappa,\vartheta}^i, i = 1,2,3,$ are the paths $\Gamma_{\kappa,\vartheta}^1 = \{te^{i\vartheta}: t \geqslant \kappa\}$, $\Gamma_{\kappa,\vartheta}^2 = \{\kappa e^{i\xi}: -\vartheta < \xi < \vartheta\}$, $\Gamma_{\kappa,\vartheta}^3 = \{te^{-i\vartheta}: t \geqslant \kappa\}$ and $\Gamma_{\kappa,\vartheta} = U_{i=1}^3 \Gamma_{\kappa,\vartheta}^i$ oriented in a positive sense. Let $\Omega(G) = \Upsilon \in C: G(\Upsilon) = \left(\Upsilon\mathfrak{I} - A - \hat{a}(\Upsilon)\right)^{-1}$ exists and $G(\Upsilon) \in \mathcal{L}(z)\}$, where $\mathfrak{I}$ is the identity operator on Y. If $\wp(.)$ is a resolvent operator for the system

$$\begin{cases} \frac{d}{dt} z(t) = Aw(t) + \int_0^t a(t-s)z(s)ds, t \epsilon I, \\ y(0) = z_0 \epsilon Y \end{cases} \qquad (2.1).$$

and $\wp(t) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{\kappa,\vartheta}} e^{\gamma t} G(\gamma) d\gamma, & t > 0, \\ \mathfrak{I}, & t = 0. \end{cases}$

We recall the following results regarding $\wp(.)$.

Lemma 2.1: The map $\wp: [0, \infty) \to \mathcal{L}(z)$ is strongly continuous and exponential bounded, and there is a constant $m_\alpha > 0$ such that $||A^\alpha \wp(t)u|| \leq m_\alpha t^{-\alpha}||u||$ for each $u \in Y$ and $0 \leq \alpha < 1$.

Lemma 2.2: The operator $\wp(t)$ is compact for all $t > 0$ if $R(\gamma_0, A)$ is compact for some $\gamma_0 \in \rho(A)$.

Lemma 2.3: The operator $\wp(t)$ is continuous in the uniform operator topology of $\mathcal{L}(z)$ for $t > 0$.

Lemma 2.4(Krasnoselskii, [32]) ⇒ Let Ω be a closed convex and nonempty subset of a Banach space X . The operators A and B such that

(i) Ax + By ∈ Ω, whenever $x, y \in \Omega$;

(ii) A is compact and continuous;

(iii) B is a contraction mapping.

Then there exists $z \in \Omega$ such that $z = Az + Bz$.

Definition 2.1: A piecewise continuous function $z: [-\tau, q] \to Y$ is called a mild solution of the impulsive integro-differential system (1.1) of finite delay if $z(t) = \xi(t) + \phi(z)(t), t \in [-\tau, 0]$, and it satisfies the following integral equation for $t \in I$:

$$z(t) = \wp(t)[\,\xi(0) + \phi(z)(0)] + \sum_{0<t_i<t} \wp(t - t_i)\,\mathfrak{X}_i(z_{t_i}) + \int_0^t \wp(t-s)[h(s, z_s) + Bw(s)]ds. \qquad (2.2)$$

Definition 2.2 : The system (1.1) is said to be approximately controllable on I if for each final state $x_q \in Y$ and for each $\epsilon > 0$ there exist a control $w(.) \in L^2(I, W)$ such that the mild solution $z(., w)$ of (1.1) satisfies that $||z(q, w) - x_q|| < \epsilon$.

Theorem 2.1 : Every completely continuous operator which maps a closed bounded convex set of a Banach space into itself has atleast one fixed point .

Define the controllability operator $\Gamma_0^q: Y \to Y$ and the resolvent operator $S(\epsilon, \Gamma_0^q) : Y \to Y$ as

$\Gamma_0^q = \int_0^q \wp(q-s)BB^*\wp^*(q-s)\,ds$, $S(\epsilon, \Gamma_0^q) = (\epsilon\mathfrak{I} + \Gamma_0^q)^{-1}$, $\epsilon > 0$, where B^* *and* $\wp^*$ are the adjoint of the operators B and $\wp$ respectively.

