Neutrosophic Spherical Cubic Soft Set And Their Applications In Decision Making

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Abstract: We introduce the concept of Neutrosophic spherical cubic soft sets (NSCSSs) which can be considered as a generalization of Neutrosophic cubic soft sets. The notions of internal Neutrosophic spherical cubic soft sets (INSCSSs), external Neutrosophic spherical cubic soft sets (ENSCSSs), P-order, P-union, P-intersection, P-AND, P-OR and R-order, R-union, R-intersection, R-AND, R-OR have been defined for Neutrosophic spherical cubic soft sets (NSCSSs). We also investigate structural properties of these operations on Neutrosophic spherical cubic soft sets (NSCSSs). It has also been proved that Neutrosophic spherical cubic soft sets (NSCSSs) satisfy commutative, associative, De Morgan’s, distributive, idempotent and absorption laws. In last section, we provide the application of a Neutrosophic spherical cubic soft sets (NSCSSs) in multi-criteria decision making problem. We present a numerical examples of Neutrosophic spherical cubic soft set.

Keywords: Neutrosophic spherical Cubic soft sets, Neutrosophic spherical internal cubic soft sets, Neutrosophic spherical external cubic soft sets, interval valued spherical fuzzy sets

1 Introduction
Fuzzy set is first introduced by Zadeh[31] to represent the degree of certainty of expert’s in different statements. Zadeh also proposed the concept of a linguistic variable with application in [32]. Then Peng et al.[18, 17] presented an application in multi-criteria decision-making problems. After Zadeh, Turksen[26, 27, 28] extend fuzzy set to an interval valued fuzzy set. Interval value fuzzy sets have many applications in real life such as Sambuc[19], Kohout[9], Mukherjee and Sarkar[11, 12] also gave its applications in Medical, Turksen[26, 27] in interval valued logic. Molodstov pointed out that the important existing theories viz: probability theory, fuzzy set theory, intuitionistic fuzzy set theory, rough set theory etc, which can be considered as mathematical tools for dealing with uncertainties, have their own difficulties. He further pointed out that the reason for these difficulties is, possibly, the inadequacy of the parameterization tool of the theory. Molodtsov[14] has been given soft sets technique and its applications. In 2003, P. K .Maji, R. Biswas and A. R. Roy[13] studied the theory of soft sets initiated by Molodstov.

In 2009, M. Irfan Ali et al.[1] gave some new notions such as the restricted intersection, the restricted union, the restricted difference and the extended intersection of two soft sets along with a new notion of complement of a soft set. Sezgin and Atagun[20] studied on soft set operations. Pei and Miao[16] and Chen et al. improved the work of Maji. Soft set theory has been applied to decision making problems. Jun et al.[6, 7, 8] introduced cubic set which is basically the combination of fuzzy sets with interval valued fuzzy sets. In [14], Muhiuuddin and Al-roqi[15] have introduced the concept of cubic soft sets with applications in BCI/BCK-algebras.

Sezgin and Atagun[20] studied on soft set operations. Majumdar and Samanta, worked on soft mappings[28] were proposed and many related concepts were discussed too. Moreover, the theory of soft sets has gone through remarkably rapid strides with a wide-ranging applications especially in soft decision making and some other fields such as [11, 12, 13].
Motivating from the realisms of physical life phenomenon, i.e., different sports (win/ tie/ defeat), votes like yes/ NA/ NO and making a decision. In 1999, Smrändache[22, 23, 24, 25, 25] presented a fresh idea of NSs and neutrosophic logic, which is the generality of an FS and IFS, NS is defined by (truth-membership, indeterminacy membership and falsity-membership degrees). This idea of NS creates the NS theory by providing the illustration to indeterminate. This theory is well throughout as the whole demonstration of nearly each model of all actual difficulties. Thus, vagueness is complicated in problematical questions we use FS whereas, commerce indeterminacy, we must have a neutrosophic theory. This theory has numerous applications in countless fields such as control theory, records, medicinal judgment difficulties and decision-making questions. Such types of models have been studied by several authors[2, 3, 4, 30].

Spherical fuzzy sets (SFS) were introduced by Kahraman and Gndogdu[10] as an extension of Pythagorean, neutrosophic and picture fuzzy sets. The idea behind SFS is to let decision makers to generalize other extensions of fuzzy sets by defining a membership function on a spherical surface and independently assign the parameters of that membership function with a larger domain.

In this paper, we introduce neutrosophic spherical cubic soft set and define some new notions such as internal (external) neutrosophic spherical cubic soft sets. The notion of neutrosophic spherical cubic soft set generalizes the concept of neutrosophic cubic soft set. We also investigate some of the core properties of neutrosophic spherical cubic soft set. By using these new notions we then construct a decision making method called neutrosophic spherical cubic soft method. We finally present an application which shows that the methods can be successfully applied to many problems containing uncertainties. Finally we present an application of a neutrosophic spherical cubic soft set in decision making.

**Definition 1.1** Let $U$ be a finite universe set containing $n$ alternatives, $E$; a set of criteria and $X$; a set of experts (or decision makers).

A pair $(\beta_s, E_s, X_s)$ is called a Neutrosophic spherical cubic soft set over $U$ if and only if $\beta_s: E_s \times X_s \rightarrow NSCP(U)$ is a mapping into the set of all Neutrosophic spherical cubic sets in $U$. Neutrosophic spherical cubic soft set is defined as

$$(\beta_s, E_s, X_s) = \{(u, A_s(\vec{e}, x)) \mid (u, \lambda_s(\vec{e}, x)(u)) : u \in U, \vec{e}, x \in E_s \times X_s\}$$

where $A_s(\vec{e}, x)(u)$ is an interval valued Neutrosophic spherical set and $\lambda_s(\vec{e}, x)(u)$ is a Neutrosophic spherical set.

### 2 Interval valued Neutrosophic spherical set

Let $U$ be a non-empty set. An interval valued Neutrosophic spherical set in $U$ is of the form,

$$A_s(\vec{e}, x) = \bigg \{ x : \left[ T_{A_s(\vec{e}, x)}^{-}, T_{A_s(\vec{e}, x)}^{+}, \right] \bigg\}
\begin{array}{c}
\left[ I_{A_s(\vec{e}, x)}^{-}, I_{A_s(\vec{e}, x)}^{+}, \right] \\
\bigg] / x \in U \bigg \}
\end{array}
$$

where

$$T_{A_s(\vec{e}, x)}^{-}(x), I_{A_s(\vec{e}, x)}^{-}(x), F_{A_s(\vec{e}, x)}^{-}(x) / x \in U[0,1],$$

and

$$T_{A_s(\vec{e}, x)}^{+}(x), I_{A_s(\vec{e}, x)}^{+}(x), F_{A_s(\vec{e}, x)}^{+}(x) / x \in U[0,1],$$

where

$$I_{A_s(\vec{e}, x)}^{-}(x) = \min \left\{ 1, 1 - \left[ T_{A_s(\vec{e}, x)}^{-}(x) \right] - \left[ F_{A_s(\vec{e}, x)}^{-}(x) \right] \right\}.$$

$$I_{A_s(\vec{e}, x)}^{+}(x) = \max \left\{ 1, 1 - \left[ T_{A_s(\vec{e}, x)}^{+}(x) \right] - \left[ F_{A_s(\vec{e}, x)}^{+}(x) \right] \right\}.$$

**Example 2.1** Let $U = \{u_1, u_2, u_3\}$ be the set of countries, $E_s = \{e_1 = \text{physiological natality}, e_2 = \text{potential}$
be the set of factors affecting population, \( X_s = \{x_1, x_2\} \) be the set of experts. Let
\[
E_s \times X_s = \{(e_1, x_1), (e_1, x_2), (e_2, x_1), (e_2, x_2)\}.
\]
Then the Neutrosophic spherical cubic soft set \((\beta_s, E_s, X_s)\) in \( U \) is given by,
\[
\beta_s(e_1, x_1) = \begin{cases} 
  u_1, & \text{([0.3,0.4], [0.5,1.0], [0.2,0.4]), } (0.3,0.4,0.5) \\
  u_2, & \text{([0.4,0.5], [0.3,1.0], [0.3,0.5]), } (0.5,0.4,0.5) \\
  u_3, & \text{([0.1,0.3], [0.7,1.0], [0.2,0.4]), } (0.7,0.7,0.3) 
\end{cases}
\]
\[
\beta_s(e_2, x_1) = \begin{cases} 
  u_1, & \text{([0.4,0.1], [0.5,1.0], [0.1,0.3]), } (0.7,0.7,0.5) \\
  u_2, & \text{([0.3,0.4], [0.6,1.0], [0.1,0.3]), } (0.6,0.7,0.6) \\
  u_3, & \text{([0.6,0.9], [0.1,1.0], [0.1,0.2]), } (0.8,0.8,0.4) 
\end{cases}
\]
\[
\beta_s(e_1, x_2) = \begin{cases} 
  u_1, & \text{([0.3,0.5], [0.7,1.0], [0.0,0.2]), } (0.7,0.5,0.6) \\
  u_2, & \text{([0.4,0.6], [0.4,1.0], [0.2,0.4]), } (0.6,0.5,0.7) \\
  u_3, & \text{([0.4,0.5], [0.6,1.0], [0.0,0.5]), } (0.8,0.6,0.6) 
\end{cases}
\]
\[
\beta_s(e_2, x_2) = \begin{cases} 
  u_1, & \text{([0.1,0.4], [0.5,1.0], [0.4,0.6]), } (0.5,0.4,0.5) \\
  u_2, & \text{([0.3,0.6], [0.5,1.0], [0.2,0.1]), } (0.6,0.7,0.3) \\
  u_3, & \text{([0.3,0.5], [0.6,1.0], [0.1,0.4]), } (0.5,0.4,0.6) 
\end{cases}
\]

In this example, interval valued Neutrosophic spherical set indicates the experts opinion for future time period and Neutrosophic spherical set indicates the experts opinion for present time period under the different circumstances related to the given problem.

