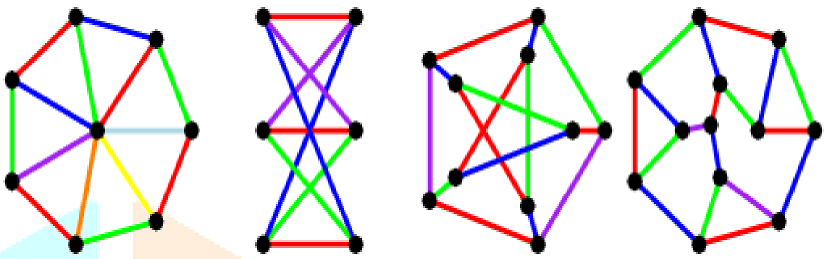
IJCRT.ORG ISSN : 2320-2882

INTERNATIONAL JOURNAL OF CREATIVE RESEARCH THOUGHTS (IJCRT)

An International Open Access, Peer-reviewed, Refereed Journal

A STUDY ON GRAPH EDGE COLORING



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ABSTRACT: Let G = (V, E) be a graph where V is the vertex set and E is the edge set. In **graph theory**, **edge coloring** of a graph is an assignment of "colors" to the **edges** of the graph so that no two adjacent **edges** have the same. Normally the aim is to use the smallest number of colors, which is denoted by χ0(G). By Vizing's theorem, the number of colors needed to edge color a simple graph is either its maximum degree Δ or Δ+1. The **edge-coloring** problem is one of the fundamental problems on **graphs**. Edge colorings have appeared in a variety of contexts in graph theory. In this paper Graph coloring, edgecoloring, chromatic number of edge coloring, Konig's, Vizing'theorems and Lemma's, Peterson Graphs, Tait's theorem are studied.

KEY WORDS: Graph Coloring, Proper coloring, *k*-Coloring, chromatic index, Edge Chromatic Number, edge covering number , Labeled Graph, Minimum Edge Coloring

INTRODUCTION TO GRAPH THEORY: A graph is an abstract structure which consists of vertices and edges; each edge joins two vertices called ends of the edge. It can be used to represent various combinatorial or topological structures that can be modelled as objects and connections between those objects. Since then graph theory has been expanding its branches quite enormously, obviously due to its well-defined and interesting applications in various fields. A graph can be used to represent almost any physical situation involving discrete objects and a relationship among them. A graph structure is very suitable for representing relationships between objects in the abstract, and a large number of combinatorial problems can be modelled as problems on the graph structure. Many Mathematicians have contributed to the growth of this theory and EULER (1707-1782) became the father of graph theory when he settled a famous unsolved problem of his days called the Konigsberg Bridge Problem.

Def: A graph G = (V, E) consists of two sets: a non-empty finite set V and a finite set E. The elements of V are called vertices (or points or nodes) and the elements of E are called edges (or lines). Each edge is identified with a pair of vertices. The set V (G) is called the vertex set of G, and the set E(G) is called the edge set of E(G). If $e = \{u, v\} \in E(G)$ then we say that e joins u and v. The vertices u and v are called the ends of the edge uv.

The order of a graph, denoted by n(G), is the number of vertices and the size of a graph, denoted by m(G), is the number of edges. Graphs are finite or infinite according to their order; however the graphs we consider are all finite. If a graph allows more than one edge (but yet a finite number) between the same pair of vertices in a graph, the resulting structure is a multi-graph. Such edges are called parallel or multiple edges. An edge that joins a single endpoint to itself is known as a loop. Graphs that allow parallel edges and loops are called pseudographs. A simple graph is a graph with no parallel edges and loops. For an edge x = u v in G deg(x)= deg u + deg v-2. The minimum and maximum edge degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively.

INTRODUCTION TO GRAPH COLORING:

In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color, called a vertex coloring. Similarly, an edge coloring assigns a color to each edge so that no two incident edges share the same color, and a face coloring of a planar graph assigns a color to each face or region so that no two faces that share a boundary share the same color.

Graph coloring enjoys many practical applications as well as theoretical challenges. Beside the classical types of problems, different limitations can also be set on the graph, or on the way a color is assigned, or even on the color itself. It has even reached popularity with the general public in the form of the popular number puzzle. Graph coloring is still a very active field of research.

A k-coloring of a graph is a proper coloring involving a total of k colors. A graph that has a k-coloring is said to be k-colorable.