3. EXISTENCE AND CONTROLLABILITY RESULTS

In this section, we shall apply the resolvent operator theory and the fixed point theorems to show the existence of a mild solution of the system (1.1) and then establish the approximate controllability of the system.

Since the semigroup $T(t)$ is compact and lemma (2.2) show that the resolvent operator $\wp(t)$ is compact for each $t > 0$. Let $sup_{t\in I}||\wp(t)|| \leq M$ for some $M > 1$. We now make the following assumptions:

H0 $\epsilon S(\epsilon, \Gamma_0^q) \to 0$ as 0 in $\epsilon \to 0^+$ is strong operator topology.

H1 The function $h: [0, b] \times \mathfrak{Y} \to$ Y satisfies that $h(t, .)$ is continuous from $\mathfrak{Y}$ to Y for all $t \in I, h(., \varphi)$ is strongly measurable for each $\varphi \in \mathfrak{Y}$, and there is also a $\psi_k(.) \in L^2([0, b], R^+)$ for each $k > 0$ such that

$\sup\{||h(t,\varphi)||: ||\varphi||_{\mathfrak{Y}} \leq k\} \leq \psi_k(t)$ for a.e. $t \in I$, and $\lim_{k\to\infty} inf \frac{1}{k}||\psi_k||_{L^2} = b < +\infty$.

H2 The functions $\mathfrak{X}_i$: $\mathfrak{Y} \to$ Y, i = 1,2, …, r, are continuous operators. There exist nondecreasing functions $\zeta_i: R^+ \to R^+(i = 1,2,3 \ldots r)$ such that $||\mathfrak{X}_i(\varphi)|| \leq \zeta_i(||\varphi||_{\mathfrak{Y}}), \varphi \in \mathfrak{Y}$, and $\lim_{k\to\infty} inf \frac{\zeta_i(k)}{k} = b_i < \infty$.

H3 The nonlocal function ϕ: $\mathfrak{P}([-\tau, b], Y) \to \mathfrak{Y}$ is continuous, and it satisfies that there is a constant $\lambda_\phi > 0$ such that $||\phi(z^{(1)}) - \phi(z^{(2)})||_{\mathfrak{Y}} \leq \lambda_\phi ||z^{(1)} - z^{(2)}||_{\mathfrak{P}}$, where $z^{(1),} z^{(2)} \in \mathfrak{P}$.

Let $x_q \in Y$ be any arbitrary final state. Define a control $w_\epsilon(t,.)$ as $w_\epsilon(t,.) = B^* \mathfrak{P}^*(q - t) S(\epsilon, \Gamma_0^q)\{x_q - \mathfrak{P}(q)\{\zeta(0) + \phi(z)(0)\} - \sum_{i=1}^{i=r} \mathfrak{P}(q - t_i)\mathfrak{X}_i(y_{t_i}) - \int_0^q \mathfrak{P}(q - s)h(s, z_s)ds\}$, where $\epsilon > 0$ is any arbitrary number.

Theorem 3.1: If the hypothesis (H1)-(H3) hold, and $M(\lambda_\phi + \sum_{i=1}^r b_i + b)\left(1 + \frac{(M||B||)^2}{\epsilon}\right) < 1$, then the nonlocal and impulsive neutral integro-differential system (1.1) of finite delay corresponding to the control function $w_\epsilon(t,z)$ has a mild solution for each $\epsilon > 0$.