**Definition 2.2** A Neutrosophic spherical cubic soft set is said to be an internal Neutrosophic spherical cubic soft (INSCS) set if
\[
A_s(\tilde{e}, x)(u) \leq \lambda_s(\tilde{e}, x)(u) \leq A^+_s(\tilde{e}, x)(u)
\]
for all \((\tilde{e}, x) \in E_s \times X_s\) and for all \( u \in U \).

**Definition 2.3** A Neutrosophic spherical cubic soft set is said to be external Neutrosophic spherical cubic soft (ENSCS) set if
\[
\lambda_s(\tilde{e}, x)(u) \not\leq A^-_s(\tilde{e}, x)(u), A^+_s(\tilde{e}, x)(u)
\]
for all \((\tilde{e}, x) \in E_s \times X_s\) and for all \( u \in U \).

**Definition 2.4** Let \((\beta_s, E_s, X_s)\) be a Neutrosophic spherical cubic soft set over \( U \). For any \( e_1, e_2 \in E_s, x_1, x_2 \in X_s \)
\[
\beta_s(e_1, x_1) = \{(u, A_{1s}(e_1, x_1)(u), \lambda_{1s}(e_1, x_1)(u)) : u \in U\}
\]
and
\[
\beta_s(e_2, x_2) = \{(u, A_{2s}(e_2, x_2)(u), \lambda_{2s}(e_2, x_2)(u)) : u \in U\}
\]
the \( P \)-order (\( R \)-order) denoted by
\[
\beta_s(e_1, x_1) \subseteq_{R_s} \beta_s(e_2, x_2)
\]
\[
\beta_s(e_1, x_1) \subseteq_{P_s} \beta_s(e_2, x_2)
\]
are defined as follows:
1. \( A_{1s}(e_1, x_1)(u) \leq A_{2s}(e_2, x_2)(u), A_{1s}(e_1, x_1)(u) \leq A_{2s}(e_2, x_2)(u) \) for all \( u \in U \)
2. \( \lambda_{1s}(e_1, x_1)(u) \leq \lambda_{2s}(e_2, x_2)(u), \lambda_{1s}(e_1, x_1)(u) \geq \lambda_{2s}(e_2, x_2)(u) \) for all \( u \in U \)

**Definition 2.5** A Neutrosophic spherical cubic soft set over \( U \) is said to be \( P \)-order (\( R \)-order) contained in another Neutrosophic spherical cubic soft set \((\beta_{2s}, E_{2s}, X_{2s})\) over \( U \) denoted by
\[
(\beta_{1s}, E_{1s}, X_{1s}) \subseteq_{P_s} (\beta_{2s}, E_{2s}, X_{2s}),
\]
\[
\beta_s(e_1, x_1) \subseteq_{R_s} \beta_s(e_2, x_2)
\]
are defined as below:
1. $E_{1s} \subseteq E_{2s}, (E_{1s} \subseteq E_{2s})$
2. $X_{1s} \subseteq X_{2s}, (X_{1s} \subseteq X_{2s})$
3. $\beta_{1s}(\bar{e}, x) \leq_{P_s} \beta_{2s}(\bar{e}, x), \beta_{1s}(\bar{e}, x) \leq_{R_s} \beta_{2s}(\bar{e}, x)$ for all $\bar{e} \in E_{1s}, x \in X_{1s}$

**Definition 2.6** Two Neutrosophic spherical cubic soft sets $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over $U$ are equal, denoted by

$(\beta_{1s}, E_{1s}, X_{1s}) = (\beta_{2s}, E_{2s}, X_{2s})$

if $E_{1s} = E_{2s}, X_{1s} = X_{2s}$ and

$\beta_{1s}(\bar{e}, x) = \beta_{2s}(\bar{e}, x)$

for all $\bar{e} \in E_{1s} = E_{2s}, x \in X_{1s} = X_{2s}$.

**Corollary 2.7** For any two Neutrosophic spherical cubic soft sets $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over $U$. If

$(\beta_{1s}, E_{1s}, X_{1s}) \subseteq_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$

and

then

$(\beta_{1s}, E_{1s}, X_{1s}) \subseteq_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$

Similar result holds for $R$ -order.

**Definition 2.8** The $P$ - union of two Neutrosophic spherical cubic soft sets(NSCSS$_s$) $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over $U$ is denoted by

$(\beta_{3s}, F_s, Y_s) = (\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$

where $F = E_{1s} \cup E_{2s}, Y = X_{1s} \cup X_{2s}$ and for all $f \in F_s$ and $z \in Y_s$ it is defined as

$\beta_{3s}(f, z) = \begin{cases} 
\{(u, A_{1s}(f, z)(u), \lambda_{1s}(f, z)(u)) \} & \text{if } (f, z) \in (E_{1s} \times X_{1s}) \setminus (E_{2s} \times X_{2s}) \\
\{(u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u)) \} & \text{if } (f, z) \in (E_{2s} \times X_{2s}) \setminus (E_{1s} \times X_{1s}) \\
\{(u, \sup(A_{1s}(f, z)(u), A_{2s}(f, z)(u)), \sup(\lambda_{1s}(f, z)(u), \lambda_{2s}(f, z)(u))) \} & \text{if } (f, z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s}) \\
\end{cases}$

whenever $\beta_{1s}(f, z) = \{(u, A_{1s}(f, z)(u), \lambda_{1s}(f, z)(u)) ; u \in U \}$

and $\beta_{2s}(f, z) = \{(u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u)) ; u \in U \}$

**Definition 2.9** The $P$ - intersection of two Neutrosophic spherical cubic soft sets $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over $U$ is denoted by

$(\beta_{3s}, F_s, Y_s) = (\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$

where $F_s = E_{1s} \cap E_{2s}, Y_s = X_{1s} \cap X_{2s}$ and for all $f \in F_s$ and $z \in Y_s$. It is defined as

$\beta_{3s}(f, z) = \{(u, \inf(A_{1s}(f, z)(u), A_{2s}(f, z)(u)), \inf(\lambda_{1s}(f, z)(u), \lambda_{2s}(f, z)(u))) \}$

**Definition 2.10** The $R$ - union of two Neutrosophic spherical cubic soft sets $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over $U$ is denoted by

$(\beta_{3s}, F_s, Y_s) = (\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s})$

where $F_s = E_{1s} \cup E_{2s}, Y_s = X_{1s} \cup X_{2s}$ and for all $f \in F_s$ and $z \in Y_s$. It is defined as

$\beta_{3s}(f, z) = \begin{cases} 
\{(u, A_{1s}(f, z)(u), \lambda_{1s}(f, z)(u)) \} & \text{if } (f, z) \in (E_{1s} \times X_{1s}) \setminus (E_{2s} \times X_{2s}) \\
\{(u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u)) \} & \text{if } (f, z) \in (E_{2s} \times X_{2s}) \setminus (E_{1s} \times X_{1s}) \\
\{(u, \sup(A_{1s}(f, z)(u), A_{2s}(f, z)(u)), \sup(\lambda_{1s}(f, z)(u), \lambda_{2s}(f, z)(u))) \} & \text{if } (f, z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s}) \\
\end{cases}$

and $\beta_{2s}(f, z) = \{(u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u)) ; u \in U \}$

$\beta_{3s}(f, z) = \{(u, A_{1s}(f, z)(u), \lambda_{1s}(f, z)(u)) ; u \in U \}$
Definition 2.11  The $\cap -$intersection of two Neutrosophic spherical cubic soft sets $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over $U$ is denoted by $$ (\beta_{3s}, F_s, Y_s) = (\beta_{1s}, E_{1s}, X_{1s}) \cap_{Ps} (\beta_{2s}, E_{2s}, X_{2s}) $$ where $F_s = E_{1s} \cap E_{2s}$ and for all $f \in F_s$ and $z \in Y_s$ it is defined as $\beta_{3s}(f, z) = \{(u, \inf\{A_{1s}(f, u)A_{2s}(f, u)\}, \sup\{\lambda_{1s}(f, u), \lambda_{2s}(f, u)\})\}.$

Definition 2.12 The complement of a NSCSS $(\beta_s, E_s, X_s)$ is denoted and defined as $$(\beta^c_s, E^c_s, X^c_s) = (\beta_s, E_s, X_s)^c$$ where $\beta^c_s: E^c_s \times X_s \rightarrow NSCP(U)$ is a mapping given as, $$\beta^c_s(e^c, x) = \{(u, A^c_s(\bar{e}, x)(u), \lambda^c_s(\bar{e}, x)(u)) : u \in U, (e^c, x) \in E^c \times X\}$$ where $$A^c_s(\bar{e}, x)(u) = [1 - A^+_s(\bar{e}, x)(u), 1 - A^-_s(\bar{e}, x)(u)]$$ and $$\lambda^c_s(\bar{e}, x)(u) = 1 - \lambda_s(\bar{e}, x)(u)$$ whenever $$\beta(\bar{e}, x) = \{(u, A_s(\bar{e}, x)(u), \lambda_s(\bar{e}, x)(u)) : u \in U\}.$$.