Graph coloring problem is to assign colors to certain elements of a graph subject to certain constraints. The graph coloring problem has huge number of applications.

EDGE COLORING:

An edge coloring of a graph G is a coloring of the edges of G such that adjacent edges (or the edges bounding different regions) receive different colors. The chromatic number $\chi(G)$ of graph G is the minimum number of colors required to assign to the vertices of G in such a way that no two adjacent vertices in G receive the same color. The edge chromatic number / chromatic index of a graph G, denoted $\chi 0(G)$**,** minimum number of different colors required for a proper edge coloring of G.

History of Edge Coloring: The edge-coloring problem is to color all edges of a given graph with the minimum number of colors so that no two adjacent edges are assigned the same color. The edge-coloring problem was appeared in 1880 in relation with the four-color problem. The problem is that every map could be colored with four colors so that any neighboring countries have different colors. It took more than 100 years to prove the problem affirmatively in 1976 with the help of computers. The first paper dealing with the edge-coloring problem was written by Tait in 1880. In this paper Tait proved that if the four-color conjecture is true, then the edges of every 3-connected planar graph can be properly colored using only three colors. Several years later, in 1891 Petersen pointed out that there are 3-connected, cubic graphs which are not 3 colorable. The minimum number of colors needed to color edges of G is called the chromatic index $\chi 0(G)$ of G. Obviously $\chi 0(G) \geq \Delta(G)$, since all edges incident to the same vertex must be assigned different colors. In 1916, K¨nig has proved his famous theorem which states that every bipartite graph can be edge-colored with exactly $\Delta(G)$ colors, that is $\chi 0(G) = \Delta(G)$. In 1949, Shannon proved that every graph can be edge-colored with at most $3\Delta(G)/2$ colors, that is $\chi 0(G) \leq 3\Delta(G)/2$. In 1964, Vizing proved that $\chi 0(G) \leq \Delta(G) + 1$ for every simple graph i.,ethe number of colors needed to edge color a simple graph is either its maximum degree Δ or $\Delta+1$.

Def: An edge coloring of a graph G is a function $f : E(G) \to C$, where C is a set of distinct colors. For any positive integer k, a k-edge coloring is an edge coloring that uses exactly k different colors.

Def: A proper edge coloring of a graph is an edge coloring such that no two adjacent edges are assigned the same color. Thus a proper edge coloring f of G is a function $f : E(G) \to C$ such that $f(e) \neq f(e')$ whenever edges e and e' are adjacent in G.

Def: An edge coloring containing the smallest possible number of colors for a given graph is known as a minimum edge coloring.

Def: The edge chromatic number / chromatic index of a graph G, denoted $\chi 0(G)$**,** minimum number of different colors required for a proper edge coloring of G.

Theorem: For any graph G,

$$\Delta(G) \leq \chi 0(G) \leq 2\Delta(G) - 1$$

Proof: An obvious lower bound for $\chi 0(G)$ is the maximum degree $\Delta(G)$ of any vertex in G. This is of course, because the edges incident one vertex must be differently colored. It follows that $\Delta(G) \leq \chi 0(G)$. The upper bound can be found by using adjacency of edges. Each edge is adjacent to at most $\Delta(G) - 1$ other edges at each of its endpoints. Thus,

$$1 + (\Delta(G) - 1) + (\Delta(G) - 1) = 2\Delta(G) - 1$$

colors will always suffice for a proper edge coloring of G.

Chromatic Index for Common Graph Families:

Path Graphs: $\chi 0(Pn) = 2$, for $n \geq 3$.

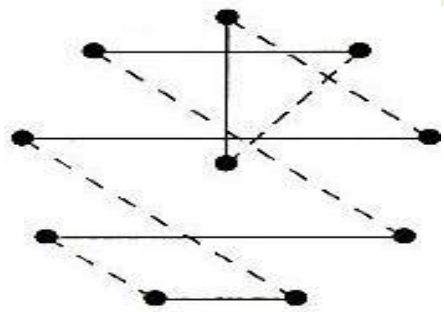
Cycle Graphs: $\chi 0(Cn) = \begin{cases} 2, & \text{if n is even;} \\ 3, & \text{if n is odd.} \end{cases}$

Trees: $\chi 0(T) = \Delta(T)$, for any tree T.

Wheel Graphs: $\chi 0(G) = n - 1$, for $n \geq 4$.