Proof :We define an operator $\mathfrak{F}_\epsilon$ from $\mathfrak{P}([-\tau, q], Y)$ to $\mathfrak{P}([-\tau, q], Y)$ by

$$\mathfrak{F}_\epsilon z(t) = \begin{cases} \frac{d}{dt}z(t) = \wp(t)[\xi(0) + \phi(z)(0)] + \sum_{0<t_i<t} \wp(t - t_i)\mathfrak{X}_i(z_{t_i}) + \int_0^t \wp(t - s)[h(s, z_s) + Bw_\epsilon(s, z)]ds, & t \epsilon I, \\ \xi(t) + \phi(z)(t), & t \in [-\tau, 0]. \end{cases}$$

Obviously, any fixed point of $\mathfrak{F}_\epsilon$ is a solution of impulsive neutral integro-differential system (1.1) of finite delay. Let us decompose the operator $\mathfrak{F}_\epsilon$ into the following two operators $\mathfrak{F}_{1,\epsilon}$ and $\mathfrak{F}_{2,\epsilon}$ as:

$$\mathfrak{F}_{1,\epsilon}z(t) = \begin{cases} \frac{d}{dt}z(t) = \wp(t)[\xi(0) + \phi(z)(0)] & t \epsilon I, \\ \xi(t) + \phi(z)(t), & t \in [-\tau, 0]. \end{cases}$$

and

$$\mathfrak{F}_{2,\epsilon}z(t) = \begin{cases} \sum_{0<t_i<t} \wp(t - t_i)\mathfrak{X}_i(z_{t_i}) + \int_0^t \wp(t - s)[h(s, z_s) + Bw_\epsilon(s, z)]ds, & t \epsilon I, \\ 0, & t \in [-\tau, 0]. \end{cases}$$

First we would like to show that there is a $k > 0$ such that $\mathfrak{F}_\epsilon Z_k \subset Z_{k,}$ where $Z_k = \{z(.) \in \mathfrak{P}: ||z||_{\mathfrak{P}} \leq k\}$. Suppose that this is false, then there would exist $z^k \in Z_k$ for each $k > 0$ such that $||\mathfrak{F}_\epsilon z^k(t)|| > k$ for some $t \in [0, q]$. Therefore, we have

$$k < ||\mathfrak{F}_\epsilon z^k(t)|| \leq [||\xi||_{\mathfrak{Y}} + ||\phi(z^k)||_{\mathfrak{Y}}] + ||\sum_{0<t_{i<t}} \wp(t - t_i)\mathfrak{X}_i(z_{t_i}^k)||$$
$$+ ||\int_0^t \wp(t - s)[h(s, z_s^k) + Bw_\epsilon(s, z^k)]ds||$$

$$\leq M [||\xi||_{\mathfrak{Y}} + \lambda_\phi||(z^k)||_{\mathfrak{P}} + ||\phi(0)||_{\mathfrak{Y}}] + M\sum_{0<t_i<t} \zeta_i(||z_{t_i}^k||_{\mathfrak{Y}}) + M\int_0^t \psi_k(s)ds + M||B||\int_0^t ||w_\epsilon(s, z^k)||ds \quad (3.1)$$

and

$$||w_\epsilon(s, z^k)|| \leq \frac{M||B||}{\epsilon}[||x_q|| + M\{||\xi||_{\mathfrak{Y}} + \lambda_\phi||(z^k)||_{\mathfrak{P}} + ||\phi(0)||_{\mathfrak{Y}}]\} + M\sum_{0<t_i<t} \zeta_i(||z_{t_i}^k||_{\mathfrak{Y}}) + M\int_0^t \psi_k(s)ds] = k^* \text{ (say)}$$

We divide both sides of the equation (3.1) by k, and then letting $k \to \infty$, we get

$$1 < M(\lambda_\phi + \sum_{i=1}^r b_i + b)\left(1 + \frac{(M||B||)^2}{\epsilon}\right).$$

This gives us a contradiction. We can now say that there is a number $k > 0$ such that the map $\mathfrak{F}_\epsilon$ is from Z_k to Z_k.