Theorem 2.13 For any NSCSSs, $(\beta_{1s}, E_{1s}, X_{1s}), (\beta_{2s}, E_{2s}, X_{2s}), (\beta_{3s}, E_{3s}, X_{3s})$ and $(\beta_{4s}, E_{4s}, X_{4s})$ over $U$ the following properties hold:

1. **Idempotent** $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{Ps} (\beta_{1s}, E_{1s}, X_{1s}) = (\beta_{1s}, E_{1s}, X_{1s}) \cap_{Ps} (\beta_{1s}, E_{1s}, X_{1s})$

2. **Commutative** $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{Ps} (\beta_{2s}, E_{2s}, X_{2s}) = (\beta_{2s}, E_{2s}, X_{2s}) \cup_{Ps} (\beta_{1s}, E_{1s}, X_{1s})$

3. **Associative** $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{Ps} (\beta_{2s}, E_{2s}, X_{2s}) \cup_{Ps} (\beta_{3s}, E_{3s}, X_{3s}) = (\beta_{1s}, E_{1s}, X_{1s}) \cup_{Ps} (\beta_{2s}, E_{2s}, X_{2s}) \cup_{Ps} (\beta_{3s}, E_{3s}, X_{3s})$

4. **Distributive** $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{Ps} (\beta_{2s}, E_{2s}, X_{2s}) \cup_{Ps} (\beta_{3s}, E_{3s}, X_{3s}) = (\beta_{1s}, E_{1s}, X_{1s}) \cup_{Ps} (\beta_{2s}, E_{2s}, X_{2s}) \cup_{Ps} (\beta_{3s}, E_{3s}, X_{3s})$

5. **De Morgan’s laws** $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{Ps} (\beta_{2s}, E_{2s}, X_{2s}) = (\beta_{1s}, E_{1s}, X_{1s}) \cap_{Ps} (\beta_{2s}, E_{2s}, X_{2s})$

6. **Involution law** $(\beta_{1s}, E_{1s}, X_{1s})^c = (\beta_{1s}, E_{1s}, X_{1s})$

Similar results hold for $R$—order, $R$—union and $R$—intersection.

Proof: These properties can be verified by using definitions 2.8,2.9,2.10,2.11 and 2.12.

Proposition 2.14 For any two NSCSS $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ over $U$ the following are equivalent

1. $(\beta_{1s}, E_{1s}, X_{1s}) \subseteq_{Ps} (\beta_{2s}, E_{2s}, X_{2s})$

2. $(\beta_{1s}, E_{1s}, X_{1s}) \cap_{Ps} (\beta_{2s}, E_{2s}, X_{2s}) = (\beta_{1s}, E_{1s}, X_{1s})$

3. $(\beta_{1s}, E_{1s}, X_{1s}) \cup_{Ps} (\beta_{2s}, E_{2s}, X_{2s}) = (\beta_{2s}, E_{2s}, X_{2s})$

Proof. (2.14)$\Rightarrow$(2.14)

By definition 2.9, we have $$(\beta_{1s}, E_{1s}, X_{1s}) \cap_{Ps} (\beta_{2s}, E_{2s}, X_{2s}) = (\beta_{1s} \cap_{Ps} \beta_{2s}, E_{1s} \cap_{Ps} E_{2s}, X_{1s} \cap_{Ps} X_{2s})$$ as $E_1 \subseteq E_2$ and $X_1 \subseteq X_2$ by hypothesis.

Now for any $(\bar{e}, x) \in (E_{1s} \times X_{1s})$, since $$\beta_{1s}(\bar{e}, x) \subseteq_{Ps} \beta_{2s}(\bar{e}, x)$$

definition 2.4 implies that $$A_{1s}(\bar{e}, x)(u) \leq A_{2s}(\bar{e}, x)(u)$$

and
By definition 2.9 we have
\[ \lambda_{1S}(\varepsilon, x)(u) \leq \lambda_{2S}(\varepsilon, x)(u) \]
for any \( u \in U \) where
\[ \beta_{1S}(\varepsilon, x) = \{(u, A_{1S}(\varepsilon, x)(u), \varepsilon_{1S}(\varepsilon, x)(u)) : u \in U\} \]
By definition 2.4, we have
\[ A_{1S}(\varepsilon, x)(u) \leq A_{2S}(\varepsilon, x)(u) \]
and \( A_{1S}(\varepsilon, x)(u) \leq A_{2S}(\varepsilon, x)(u) \)
Thus,
\[
\text{inf}\{A_{1S}(\varepsilon, x)(u), A_{2S}(\varepsilon, x)(u)\} = \text{inf}\{A_{1S}(\varepsilon, x)(u) \leq A_{2S}(\varepsilon, x)(u)\}, \text{inf}\{A_{1S}(\varepsilon, x)(u) \leq A_{2S}(\varepsilon, x)(u)\} = [A_{1S}(\varepsilon, x)(u) \leq A_{1S}(\varepsilon, x)(u)]
\]
and
\[ \text{inf}\{\lambda_{1S}(\varepsilon, x)(u), \lambda_{2S}(\varepsilon, x)(u)\} = \lambda_{1S}(\varepsilon, x)(u). \]
By using definition 2.9,
\[ \beta_{1S}(\varepsilon, x) \cap P_s \beta_{2S}(\varepsilon, x) = \{(u, \text{inf}\{A_{1S}(\varepsilon, x)(u), A_{2S}(\varepsilon, x)(u)\}, \text{inf}\{A_{1S}(\varepsilon, x)(u), \lambda_{2S}(\varepsilon, x)(u)\}): u \in U\} \]
\[ =\{(u, A_{1S}(\varepsilon, x)(u), \lambda_{1S}(\varepsilon, x)(u))): u \in U\} \]
Hence, \( \beta_{1S}(\varepsilon, x) \cap P_s \beta_{2S}(\varepsilon, x) = \beta_{1S}(\varepsilon, x) \).
By definition 2.8, we have
\[ (\beta_{1S}, E_{1S}, X_{1S}) \cup P_s (\beta_{2S}, E_{2S}, X_{2S}) = (\beta_{1S} \cup P_s \beta_{2S}, E_{1S} \cup P_s E_{2S}, X_{1S} \cup P_s X_{2S}) \]
\[ = (\beta_{1S} \cup P_s \beta_{2S}, E_{1S}, X_{1S}) \]
as \( E_{1S} \cap E_{2S} = E_{1S} \) and \( X_{1S} \cap X_{2S} = X_1 \) by hypothesis.
Now for any \( (\varepsilon, x) \in (E_{1S} \times X_{1S}) \). Since
\[ \beta_{1S}(\varepsilon, x) \cap P_s \beta_{2S}(\varepsilon, x) = \beta_{1S}(\varepsilon, x). \]
By definition 2.9 we have
\[ \text{inf}\{A_{1S}(\varepsilon, x)(u), A_{2S}(\varepsilon, x)(u)\} = A_{1S}(\varepsilon, x)(u) \]
and
\[ \text{inf}\{\lambda_{1S}(\varepsilon, x)(u), \lambda_{2S}(\varepsilon, x)(u)\} = \lambda_{1S}(\varepsilon, x)(u). \]
This implies that
\[ \text{sup}\{A_{1S}(\varepsilon, x)(u), A_{2S}(\varepsilon, x)(u)\} = A_{2S}(\varepsilon, x)(u) \]
and
\[ \text{sup}\{\lambda_{1S}(\varepsilon, x)(u), \lambda_{2S}(\varepsilon, x)(u)\} = \lambda_{2S}(\varepsilon, x)(u). \]
Thus we have,
\[ \beta_{1S}(\varepsilon, x) \cup P_s \beta_{2S}(\varepsilon, x) = \{(u, \text{sup}\{A_{1S}(\varepsilon, x)(u), A_{2S}(\varepsilon, x)(u)\}, \text{sup}\{\lambda_{1S}(\varepsilon, x)(u), \lambda_{2S}(\varepsilon, x)(u)\}): u \in U\} \]
\[ =\{(u, A_{2S}(\varepsilon, x)(u), \lambda_{2S}(\varepsilon, x)(u))): u \in U\} \]
\[ = \beta_{2S}(\varepsilon, x). \]
By hypothesis, we have
\[ (\beta_{1S}, E_{1S}, X_{1S}) \cup P_s (\beta_{2S}, E_{2S}, X_{2S}) = (\beta_{1S} \cup P_s \beta_{2S}, E_{1S} \cup P_s E_{2S}, X_{1S} \cup P_s X_{2S}) \]
\[ = (\beta_{1S} \cup P_s \beta_{2S}, E_{1S}, X_{1S}) \]
as \( E_{1S} \cup E_{2S} = E_{2S} \) and \( X_{1S} \cup X_{2S} = X_{2S} \) \( \Rightarrow E_{1S} \subseteq E_{2S} \) and \( X_{1S} \subseteq X_{2S} \).
Also,
\[ \beta_{1S}(\varepsilon, x) \cup P_s \beta_{2S}(\varepsilon, x) = \{(u, \text{sup}\{A_{1S}(\varepsilon, x)(u), A_{2S}(\varepsilon, x)(u)\}, \text{sup}\{\lambda_{1S}(\varepsilon, x)(u), \lambda_{2S}(\varepsilon, x)(u)\}): u \in U\} \]
\[ =\{(u, A_{2S}(\varepsilon, x)(u), \lambda_{2S}(\varepsilon, x)(u))): u \in U\} \]
\[ \beta_{2S}(\varepsilon, x) \Rightarrow A_{1S}(\varepsilon, x)(u) \leq A_{2S}(\varepsilon, x)(u) \]
and \( \lambda_{1S}(\varepsilon, x)(u) \leq \lambda_{2S}(\varepsilon, x)(u) \) for any \( u \in U \).
Hence, \( (\beta_{1S}, E_{1S}, X_{1S}) \subseteq P_s (\beta_{2S}, E_{2S}, X_{2S}) \).
Corollary 2.15 If we take \( X_{1s} = X_{2s} = X_s \) in above proposition then the following are equivalent.
1. \((\beta_{1s}, E_{1s}, X_s) \subseteq P_s (\beta_{2s}, E_{2s}, X_s)\)
2. \((\beta_{1s}, E_{1s}, X_s) \cap P_s (\beta_{2s}, E_{2s}, X_s) = (\beta_{1s}, E_{1s}, X_s)\)
3. \((\beta_{1s}, E_{1s}, X_s) \cup P_s (\beta_{2s}, E_{2s}, X_s) = (\beta_{2s}, E_{2s}, X_s)\)
4. \((\beta_{2s}, E_{2s}, X_s)^c \subseteq P_s (\beta_{1s}, E_{1s}, X_s)^c\)

Definition 2.16 Let \( \{L_i\}_{i \in I} \) be a family of Neutrosophic spherical cubic soft sets over \( U \), where
\[
\beta_{is}(\tilde{e}, x) = \{(u, A_{is}(\tilde{e}, x)(u), \lambda_{is}(\tilde{e}, x)(u)) : u \in U\}
\]
for any \( \tilde{e} \in E_i \), \( x \in X_i \). Then \( P \) — union, \( P \) — intersection, \( R \) — union and \( R \) — intersection are defined as below:
1. \( \bigcup_{P_s \in I} L_i = \{(u, (\sup_{E_i} A_{is}(\tilde{e}, x)) (u), (\vee_{E_i} \lambda_{is}(\tilde{e}, x)) (u)) : u \in U\}\)
2. \( \bigcap_{P_s \in I} L_i = \{(u, (\inf_{E_i} A_{is}(\tilde{e}, x)) (u), (\wedge_{E_i} \lambda_{is}(\tilde{e}, x)) (u)) : u \in U\}\)
3. \( \bigcup_{E_i \in J} L_i = \{(u, (\sup_{E_i} A_{is}(\tilde{e}, x)) (u), (\vee_{E_i} \lambda_{is}(\tilde{e}, x)) (u)) : u \in U\}\)
4. \( \bigcap_{E_i \in J} L_i = \{(u, (\inf_{E_i} A_{is}(\tilde{e}, x)) (u), (\wedge_{E_i} \lambda_{is}(\tilde{e}, x)) (u)) : u \in U\}\)

Theorem 2.17 Let \( \{L_i\}_{i \in I} = \{\beta_{1s}, E_{1s}, X_{is}\}_{i \in I} \) be a family of Internal Neutrosophic spherical cubic soft sets (INSCSSs) over \( U \), where
\[
\beta_{is}(\tilde{e}, x) = \{(u, A_{is}(\tilde{e}, x)(u), \lambda_{is}(\tilde{e}, x)(u)) : u \in U\}
\]
for any \( \tilde{e} \in E_i \), \( x \in X_i \). Then the \( \bigcup_{P_s \in I} L_i \) and \( \bigcap_{P_s \in I} L_i \) are INSCSSs over \( U \).

Proof. By using definitions 2.16 and 2.2, we can easily prove this theorem.

Theorem 2.18 Let \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) are two INSCSSs over \( U \), where
\[
\beta_{1s}(\tilde{e}, x) = \{(u, A_{1s}(\tilde{e}, x)(u), \lambda_{1s}(\tilde{e}, x)(u)) : u \in U\}
\]
for any \((\tilde{e}, x) \in E_1 \times X_1\) and
\[
\beta_{2s}(f, z) = \{(u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u)) : u \in U\}
\]
for any \((f, z) \in (\beta_{2s}, E_{2s}, X_{2s})\) is also an INSCSSs over \( U \).

Proof. Since \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) are INSCSSs over \( U \),
\[
A_{1s}(f, z)(u) \leq \lambda_{1s}(f, z)(u) \leq A_{1s}^+(f, z)(u)
\]
for all \( u \in U \) and
\[
A_{2s}(f, z)(u) \leq \lambda_{2s}(f, z)(u) \leq A_{2s}^+(f, z)(u)
\]
for all \( u \in U \). Then we have
\[
\sup[A_{1s}(f, z)(u), A_{2s}(f, z)(u)] \leq \lambda_{1s}(f, z)(u) \leq \sup[A_{1s}^+(f, z)(u), A_{2s}^+(f, z)(u)]
\]
for all \( u \in U \) and \((f, z) \in (E_1 \cup E_2 \times X_1 \cup X_2)\).

By definition 2.8, we have
\[
(\beta_{1s}, F_s, Y_s) = (\beta_{1s}, E_{1s}, X_{1s}) \cup P_s (\beta_{2s}, E_{2s}, X_{2s})
\]
where \( F_s = E_1 \cup E_2 \) and \( Y_s = X_1 \cup X_2 \) and for any \( f \in F \) and \( z \in Y \).
\[
\beta_{3s}(f, z) = \begin{cases} 
\{(u, A_{1s}(f, z)(u), \lambda_{1s}(f, z)(u))\} & \text{if } (f, z) \in (E_1 \times X_1) \setminus (E_2 \times X_2) \\
\{(u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u))\} & \text{if } (f, z) \in (E_2 \times X_2) \setminus (E_1 \times X_1) \\
\{(u, \sup[A_{1s}(f, z)(u), A_{2s}(f, z)(u)]), \sup[A_{1s}(f, z)(u), \lambda_{2s}(f, z)(u)]\} & \text{if } (f, z) \in (E_1 \cap E_2) \times (X_1 \cap X_2) 
\end{cases}
\]
If \((f, z) \in (E_1 \cap E_2 \times X_1 \cap X_2)\) then,
\[
\beta_{3s}(f, z) = \{(u, \sup[A_{1s}(f, z)(u), A_{2s}(f, z)(u)], \lambda_{1s}(f, z)(u) \vee \lambda_{2s}(f, z)(u)) : u \in U\}.
\]
Thus
\[
(\beta_{1s}, E_{1s}, X_{1s}) \cup P_s (\beta_{2s}, E_{2s}, X_{2s})
\]
is INSCSS.

If \((f, z) \in (E_{1s} \times X_{1s}) \setminus (E_{2s} \times X_{2s})\) or if \((f, z) \in (E_{2s} \times X_{2s}) \setminus (E_{1s} \times X_{1s})\), then the result is trivial. Hence \((\beta_{1s}, E_{1s}, X_{1s}) \cup_{p_{s}} (\beta_{2s}, E_{2s}, X_{2s})\) is an INSCSS over \(U\).

**Theorem 2.19** Let \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) are two INSCSSs over \(U\), where

\[\beta_{1s} (\tilde{e}, x) = \{(u, A_{1s}(\tilde{e}, x)(u), \lambda_{1s}(\tilde{e}, x)(u)) : u \in U\}\]

for any \((\tilde{e}, x) \in E_{1s} \times X_{1s}\) and

\[\beta_{2s} (f, z) = \{(u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u)) : u \in U\}\]

for any \((f, z) \in E_{2s} \times X_{2s}\). Then the \(P\) – intersection of \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) is also an INSCSS.

**Proof.** By similar way as theorem 2.18, we can prove this theorem.

The following theorem gives the condition that \(R\) – union of INSCSSs is also an INSCSS.

**Theorem 2.20** Let \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) are two INSCSSs over \(U\), where

\[\beta_{1s} (\tilde{e}, x) = \{(u, A_{1s}(\tilde{e}, x)(u), \lambda_{1s}(\tilde{e}, x)(u)) : u \in U\}\]

for any \((\tilde{e}, x) \in E_{1s} \times X_{1s}\) and

\[\beta_{2s} (f, z) = \{(u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u)) : u \in U\}\]

for any \((f, z) \in E_{2s} \times X_{2s}\) such that

\[\sup \{A_{1s}(f, z)(u), A_{2s}(f, z)(u)\} \leq (\lambda_{1s}(f, z)(u) \land \lambda_{2s}(f, z)(u))\]

for all \(u \in U\) and \((f, z) \in (E_{1s} \cap E_{2s} \times X_{1s} \cap X_{2s})\).

Then the \(R\) – union of \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) is also an INSCSS.

**Proof.** By definition 2.10 and 2.2 we can easily prove this theorem.

The following theorem gives the conclusion that \(R\) – intersection of two INSCSSs is also the ±INCSS.

**Theorem 2.21** Let \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) are two INSCSSs over \(U\), where

\[\beta_{1s} (\tilde{e}, x) = \{(u, A_{1s}(\tilde{e}, x)(u), \lambda_{1s}(\tilde{e}, x)(u)) : u \in U\}\]

for any \((\tilde{e}, x) \in E_{1s} \times X_{1s}\) and

\[\beta_{2s} (f, z) = \{(u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u)) : u \in U\}\]

for any \((f, z) \in E_{2s} \times X_{2s}\) such that

\[\inf \{A_{1s}^{+}(f, z)(u), A_{2s}^{-}(f, z)(u)\} \geq (\lambda_{1s}(f, z)(u) \lor \lambda_{2s}(f, z)(u))\]

for all \(u \in U\) and \((f, z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s})\).

Then the \(R\) – intersection of \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) is also an INSCSS over \(U\).