It is easy to conclude from the assumption (H3) and the condition of the theorem that the map $\mathfrak{F}_{1,\epsilon}$ is a contraction on z_k. We shall now show that the $\mathfrak{F}_{2,\epsilon}$ is completely continuous on Z_k. For this we will have to show that the operator $\mathfrak{F}_{2,\epsilon}$ is continuous on Z_k, and the set $\mathfrak{F}_{2,\epsilon}(Z_k)$ is relatively compact in $\mathfrak{P}$. Further, for the relatively compactness of $\mathfrak{F}_{2,\epsilon}(Z_k)$ we shall use the Arzelà-Ascoli theorem. Therefore, we will have to show that the family

$\mathfrak{F}_{2,\epsilon}(Z_k)$ is piecewise equicontinuous on $[-\tau, q]$, and the set $\mathfrak{F}_{2,\epsilon}(Z_k)(t)$ is relatively compact in Y for each $t \in [-\tau, q]$.

First, we would like to show the continuity of the operator $\mathfrak{F}_{2,\epsilon} : Z_k \subset \mathfrak{P} \to Z_k$. For this we consider a sequence $\{z^{(n)}\} \subset Z_k$ such that $z^{(n)} \to y \in Z_k$ as $n \to \infty$. Therefore, for any $t \in I$, we get

$$||\mathfrak{F}_{2,\epsilon}z^{(n)}(t) - \mathfrak{F}_{2,\epsilon}z(t)|| \leq M\sum_{i=1}^{r} ||\mathfrak{X}_i\left(z_{t_i}^{(n)}\right) - \mathfrak{X}_i(z_{t_i})|| + M\int_0^q ||h\left(s, z_s^{(n)}\right) - h(s, z_s)|| + M||B||\int_0^q ||w_\epsilon(s, z^{(n)}) - w_\epsilon(s, z)||\, ds, \qquad (3.2)$$

and

$$||w_\epsilon(t, z^{(n)}) - w_\epsilon(t, z)|| \leq \frac{M||B||}{\epsilon}\left[M||\phi(z^n) - \phi(z)|| + M\sum_{i=1}^{r} ||\mathfrak{X}_i\left(z_{t_i}^{(n)}\right) - \mathfrak{X}_i(z_{t_i})|| + M\int_0^q ||h\left(s, z_s^{(n)}\right) - h(s, z_s)||ds\right]. \qquad (3.3)$$

It is easy to check that $||z_{t_i}^{(n)} - z_{t_i}||_{\mathfrak{Y}} \leq ||z^{(n)} - z||_{\mathfrak{P}}$ and $||h\left(t, z_t^{(n)}\right) - h(t, z_t)|| \leq 2\psi_k(t)$. Using the hypotheses of the theorem and Lebesgue's Dominated Convergence theorem, we obtain from equations (3.2) and (3.3) that

$$||\mathfrak{F}_{2,\epsilon}z^{(n)} - \mathfrak{F}_{2,\epsilon}z||_{\mathfrak{P}} \to 0 \text{ as } n \to \infty.$$

The next aim of us is to show the piecewise equicontinuity of the family $\mathfrak{F}_{2,\epsilon}(Z_k)$ in $\mathfrak{P}$. Let $I_0 = [0, t_1]$, $I_i = (t_i, t_{i+1}]$, $i = 1,2, \ldots, r$. For any $\sigma_1, \sigma_2 \in I_i$ with $\sigma_1 < \sigma_2$ and $y \in Z_k$, we get

$$\begin{aligned} ||\mathfrak{F}_{2,\epsilon}z(\sigma_2) - \mathfrak{F}_{2,\epsilon}z(\sigma_1)|| &\leq ||\sum_{0<t_i<\sigma_1} [\wp(\sigma_2 - t_i) - \wp(\sigma_1 - t_i)]I_i(y_{t_i})|| + ||\int_0^{\sigma_1} [\wp(\sigma_2 - s) - \wp(\sigma_1 - s)] \\ &\times [h(s, z_s) + Bw_\epsilon(s, z)]ds|| + ||\int_{\sigma_1}^{\sigma_2} \wp(\sigma_2 - s)[h(s, z_s) + Bw_\epsilon(s, z)]ds|| \\ &\leq J_1 + J_2 + J_3. \end{aligned}$$

Let $\hbar \in (0, \sigma_1)$. From the hypotheses of the theorem, we obtain that

$$J_1 \leq \sum_{0<t_i<\sigma_1} \zeta_i(k)||\wp(\sigma_2 - t_i) - \wp(\sigma_1 - t_i)||$$

$$J_2 \leq \int_0^{\sigma_1-\hbar} [\psi_k(s) + k^*]ds \times sup_{s\in[0,\sigma_1-\hbar]}||\wp(\sigma_2 - s) - \wp(\sigma_1 - s)|| + 2M\int_{\sigma_1-\hbar}^{\sigma_1} [\psi_k(s) + k^*]ds.$$

$$J_3 \leq M\int_{\sigma_1}^{\sigma_2} [\psi_k(s) + k^*]ds.$$

We follow from Lemma 2.3 that the resolvent operator $\wp(t)$ is uniformly continuous for $t > 0$. Therefore, we obtain that $J_1 \to 0, J_2 \to 0, J_3 \to 0$ as $t_2 \to t_1$ and $\hbar \to 0$ independent of $y \in Z_k$. Since $\mathfrak{F}_{2,\epsilon}(Z_k) = 0$ on $-\tau, 0]$, the family $\mathfrak{F}_{2,\epsilon}(Z_k)$ is piecewise equicontinuous in $\mathfrak{P}$ over the interval $[-\tau, q]$.

Finally, we show that the set $F(t) = \mathfrak{F}_{2,\epsilon}(Z_k)(t)$ is relatively compact in Y for each $t \in [-\tau, q]$. Since the resolvent operator $\wp(t)$ is compact for $t > 0$, the set $\{\sum_{0<t_i<t} \wp(t - t_i)\, I_i\,(z_{t_i})\}$ is compact for each $t \in [0, q]$. Let $0 < \alpha < \frac{1}{2}$ and $t \in [0, q]$. Consider the following

$$\begin{aligned} ||A^\alpha \int_0^t \wp(t - s)[h(s, z_s) + Bw_\epsilon(s, z)]ds|| &\leq \int_0^t ||A^\alpha \wp(t - s)||\,[\,||h(s, z_s)|| + ||Bw_\epsilon(s, z)||]ds \\ &\leq m_\alpha \int_0^t (t - s)^{-\alpha}\,[\psi_k(s) + ||B||\,||w_\epsilon(s, z)||]ds \end{aligned}$$

$$\leq m_\alpha \frac{q^{1-2\alpha}}{1-2\alpha}\,[||\psi_k||_{L^2} + |B|| \, ||w||_{L^2}].$$

Thus the set $\{ A^\alpha \int_0^t \wp(t-s)\,[h(s,z_s) + Bw_\epsilon(s,z)]ds \}$ is bounded in Y. We conclude from the compactness of embedding $Y_\alpha \hookrightarrow Y$ that $\{ \int_0^t \wp(t-s)\,[h(s,z_s) + Bw_\epsilon(s,z)]ds \}$ is relatively compact for each $t \in [0,q]$. Since $F(t) = 0, t \in [-\tau, 0]$ and the sum of two compact operators is again compact, the set $F(t)$ is relatively compact in Y for each $t \in [-\tau, q]$.

Hence, we obtain that the map $\mathfrak{F}_{1,\epsilon}$ is contraction and $\mathfrak{F}_{2,\epsilon}$ is completely continuous on Z_k. We now conclude from krasnoselskii's fixed point theorem that the operator $\mathfrak{F}_\epsilon : Z_k \to Z_k$ has a fixed point $z(.) \in Z_k$. That is, $y(.)$ is a mild solution of the nonlocal and impulsive integro-differential system (1.1) of finite delay.