**Proof.** By definition 2.11, we have

\[(\beta_{3s}, E_{3s}, X_{3s}) = (\beta_{1s}, E_{1s}, X_{1s}) \cap_{R} (\beta_{2s}, E_{2s}, X_{2s})\]

where, \(E_{3s} = E_{1s} \cap E_{2s}\) and \(X_{3s} = X_{1s} \cap X_{2s}\), \(f \in E_{3s}\) and \(z \in X_{3s}\).

\[\tilde{a}_{3s} (f, z) = \{(u, \inf \{A_{1s}(f, z)(u), A_{2s}(f, z)(u)\}, \sup \{\lambda_{1s}(f, z)(u), \lambda_{2s}(f, z)(u)\})\}\]

since \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) are INSCSSs over \(U\). So we have,

\[A_{1s}^{-}(f, z)(u) \leq \lambda_{1s}(f, z)(u) \leq A_{1s}^{+}(f, z)(u)\]

for all \(u \in U\) and

\[A_{2s}^{-}(f, z)(u) \leq \lambda_{2s}(f, z)(u) \leq A_{2s}^{+}(f, z)(u)\]

for all \(u \in U\). Also

\[\inf \{A_{1s}^{+}(f, z)(u), A_{2s}^{-}(f, z)(u)\} \leq \lambda_{1s}(f, z)(u) \lor \lambda_{2s}(f, z)(u) \leq \inf \{A_{1s}^{-}(f, z)(u), A_{2s}^{+}(f, z)(u)\}\]

for all \(u \in U\) and \((f, z) \in (E_{1s} \cap E_{2s} \times X_{1s} \cap X_{2s})\).

Hence \((\beta_{1s}, E_{1s}, X_{1s}) \cap_{R_{S}} (\beta_{2s}, E_{2s}, X_{2s})\) is an INSCSS over \(U\).
Theorem 2.22  Let $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ are two ENSCSSs over $U$, where
\[ \beta_{1s}(\tilde{e}, x) = \{(u, A_{1s}(\tilde{e}, x)(u), \lambda_{1s}(\tilde{e}, x)(u)): u \in U\} \]
for any $(\tilde{e}, x) \in E_{1s} \times X_{1s}$ and
\[ \beta_{2s}(f, z) = \{(u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u)): u \in U\} \]
for any $(f, z) \in E_{2s} \times X_{2s}$ such that
\[ \lambda_{1s}(f, z)(u) \land \lambda_{2s}(f, z)(u) \in \{\inf(\sup(A_{1s}^+(f, z)(u), A_{2s}^-(f, z)(u)), \sup(A_{1s}^-(f, z)(u), A_{2s}^+(f, z)(u)), \inf(A_{1s}^-(f, z)(u), A_{2s}^+(f, z)(u)))\} \]
for all $u \in U$ and $(f, z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s})$. Then
\[ (\beta_{1s}, E_{1s}, X_{1s}) \cup_{\beta_{2s}} (\beta_{2s}, E_{2s}, X_{2s}) \]
is also an ENSCSS over $U$.

Proof. By definition 2.10, we have
\[ (\beta_{3s}, E_{3s}, X_{3s}) = (\beta_{1s}, E_{1s}, X_{1s}) \cup_{\beta_{2s}} (\beta_{2s}, E_{2s}, X_{2s}) \]
where,
\[ \beta_{3s}(f, z) = \begin{cases} \{(u, A_{1s}(f, z)(u), \lambda_{1s}(f, z)(u))\} & \text{if } (f, z) \in (E_{1s} \times X_{1s}) \setminus (E_{2s} \times X_{2s}) \\ \{(u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u))\} & \text{if } (f, z) \in (E_{2s} \times X_{2s}) \setminus (E_{1s} \times X_{1s}) \\ \{(u, \sup(A_{1s}(f, z)(u), A_{2s}(f, z)(u)), \inf(A_{1s}(f, z)(u), A_{2s}(f, z)(u)))\} & \text{if } (f, z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s}) \end{cases} \]
and
\[ h = \inf(\sup(A_{1s}^+(f, z)(u), A_{2s}^-(f, z)(u)), \sup(A_{1s}^-(f, z)(u), A_{2s}^+(f, z)(u))) \]
Then $h$ on of $A_{1s}^+(f, z)(u)$, $A_{2s}^-(f, z)(u)$, $A_{1s}^-(f, z)(u)$, $A_{2s}^+(f, z)(u)$, we only consider $h = A_{2s}^-(f, z)(u)$. Or $A_{2s}^+(f, z)(u)$ because remaining cases are similar to this one. If $h = A_{2s}^-(f, z)(u)$ then
\[ A_{1s}^-(f, z)(u) \leq A_{1s}^+(f, z)(u) \leq A_{2s}^-(f, z)(u) \leq A_{2s}^+(f, z)(u) \]
and $k = A_{1s}^+(f, z)(u)$.
Thus
\[ (\sup(A_{1s}(f, z), A_{2s}(f, z))^\circ)(u) = A_{2s}(f, z)^\circ(u) = h > (\lambda_{1s}(f, z) \land \lambda_{2s}(f, z))(u). \]
Hence
\[ (\lambda_{1s}(f, z) \land \lambda_{2s}(f, z))(u) \notin \{\sup(A_{1s}(f, z), A_{2s}(f, z))^\circ(u), \sup(A_{1s}(f, z), A_{2s}(f, z))^+(u)\}. \]
If $h = A_{2s}(f, z)(u)$ then
\[ A_{1s}^-(f, z)(u) \leq A_{2s}^+(f, z)(u) \leq A_{1s}^+(f, z)(u) \]
so
\[ k = \sup(A_{1s}^-(f, z)(u), A_{2s}^-(f, z)(u)) \].
Assume $k = A_{2s}^-(f, z)(u)$, then we have
\[ A_{2s}^+(f, z)(u) \leq A_{1s}^-(f, z)(u) \leq (\lambda_{1s}(f, z) \land \lambda_{2s}(f, z))(u) < A_{2s}^+(f, z)(u) \leq A_{1s}^+(f, z)(u). \]
So we can write
\[ A_{2s}(f, z)(u) \leq A_{1s}^-(f, z)(u) < (\lambda_{1s}(f, z) \land \lambda_{2s}(f, z))(u) < A_{2s}^+(f, z)(u) \leq A_{1s}^+(f, z)(u) \]
or
\[ A_{2s}^-(f, z)(u) \leq A_{1s}^-\land \lambda_{2s}(f, z))(u) \leq A_{2s}^+(f, z)(u) \leq A_{1s}^+(f, z)(u). \]
For the case
\[ A_{2s}^-(f, z)(u) \leq A_{1s}^-(f, z)(u) < (\lambda_{1s}(f, z) \land \lambda_{2s}(f, z))(u) < A_{2s}^+(f, z)(u) \leq A_{1s}^+(f, z)(u) \]
which contradicts the fact that $(\beta_{1s}, E_{1s}, X_{1s})$ and $(\beta_{2s}, E_{2s}, X_{2s})$ are ENSCSSs.
For the case
\[ A_{\beta_{2s}(f,z)}(u) < A_{\lambda_{2s}(f,z)}(u) = (\lambda_{1s}(f,z) \land \lambda_{2s}(f,z))(u) \leq A_{\alpha_{2s}(f,z)}(u) \leq A_{\alpha_{1s}(f,z)}(u). \]

We have
\[ (\lambda_{1s}(f,z) \land \lambda_{2s}(f,z))(u) \neq \sup \{ A_{1s}(f,z), A_{2s}(f,z) \}^{-}(u), \sup \{ A_{1s}(f,z), A_{2s}(f,z) \}^{+}(u) \]
because
\[ \sup \{ A_{1s}(f,z), A_{2s}(f,z) \}^{-}(u) = A_{\lambda_{2s}(f,z)}(u) = (\lambda_{1s}(f,z) \land \lambda_{2s}(f,z))(u). \]

If \((f, z) \in (E_{1s} \times X_{1s}) \setminus (E_{2s} \times X_{2s})\) or \((f, z) \in (E_{2s} \times X_{2s}) \setminus (E_{1s} \times X_{1s})\) then the result holds trivially. Hence \((\beta_{1s}, E_{1s}, X_{1s}) \cup_{R_s} (\beta_{2s}, E_{2s}, X_{2s})\) is an ENSCSS over \(U\).

**Theorem 2.23** Let \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) are two ENSCSSs over \(U\), where
\[ \beta_{1s}(\hat{e}, x) = \{ (u, A_{1s}(\hat{e}, x)(u), \lambda_{1s}(\hat{e}, x)(u)) : u \in U \} \]
for any \((\hat{e}, x) \in E_{1s} \times X_{1s}\) and
\[ \beta_{2s}(f, z) = \{ (u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u)) : u \in U \} \]
for any \((f, z) \in E_{2s} \times X_{2s}\) such that
\[ \lambda_{1s}(f, z)(u) \lor \lambda_{2s}(f, z)(u) \in \{ \inf \{ \sup \{ A_{1s}(f, z)(u), A_{2s}(f, z)(u) \} \}, \sup \{ A_{1s}(f, z)(u), A_{2s}(f, z)(u) \} \}, \inf \{ A_{1s}(f, z)(u), A_{2s}(f, z)(u) \} \} \]
for all \(u \in U\) and \((f, z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s})\).

Then
\[ (\beta_{1s}, E_{1s}, X_{1s}) \cap_{R_s} (\beta_{2s}, E_{2s}, X_{2s}) \]
is also an ENSCSS over \(U\).

**Proof.** We can prove this theorem similar to theorem 2.22.

In the next theorem we derive condition that \(P \leftarrow \)union of two ENSCSSs are ENSCSS over \(U\).

**Theorem 2.24** Let \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) are two ENSCSSs over \(U\), where
\[ \beta_{1s}(\hat{e}, x) = \{ (u, A_{1s}(\hat{e}, x)(u), \lambda_{1s}(\hat{e}, x)(u)) : u \in U \} \]
for any \((\hat{e}, x) \in E_{1s} \times X_{1s}\) and
\[ \beta_{2s}(f, z) = \{ (u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u)) : u \in U \} \]
for any \((f, z) \in E_{2s} \times X_{2s}\) such that
\[ \lambda_{1s}(f, z)(u) \lor \lambda_{2s}(f, z)(u) \in \{ \inf \{ \sup \{ A_{1s}(f, z)(u), A_{2s}(f, z)(u) \} \}, \sup \{ A_{1s}(f, z)(u), A_{2s}(f, z)(u) \} \}, \inf \{ A_{1s}(f, z)(u), A_{2s}(f, z)(u) \} \} \]
for all \((f, z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s})\) and \(u \in U\).

Then
\[ (\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) \]
is also an ENSCSS over \(U\).

**Proof.** By definition 2.8, we can prove this theorem.

In the next theorem we derive condition that \(P \leftarrow \)intersection of two NSCSSs are ENSCSS(INSCSS).
Theorem 2.25 Let \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) are two NSCSSs over \(U\), where 
\[ \beta_{1s}(\bar{e}, x) = \{(u, A_{1s}(\bar{e}, x)(u), \lambda_{1s}(\bar{e}, x)(u)): u \in U\} \]
for any \((\bar{e}, x) \in E_{1s} \times X_{1s}\) and 
\[ \beta_{2s}(f, z) = \{(u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u)): u \in U\} \]
for any \((f, z) \in E_{2s} \times X_{2s}\) such that 
\[ \lambda_{1s}(f, z)(u) \wedge \lambda_{2s}(f, z)(u) \in \{\inf\{\sup(A_{1s}(f, z)(u), A_{2s}(f, z)(u))\}, \sup(A_{1s}(f, z)(u), A_{2s}(f, z)(u))\}, \inf(A_{1s}(f, z)(u), A_{2s}(f, z)(u)), \sup(A_{1s}(f, z)(u), A_{2s}(f, z)(u))\} \]
for all \((f, z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s})\) and \(u \in U\). Then 
\[ (\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s}) \]
is both an ENSCSS and INSCSS over \(U\).

**Proof.** We can prove this theorem by similar way to theorem 2.22.

Theorem 2.26 Let \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) are two NSCSSs over \(U\), where 
\[ \beta_{1s}(\bar{e}, x) = \{(u, A_{1s}(\bar{e}, x)(u), \lambda_{1s}(\bar{e}, x)(u)): u \in U\} \]
for any \((\bar{e}, x) \in E_{1s} \times X_{1s}\) and 
\[ \beta_{2s}(f, z) = \{(u, A_{2s}(f, z)(u), \lambda_{2s}(f, z)(u)): u \in U\} \]
for any \((f, z) \in E_{2s} \times X_{2s}\) such that 
\[ \lambda_{1s}(f, z)(u) \vee \lambda_{2s}(f, z)(u) \in \{\inf\{\sup(A_{1s}(f, z)(u), A_{2s}(f, z)(u))\}, \sup(A_{1s}(f, z)(u), A_{2s}(f, z)(u))\}, \inf(A_{1s}(f, z)(u), A_{2s}(f, z)(u)), \sup(A_{1s}(f, z)(u), A_{2s}(f, z)(u))\} \]
for all \((f, z) \in (E_{1s} \cap E_{2s}) \times (X_{1s} \cap X_{2s})\) and \(u \in U\). Then 
\[ (\beta_{1s}, E_{1s}, X_{1s}) \cap_{R_s} (\beta_{2s}, E_{2s}, X_{2s}) \]
is both an ENSCSS and INSCSS over \(U\).

**Proof.** We can prove this theorem by similar way to theorem 2.22.

Theorem 2.27 For any two NSCSS \((\beta_{1s}, E_{1s}, X_{1s})\) and \((\beta_{2s}, E_{2s}, X_{2s})\) the following absorption law hold [1]

1. \((\beta_{1s}, E_{1s}, X_{1s}) \cup_{R_s} ((\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s})) = (\beta_{1s}, E_{1s}, X_{1s})\)
2. \((\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} ((\beta_{1s}, E_{1s}, X_{1s}) \cup_{R_s} (\beta_{2s}, E_{2s}, X_{2s})) = (\beta_{1s}, E_{1s}, X_{1s})\)
3. \((\beta_{1s}, E_{1s}, X_{1s}) \cup_{R_s} ((\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s})) = (\beta_{1s}, E_{1s}, X_{1s})\)
4. \((\beta_{1s}, E_{1s}, X_{1s}) \cap_{R_s} ((\beta_{1s}, E_{1s}, X_{1s}) \cup_{R_s} (\beta_{2s}, E_{2s}, X_{2s})) = (\beta_{1s}, E_{1s}, X_{1s})\)

**Proof.** [1]

1. By definitions 2.8 and 2.9, we have 
\[(\beta_{1s}, E_{1s}, X_{1s}) \cup_{P_s} ((\beta_{1s}, E_{1s}, X_{1s}) \cap_{P_s} (\beta_{2s}, E_{2s}, X_{2s})) = (\beta_{3s}, (E_{1s} \cup_{P_s} (E_{1s} \cap_{P_s} E_{2s})), (X_{1s} \cup_{P_s} (X_{1s} \cap_{P_s} X_{2s}))) = (\beta_{3s}, E_{1s}, X_{1s})\] 
such that for any \(f \in E_{1s}\) and \(z \in X_{1s}\) we have 
\[\beta_{3s}(f, z) = \beta_{1s}(f, z) \cup_{P_s} (\beta_{1s}(f, z) \cap_{P_s} \beta_{2s}(f, z)).\]
If \((f, z) \in E_{1s} \times X_{1s}\),
\[
\beta_{15}(f, z) = \{ \{u, A_{15}(e, x)(u), \lambda_{15}(e, x)(u) \}, u \in U, (e, x) \in E_{15} \times X_{15} \}\]

\[
\beta_{25}(f, z) = \{ \{u, A_{25}(e, x)(u), \lambda_{25}(e, x)(u) \}, u \in U, (e, x) \in E_{25} \times X_{25} \}\]

\[
\Pi_P (\beta_{15}(f, z) \cap \beta_{25}(f, z)) = \{ \{u, rinf[A_{15}(e, x)(u), A_{25}(f, z)(u)], sup[\lambda_{15}(e, x)(u), \lambda_{25}(f, z)(u)] \}, u \in U, (e, x) \in E_{15} \times X_{15} \}\]

\[
= \{ \{u, rinf[A_{15}(e, x)(u), A_{25}(f, z)(u)], sup[\lambda_{15}(e, x)(u), \lambda_{25}(f, z)(u)] \}, u \in U, (e, x) \in E_{15} \times X_{15} \}\]

For two NSCSSs \((\beta_{15}, E_{15}, X_{15}) \) and \((\beta_{25}, E_{25}, X_{25})\) over \(U\), \(P - AND\) is denoted and defined as,

\[
\beta_{35}(f, z) = \{ \{u, rinf[A_{15}(e, x)(u), A_{25}(f, z)(u)], sup[\lambda_{15}(e, x)(u), \lambda_{25}(f, z)(u)] \}, u \in U, (e, x) \in E_{15} \times X_{15} \}\]

where \(\beta_{35}((\tilde{e}, f), (x, y)) = \beta_{15}(\tilde{e}, x) \cap \beta_{25}(f, z) \) for all \(((\tilde{e}, f), (x, y)) \in (E_{15} \times E_{25}) \times (X_{15} \times X_{25})\).

Similarly we can prove 2.27, 2.27) and 2.27).

Definition 2.28 For two NSCSSs \((\beta_{15}, E_{15}, X_{15}) \) and \((\beta_{25}, E_{25}, X_{25})\) over \(U\), \(R - AND\) is denoted and defined as,

\[
\beta_{35}(f, z) = \{ \{u, rinf[A_{15}(e, x)(u), A_{25}(f, z)(u)], sup[\lambda_{15}(e, x)(u), \lambda_{25}(f, z)(u)] \}, u \in U, (e, x) \in E_{15} \times X_{15} \}\]

where \(\beta_{35}((\tilde{e}, f), (x, y)) = \beta_{15}(\tilde{e}, x) \cap \beta_{25}(f, z) \) for all \(((\tilde{e}, f), (x, y)) \in (E_{15} \times E_{25}) \times (X_{15} \times X_{25})\).

Definition 2.29 For two NSCSSs \((\beta_{15}, E_{15}, X_{15}) \) and \((\beta_{25}, E_{25}, X_{25})\) over \(U\), \(P - OR\) is denoted and defined as,

\[
\beta_{35}(f, z) = \{ \{u, rinf[A_{15}(e, x)(u), A_{25}(f, z)(u)], sup[\lambda_{15}(e, x)(u), \lambda_{25}(f, z)(u)] \}, u \in U, (e, x) \in E_{15} \times X_{15} \}\]

where \(\beta_{35}((\tilde{e}, f), (x, y)) = \beta_{15}(\tilde{e}, x) \cup \beta_{25}(f, z) \) for all \(((\tilde{e}, f), (x, y)) \in (E_{15} \times E_{25}) \times (X_{15} \times X_{25})\).