Further, we shall again show the existence of a mild solution of the nonlocal and impulsive integro-differential system (1.1) of finite delay if the nonlocal function $\phi : \mathfrak{P}([-\tau, b], Y) \to \mathfrak{Y}$ satisfies the following hypothesis (H4) instead of (H3).

H4 The operator $\phi : \mathfrak{P}([-\tau, b], Y) \to \mathfrak{Y}$ is completely continuous such that

$$\lim_{||y||_{\mathfrak{P}} \to \infty} \frac{||\phi(y)||_{\mathfrak{Y}}}{||(y)||_{\mathfrak{P}}} = \lambda'_\phi.$$

Now, we shall prove that the nonlocal and impulsive integro-differential system (1.1) of finite delay is approximately controllable.

Theorem 3.2 : Assume that the hypotheses (H0), (H1),(H2) and (H4) hold. If the functions $h: [0,b] \times \mathfrak{Y} \to$ Y, $\phi : \mathfrak{P}([-\tau, b], Y) \to \mathfrak{Y}$, and $\mathfrak{X}_i : \mathfrak{Y} \to$ Y, I = 1,2,…r, are uniformly bounded, the nonlocal and impulsive integro-differential system (1.1) of finite delay is approximately controllable on $[-\tau, q]$.

Proof : It is easy to see that all hypotheses of theorem 3.1 hold. So we let z^ϵ be a mild solution of the system (1.1) in some $Z_k \subset \mathfrak{P}([-\tau, b], Y)$ for each $\epsilon > 0$ using the control

$w_\epsilon(t, z^\epsilon) = B^* \wp^*(q-t) S(\epsilon, \Gamma_0^q) \psi(z^\epsilon),$

where

$$\psi(z^\epsilon) = x_q - \wp(q)\{\zeta(0) + \phi(z)(0)\} - \sum_{i=1}^{i=r} \wp(q-t_i)\mathfrak{X}_i(z_{t_i}) - \int_0^q \wp(q-s)h(s,z_s)ds \},$$

and $x_q \in Y$ is any element. It can be easily verified that

$$z^\epsilon(q) = x_q - \epsilon S(\epsilon, \Gamma_0^q)\psi(z^\epsilon) \tag{3.4}$$

Since the resolvent operator $\wp(t)$ is compact for each $t > 0$, and the maps $\phi(.)$ and $\mathfrak{X}_i(.) (i = 1,2 \dots, r)$ are uniformly bounded, we get that the sets $\wp(q)\{\zeta(0) + \phi(z)(0)\}$ and $\{ \sum_{i=1}^{i=r} \wp(q-t_i)\mathfrak{X}_i(z_{t_i})\}$ are relatively compact in Y. Therefore, we can assume without loss of generality that $\wp(q)\{\zeta(0) + \phi(z^\epsilon)(0)\} \to \Phi$ in Y and $\{ \sum_{i=1}^{i=r} \wp(q-t_i)\mathfrak{X}_i(z^\epsilon_{t_i}) \to \tilde{I}$ as $\epsilon \to 0$.

From the hypotheses of the theorem, it is easy to check that the set $\{ A^\alpha \int_0^t \wp(t-s)\,[h(s,z_s) + Bw_\epsilon(s,z)]ds \}$ is bounded in Y for any $\alpha \in (0,1)$. We now conclude from the compactness of embedding $Y_\alpha \hookrightarrow Y$ that there exists a subsequence denoted by itself such that $\int_0^q \wp(q-s)\, h(s, z_s^\epsilon)ds \to \tilde{h} \in Y$.