Definition 2.30 For two NSCSSs \((\beta_{15}, E_{15}, X_{15}) \) and \((\beta_{25}, E_{25}, X_{25})\) over \(U\), \(P - OR\) is denoted and defined as,

\[
\beta_{35}(f, z) = \{ \{u, rinf[A_{15}(e, x)(u), A_{25}(f, z)(u)], sup[\lambda_{15}(e, x)(u), \lambda_{25}(f, z)(u)] \}, u \in U, (e, x) \in E_{15} \times X_{15} \}\]

where \(\beta_{35}((\tilde{e}, f), (x, y)) = \beta_{15}(\tilde{e}, x) \cup \beta_{25}(f, z) \) for all \(((\tilde{e}, f), (x, y)) \in (E_{15} \times E_{25}) \times (X_{15} \times X_{25})\).

Definition 2.31 For two NSCSSs \((\beta_{15}, E_{15}, X_{15}) \) and \((\beta_{25}, E_{25}, X_{25})\) over \(U\), \(R - OR\) is denoted and defined as,

\[
\beta_{35}(f, z) = \{ \{u, rinf[A_{15}(e, x)(u), A_{25}(f, z)(u)], sup[\lambda_{15}(e, x)(u), \lambda_{25}(f, z)(u)] \}, u \in U, (e, x) \in E_{15} \times X_{15} \}\]

where \(\beta_{35}((\tilde{e}, f), (x, y)) = \beta_{15}(\tilde{e}, x) \cup \beta_{25}(f, z) \) for all \(((\tilde{e}, f), (x, y)) \in (E_{15} \times E_{25}) \times (X_{15} \times X_{25})\).

Example 2.32 Let \(U = \{u_1, u_2, u_3\}\) be the initial universe. \(E_s = \{e_1, e_2\}\) be the set of attributes, \(X_s = \{x_1, x_2\}\) be the set of experts. Then the cubic set \((\beta_{15}, E_{15}, X_{15})\) over \(U\) is given below.

\[
\beta_{15}(e_1, x_1) = \{u_1, ([0.1, 0.4], [0.6, 1.0], [0.3, 0.4]), (0.2, 0.5, 0.3)\}
\]

\[
\beta_{15}(e_2, x_2) = \{u_1, ([0.1, 0.3], [0.8, 1.0], [0.1, 0.4]), (0.2, 0.7, 0.3)\}
\]

Let \(U = \{u_1, u_2, u_3\}\) be the initial universe, \(F = \{f_1, f_2\}\) be the set of attributes and \(Y_s = \{y_1, y_2\}\) be the set of experts. Then the cubic set \((\beta_{25}, F_s, Y_s)\) over \(U\) is given below.

\[
\beta_{25}(f_1) = \{u_1, ([0.1, 0.3], [0.8, 1.0], [0.1, 0.4]), (0.2, 0.7, 0.3)\}
\]

\[
\beta_{25}(f_2) = \{u_2, ([0.3, 0.4], [0.5, 1.0], [0.2, 0.3]), (0.4, 0.5, 0.3)\}
\]

\[
\beta_{25}(f_3) = \{u_3, ([0.1, 0.2], [0.6, 1.0], [0.3, 0.4]), (0.1, 0.7, 0.4)\}
\]
\[ \beta_{2s}(f_1, y_1) = \begin{cases} u_1, & \{[0.2, 0.4], [0.5, 1.0], [0.3, 0.5]\} \\
  & (0.2, 0.6, 0.4) \\
 u_2, & \{[0.3, 0.5], [0.5, 1.0], [0.2, 0.3]\} \\
  & (0.3, 0.6, 0.2) \\
 u_3, & \{[0.1, 0.2], [0.7, 1.0], [0.2, 0.4]\} \\
  & (0.1, 0.8, 0.3) \end{cases} \]

\[ \beta_{2s}(f_2, y_2) = \begin{cases} u_1, & \{[0.2, 0.4], [0.2, 1.0], [0.3, 0.5]\} \\
  & (0.3, 0.5, 0.4) \\
 u_2, & \{[0.1, 0.4], [0.8, 1.0], [0.1, 0.2]\} \\
  & (0.2, 0.8, 0.1) \\
 u_3, & \{[0.3, 0.4], [0.3, 1.0], [0.4, 0.5]\} \\
  & (0.4, 0.6, 0.5) \end{cases} \]

By using definitions 2.28, 2.29, 2.30 and 2.31, we have

\( (\beta_{1s}, E_{1s}, X_s) \land_{R_s} (\beta_{2s}, F_s, Y_s) \)
\( = \beta_{3s}(E_{1s} \times F_s) \times (X_s \times Y_s) \)
\( = \beta_{3s}((\tilde{\varepsilon}, f), (x, y)) \)
\( = \begin{cases} u_1, & \{[0.1, 0.4], [0.5, 1.0], [0.3, 0.4]\}, \\
  & (0.2, 0.5, 0.3) \\
 u_2, & \{[0.3, 0.5], [0.5, 1.0], [0.1, 0.2]\}, \\
  & (0.3, 0.4, 0.2) \\
 u_3, & \{[0.1, 0.2], [0.6, 1.0], [0.1, 0.4]\}, \\
  & (0.1, 0.4, 0.3) \end{cases} \)

\( (\beta_{1s}, E_{1s}, X_s) \lor_{P_s} (\beta_{2s}, F_s, Y_s) \)
\( = \beta_{3s}(E_{1s} \times F_s) \times (X_s \times Y_s) \)
\( = \beta_{3s}((\varepsilon, f), (x, y)) \)
\( = \begin{cases} u_1, & \{[0.1, 0.4], [0.5, 1.0], [0.3, 0.4]\}, \\
  & (0.2, 0.6, 0.4) \\
 u_2, & \{[0.3, 0.5], [0.5, 1.0], [0.1, 0.2]\}, \\
  & (0.4, 0.6, 0.2) \\
 u_3, & \{[0.1, 0.2], [0.6, 1.0], [0.1, 0.4]\}, \\
  & (0.5, 0.8, 0.5) \end{cases} \)

\( (\beta_{1s}, E_{1s}, X_s) \land_{P_s} (\beta_{2s}, F_s, Y_s) \)
\( = \beta_{3s}(E_{1s} \times F_s) \times (X_s \times Y_s) \)
\( = \beta_{3s}((\varepsilon, f), (x, y)) \)
\( = \begin{cases} u_1, & \{[0.2, 0.4], [0.6, 1.0], [0.3, 0.5]\}, \\
  & (0.2, 0.6, 0.4) \\
 u_2, & \{[0.4, 0.5], [0.5, 1.0], [0.2, 0.3]\}, \\
  & (0.4, 0.6, 0.2) \\
 u_3, & \{[0.3, 0.5], [0.7, 1.0], [0.2, 0.4]\}, \\
  & (0.5, 0.8, 0.5) \end{cases} \)

\( \beta_{3s}(e_2 f_2, (x_2, y_2)) \)
\( = \begin{cases} u_1, & \{[0.2, 0.4], [0.8, 1.0], [0.3, 0.5]\}, \\
  & (0.3, 0.5, 0.4) \\
 u_2, & \{[0.3, 0.4], [0.8, 1.0], [0.2, 0.3]\}, \\
  & (0.4, 0.8, 0.3) \\
 u_3, & \{[0.3, 0.4], [0.6, 1.0], [0.4, 0.5]\}, \\
  & (0.4, 0.7, 0.5) \end{cases} \)
Definition 2.34 Let \( A_{s}(\epsilon, x_i), \lambda_{s}(\epsilon, x_i) \in \text{NSCES}_{\epsilon} \) over \( U \), \( 1 \leq i \leq n \). The Neutrosophic Spherical Cubic soft Weighted Average Quotient Operator (NSCSWAQO) is denoted and defined as

\[
P_{w_{si}}(A_{s}(\epsilon, x_i), \lambda_{s}(\epsilon, x_i)) = \left( \prod_{i=1}^{n} \left( 1 + A_{s}(\epsilon, x_i)(u) \right)^{w_{si}} \right)^{1/n} = \left( \prod_{i=1}^{n} \left( 1 - A_{s}(\epsilon, x_i)(u) \right)^{w_{si}} \right)^{1/n}
\]

where \( w_{si} \) is the weight of experts opinion \( w_{si} \in [0, 1] \) and \( \sum_{i=1}^{n} w_{si} = 1 \).

Definition 2.35 Let \( \beta_{s} = \langle A_{s}(\epsilon, x_i), A_{s}(\epsilon, x_i), \lambda_{s}(\epsilon, x_i) \rangle \) be a NSCS value. A score function \( \tilde{S} \) of NSCSS value defined as

\[
\tilde{S}(\beta_{s}) = \frac{A_{s}(\epsilon, x_i) + A_{s}(\epsilon, x_i) - \lambda_{s}(\epsilon, x_i)}{3}
\]

where \( \tilde{S}(\beta_{s}) \in [-1, 1] \).

Theorem 2.33 Let \( (\beta_{1s}, E_{1s}, X_{1s}) \) be a NSCSS over \( U \). If \( (\beta_{1s}, E_{1s}, X_{1s}) \) is an INSCSS (ENCSS) then \( (\beta_{1s}, E_{1s}, X_{1s})^C \) INSCSS (ENCSS) respectively.

Proof: By using definition 2.2 and 2.3 we can proof this theorem.