Let $\Upsilon = x_q - \Psi - \tilde{I} - \tilde{h}$. Then we have

$$|| \Psi(z^{(\epsilon)}) - \boldsymbol{\Upsilon} ||$$

$$\leq ||\wp(q)\{\zeta(0) + \phi(z)(0)\} - \Phi|| + ||\sum_{i=1}^{i=r} \wp(q-t_i)\mathfrak{X}_i(z_{t_i}) - \tilde{I}\,||$$

$$+ || \int_0^q \wp(q-s)h\left(s, z_s^{(\epsilon)}\right) ds \} - \tilde{h}\, ||$$

$$\to 0 \; as \; \epsilon \to 0. \tag{3.5}$$

In view of the equation (3.4), we obtain that $||z^\epsilon(q) - x_q|| \leq ||\epsilon S(\epsilon, \Gamma_0^q)(\Upsilon)|| + ||\psi(z^\epsilon) - \boldsymbol{\Upsilon}\,||_\alpha$.

Therefore, from (3.5) and hypothesis (H0), we get $||z^\epsilon(q) - x_q|| \to 0 \; as \; \epsilon \to 0$.

Hence, nonlocal and impulsive integro-differential system (1.1) of finite delay is approximately controllable on $[-\tau, q]$.

4. APPLICATION

In order to illustrate the applicability of our main results, we consider the following impulsive partial integro differential equation with nonlocal conditions of the form

$$\begin{cases} \frac{\partial}{\partial t} z(t,\varpi) = \frac{\partial^2 z(t,\varpi)}{\partial \varpi^2} + \int_0^t e^{-\beta(t-s)} \frac{\partial^2 z(t,\varpi)}{\partial \varpi^2} ds + h\big(t, z(t-\tau,\varpi)\big) + w(t,\varpi), t \in [0,q], \varpi \in [0,\pi], t \neq t_i, \\ z(t,0) = z(t,\pi) = 0, \quad t \in [0,q] \\ \Delta z(t,\varpi)|_{t\neq t_i} = \int_0^\pi \eta_i(p,\varpi) \frac{|z(t_i-\tau,p)|}{1+|z(t_i-\tau,p)|} dp, \\ z(t,\varpi) = \xi(t,\varpi) + \int_0^q \rho(t,s) \sin\big(x(s,\varpi)\big) ds, t \in [-\tau, 0], \end{cases} \quad (4.1)$$

where $i = 1,2,3\,; 0 < t_1 < t_2 < t_3 < q\,;\ 0 < s_1 < s_2 < s_3 < q\,; y(.,.) : [-\tau, q] \times [0,\pi] \to R$ is a state function ; $\eta_i \in C([0,\pi] \times [0,\pi], R), i = 1,2,3\,; \rho(.,.) \in C([-\tau, 0] \times [0,q], R) (j = 1,2,3); h$ is a given function ; and $\xi \in \mathfrak{Y} = \{x : [-\tau, 0] \times [0,\pi] \to R\ | x(.,\varpi)$ is continuous at all points except at a finite number of points s_i at which $x(s_i^+,\varpi)$ and $x(s_i^-,\varpi)$ exist and $x(s_i,\varpi) = x(s_i^-,\varpi)$ for each $\varpi \in [0,\pi]\}$.

Let $Y = L^2([0,\pi], R)$. An operator A is defined from $D(A) \subset Y$ to Y as $Az = -z''$ with

$$D(A) = \{z \in Y : z, z' \text{ are absolutely continuous}, z'' \in Y \text{ and } z(0) = z(\pi) = 0\}. \quad (4.2)$$

In fact, $-A$ generates an analytic and compact semigroup $\{T(t), t \geq 0\}$ that is self-adjoint in Hilbert space Y. Moreover, the operator A is given by

$Au = \sum_{r=1}^{\infty} r^2 \langle u, e_r\rangle e_r\,,\ u \in D(A)$, and semigroup $\{T(t)\}$ is given by

$$T(t)u = \sum_{r=1}^{\infty} \exp(-r^2 t) \langle u, e_r\rangle e_r\,, \qquad u \in Y, \qquad (4.3)$$

where $e_r(\varpi) = \sqrt{\frac{2}{\pi}} \sin(r\varpi), r \in \mathbb{N}$. Obviously the set $\{e_r : r \in \mathbb{N}\}$ is an orthonormal basis for Y. It is clear that $||T(t)|| \leq 1$.