Example 2.36 Let \( U = \{ u_1 = \text{Kenya, } u_2 = \text{Uganda, } u_3 = \text{Algeria Sudan, } u_4 = \text{Morocco} \} \) be the set of countries, \( E = \{ e_1 = \text{Dry Cough, } e_2 = \text{Diarrhea, } e_3 = \text{Nausea and Vomiting, } e_4 = \text{Severe Headache} \} \) be the set of attributes, \( X_s = \{ x_1, x_2 \} \) be the set of symptoms of COVID patients \( X = (x_1, x_2, x_3) \) be the set of Physicians.

Step 1:

\[
\beta_1(e_1, x_1) = \begin{cases} 
  u_1, & ([0.2, 0.4], [0.6, 1.0], [0.2, 0.4]), \\
  u_2, & ([0.3, 0.4], [0.5, 1.0], [0.2, 0.4]), \\
  u_3, & ([0.4, 0.5], [0.4, 1.0], [0.2, 0.5]), \\
  u_4, & ([0.1, 0.3], [0.3, 1.0], [0.6, 0.7]),
\end{cases}
\]

\[
\beta_1(e_2, x_1) = \begin{cases} 
  u_1, & ([0.3, 0.5], [0.5, 1.0], [0.2, 0.3]), \\
  u_2, & ([0.2, 0.4], [0.3, 1.0], [0.5, 0.6]), \\
  u_3, & ([0.1, 0.3], [0.3, 1.0], [0.6, 0.7]), \\
  u_4, & ([0.3, 0.4], [0.2, 1.0], [0.5, 0.6]),
\end{cases}
\]

respectively.
\[ \beta_1(e_3, x_1) = \begin{cases} 
|u_1|, & ([0.1,0.4], [0.6,1.0], [0.3,0.5]), \\
|u_2|, & ([0.3,0.5], [0.6,1.0], [0.1,0.4]), \\
|u_3|, & ([0.1,0.3], [0.6,1.0], [0.3,0.4]), \\
|u_4|, & ([0.5,0.6], [0.2,1.0], [0.2,0.4]), \\
\end{cases} \]

\[ \beta_1(e_4, x_1) = \begin{cases} 
|u_1|, & ([0.3,0.6], [0.7,1.0], [0.0,0.2]), \\
|u_2|, & ([0.2,0.6], [0.7,1.0], [0.1,0.4]), \\
|u_3|, & ([0.3,0.5], [0.6,1.0], [0.1,0.4]), \\
|u_4|, & ([0.2,0.3], [0.7,1.0], [0.1,0.2]), \\
\end{cases} \]

Step 2:
\[ \beta_1(e_1, x_2) = \begin{cases} 
|u_1|, & ([0.2,0.3], [0.5,1.0], [0.3,0.5]), \\
|u_2|, & ([0.1,0.4], [0.7,1.0], [0.2,0.4]), \\
|u_3|, & ([0.5,0.6], [0.3,1.0], [0.2,0.3]), \\
|u_4|, & ([0.1,0.2], [0.7,1.0], [0.2,0.3]), \\
\end{cases} \]

\[ \beta_1(e_2, x_2) = \begin{cases} 
|u_1|, & ([0.4,0.6], [0.4,1.0], [0.2,0.3]), \\
|u_2|, & ([0.5,0.6], [0.3,1.0], [0.2,0.3]), \\
|u_3|, & ([0.2,0.4], [0.7,1.0], [0.1,0.2]), \\
|u_4|, & ([0.0,0.1], [0.7,1.0], [0.3,0.4]), \\
\end{cases} \]

\[ \beta_1(e_3, x_2) = \begin{cases} 
|u_1|, & ([0.1,0.2], [0.5,1.0], [0.4,0.4]), \\
|u_2|, & ([0.2,0.3], [0.2,1.0], [0.6,0.6]), \\
|u_3|, & ([0.2,0.3], [0.5,1.0], [0.3,0.4]), \\
|u_4|, & ([0.0,0.1], [0.8,1.0], [0.2,0.4]), \\
\end{cases} \]

Step 3:
\[ \beta_1(e_1, x_3) = \begin{cases} 
|u_1|, & ([0.5,0.5], [0.4,1.0], [0.1,0.2]), \\
|u_2|, & ([0.2,0.3], [0.4,1.0], [0.4,0.5]), \\
|u_3|, & ([0.0,0.1], [0.6,1.0], [0.4,0.5]), \\
|u_4|, & ([0.2,0.3], [0.6,1.0], [0.2,0.3]), \\
\end{cases} \]

\[ \beta_1(e_2, x_3) = \begin{cases} 
|u_1|, & ([0.0,0.1], [0.6,1.0], [0.4,0.5]), \\
|u_2|, & ([0.2,0.3], [0.5,1.0], [0.3,0.4]), \\
|u_3|, & ([0.5,0.6], [0.4,1.0], [0.1,0.2]), \\
|u_4|, & ([0.3,0.4], [0.3,1.0], [0.2,0.5]), \\
\end{cases} \]

\[ \beta_1(e_3, x_3) = \begin{cases} 
|u_1|, & ([0.2,0.4], [0.3,1.0], [0.5,0.6]), \\
|u_2|, & ([0.1,0.3], [0.8,1.0], [0.1,0.2]), \\
|u_3|, & ([0.4,0.5], [0.6,1.0], [0.0,0.2]), \\
|u_4|, & ([0.5,0.6], [0.2,1.0], [0.3,0.4]), \\
\end{cases} \]

\[ \beta_1(e_4, x_3) = \begin{cases} 
|u_1|, & ([0.2,0.3], [0.5,1.0], [0.3,0.4]), \\
|u_2|, & ([0.4,0.6], [0.2,1.0], [0.4,0.5]), \\
|u_3|, & ([0.5,0.6], [0.4,1.0], [0.1,0.2]), \\
|u_4|, & ([0.1,0.2], [0.7,1.0], [0.2,0.3]), \\
\end{cases} \]
Step 4: The cubic soft expert weighted average of each attribute.

\[
e_1, \quad (0.20, 0.34), [0.58, 1.0], [0.22, 0.39]), (0.59, 0.64, 0.30) \\
(0.29, 0.47), [0.54, 1.0], [0.17, 0.33]), (0.31, 0.57, 0.22) \\
(0.39, 0.56), [0.31, 1.0], [0.31, 0.46]), (0.43, 0.44, 0.35) \\
(0.16, 0.28), [0.46, 1.0], [0.4, 0.50]), (0.14, 0.50, 0.41) \\
(0.22, 0.38), [0.47, 1.0], [0.27, 0.37]), (0.22, 0.83, 0.31) \\
(0.27, 0.41), [0.37, 1.0], [0.37, 0.47]), (0.27, 0.52, 0.38) \\
(0.26, 0.36), [0.42, 1.0], [0.32, 0.42]), (0.28, 0.42, 0.38) \\
(0.23, 0.33), [0.33, 1.0], [0.42, 0.52]), (0.29, 0.59, 0.37) \\
(0.12, 0.36), [0.47, 1.0], [0.39, 0.51]), (0.18, 0.43, 0.37) \\
(0.21, 0.37), [0.58, 1.0], [0.21, 0.37]), (0.24, 0.57, 0.42) \\
(0.26, 0.37), [0.58, 1.0], [0.19, 0.33]), (0.49, 0.40, 0.27) \\
(0.39, 0.49), [0.33, 1.0], [0.24, 0.40]), (0.43, 0.48, 0.30) \\
(0.31, 0.47), [0.56, 1.0], [0.13, 0.27]), (0.29, 0.61, 0.33) \\
(0.21, 0.53), [0.20, 1.0], [0.21, 0.39]), (0.24, 0.33, 0.28) \\
(0.34, 0.41), [0.53, 1.0], [0.17, 0.37]), (0.31, 0.47, 0.24) \\
(0.17, 0.27), [0.68, 1.0], [0.16, 0.26]), (0.17, 0.71, 0.23)
\]

Step 5: Calculate the \( V_{ps} \) of 1st, 2nd, 3rd and 4th column of above table by using definition 3.30, so we have

\[
U_{1s} = \bigvee_{j=1}^{4} \{ e_j, u_1 \} = \{([0.39, 0.56], [0.58, 1.0], [0.41, 0.46]), (0.59, 0.69, 0.14) \}
\]

\[
U_{2s} = \bigvee_{j=1}^{4} \{ e_j, u_2 \} = \{([0.27, 0.41], [0.47, 1.0], [0.42, 0.52]), (0.29, 0.83, 0.38) \}
\]

\[
U_{3s} = \bigvee_{j=1}^{4} \{ e_j, u_3 \} = \{([0.26, 0.37], [0.58, 1.0], [0.42, 0.51]), (0.49, 0.57, 0.42) \}
\]

\[
U_{4s} = \bigvee_{j=1}^{4} \{ e_j, u_4 \} = \{([0.54, 0.53], [0.68, 0.1], [0.27, 0.37]), (0.31, 0.71, 0.33) \}
\]

Step 6: Now calculate the score of above NSCSS elements by using definition 3.34

\[
S(U_{1s}) = 0.19 \\
S(U_{2s}) = 0.01 \\
S(U_{3s}) = 0.18 \\
S(U_{4s}) = 0.61
\]

Step 7: Generate the non-decreasing order of the score of NCSES set values corresponding to \( V_p \) we have the following order

\[ u_4 > u_1 > u_3 > u_2. \]

In above example, we want to check which country is much affected by COVID-19. Here Morocco is more affected by COVID-19.
References