Furthermore, the operator $A^{1/2}$ is given by $A^{1/2} u = \sum_{r=1}^{\infty} r \langle u, e_r\rangle e_r, \quad u \in D\left(A^{1/2}\right)$, where $D\left(A^{1/2}\right) = \{u \in Y : \sum_{r=1}^{\infty} r \langle u, e_r\rangle e_r \in Y\}$. Let $B = \mathfrak{I}$ and $W = D\left(A^{1/2}\right)$ with norm $||.||_{\frac{1}{2}} = ||A^{1/2}||$.

For $\varpi \in [0,\pi]$, we define $z(t)(\varpi) = z(t,\varpi), w(t)(\varpi) = w(t,\varpi), \phi(z)(t)(\varpi) = \int_0^q \rho(t,s) \sin\big(x(s,\varpi)\big)\, ds$, $h(t, z_t)(\varpi) = h\big(t, z(t-\tau,\varpi)\big), \mathfrak{X}_i\big(z_{t_i}\big)(\varpi) = \int_0^\pi \eta_i\,(p,\varpi) \frac{|z(t_j-\tau,p)|}{1+|z(t_j-\tau,p)|} dp$, and $a(t) : D(A) \subset Y \to Y$ by $a(t)z = e^{-\beta t} Az$ for $y \in D(A)$.

Using the above notations and conditions, we can represent the system (4.1) in the abstract from (1.1). It is not difficult to check that the conditions (C1)-(C3) hold as $\hat{a}(\gamma) = \frac{1}{\gamma+\beta} A$ and $X = C_0^\infty([0,\pi])$, where $C_0^\infty([0,\pi])$ denotes the space of infinitely differentiable real valued functions vanishing at $\varpi = 0$ and $\varpi = \pi$. Then the linear system of (4.1) has a resolvent operator $\wp(.) : [0,\infty) \to \mathcal{L}(z)$ defined as $\Omega(G) = \{\Upsilon \in C : G(\Upsilon) = \big(\Upsilon\mathfrak{I} - A - \hat{a}(\Upsilon)\big)^{-1} \in \mathcal{L}(z)\}$ and $\wp(t) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{\kappa,\vartheta}} e^{\Upsilon t} G(\gamma) d\gamma, & t > 0, \\ \mathfrak{I}, & t = 0. \end{cases}$

It is clear that $\mathfrak{X}_i$, $i = 1,2,3$ are uniformly bounded functions, and satisfy the assumption $(H2)$. Let $\lambda_\phi = \sup_{t\in[-\tau,0]} \int_0^q |\rho(t,s)| ds$. Clearly ϕ satisfies the Lipschitz condition with Lipschitz constant λ_ϕ, i.e, (H3) is satisfied. For the function h we assume that it is a uniformly bounded function and satisfies the assumption $(H1)$.

Since the semigroup $T(t)$ is is compact, we conclude from (theorem 3.2 and lemma 2.2) that the resolvent operator $\wp(t)$ is compact for each $t > 0$. For each $z \in Y, u \in W$ and $t \in I$, we have

$$\langle B^* \wp^*(t)x, u\rangle = \langle \wp^*(t)x, u\rangle = \langle y, \wp(t)u\rangle.$$

If we let $B^* \wp^*(t)x = 0$, we obtain, at $t = 0$, that $\langle x, u\rangle = 0,\ \forall\, u \in W$.

Since W is dense in $Y, x = 0$. If we choose $\rho(.,.)$ in such a way so that $\lambda_\phi < 1$, then we conclude from Theorem 3.2 that the nonlocal and impulsive integro-differential system (4.1) is approximately controllable on $[-\tau, q]$.

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