Edge coloring of K_{2n} : In a league with 2n teams, we may want to schedule games so that each pair of teams plays each other, but each team plays at most once a week. Since each team must play 2n-1 others, the season losts at least 2n-1 weeks. The games of each week must form a matching. We can schedule the season in 2n-1 weeks if and only if we can partition $E(K_{2n})$ into 2n-1 disjoint matchings. Since $K_{2n}$is 2n-1–regular, these must be perfect matchings.

The figure below describes the solution. Arrange 2n-1 vertices cyclically, and let the length of an edge be the number of steps between its endpoints along the circle. This creates 2n-1 edges of each length 1, 2, …….., n-1.



In the figure, the solid matching has one edge of each length, plus an edge from the central vertex to the leftover vertex on the circle. Rotating the picture as indicated by the dashed matching yields new edges, again one of each length. The 2n-1 rotations of the figure yield the desired matchings.

Definition: A *k-edge-coloring* of G is labeling $f : E(G) \rightarrow [k]$; the labels are colors, and the set of edges with one color is a color class. A k-edge-coloring is proper if edges sharing a vertex have different colors; equivalently, each color class is a matching. A graph is k-edge-colorable if it has a proper k-edge-coloring.

The edge-chromatic number $\chi 0(G)$ of a loop less graph G is the least k such that G is k-edge-colorable. The multiplicity of an edge is the number of times its vertex pair appears in the edge set.

Chromatic index is another name used for $\chi 0(G)$. Since edges sharing a vertex need different colors, $\chi 0(G) \geq \Delta(G)$. Vizing [1964] and Gupta [1965] independently proved that $\Delta(G) + 1$ colors suffice when G is simple.

Theorem: (Konig [1916]) If G is bipartite, then $\chi 0(G) = \Delta(G)$.

Proof: We have corollary that "For k >0, every k-regular bipartite graph has a perfect matching." By this corollary every regular bipartite graph H has a 1-factor. By induction on $\Delta(H)$, this yields a proper $\Delta(H)$-edge-coloring. It therefore suffices to show that every bipartite graph G with maximum degree k has a k-regular bipartite supergraph H .

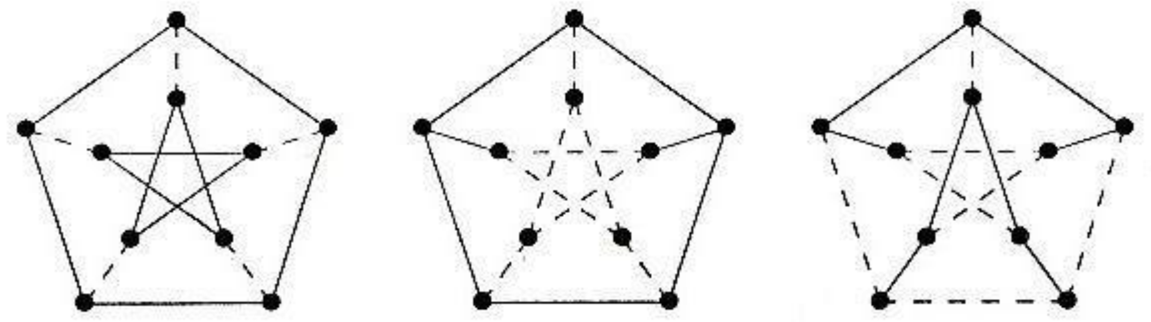
We construct such a supergraph. Add vertices to the smaller partite set of G if necessary, to equalize the sizes. If the resulting G' is not regular, then each partite set has a vertex with degree less than $\Delta(G') = \Delta(G)$. Add an edge consisting of this pair. Continue adding such edges until the graph becomes regular.

A regular graph G has a $\Delta(G)$-edge-coloring if and only if decomposes into 1-factors. Such subgraphs form a *1-factorization* of G , and we then say G is *1-factorable*. For an odd cycle, $\chi 0(G) = 3 > \Delta(G)$.

The Petersen graph also requires an extra color, but only one extra color.

Example:The Petersen graph is 4-edge-chromatic (Petersen [1898]). Consider the drawing of the Petersen graph consisting of an outer 5-cycle, an inner (twisted) 5-cycle, and a matching between them (the cross edges). Since C_5 is 3-edge-colorable, giving the five cross edges the same color produces a 4-edge-coloring. To prove that the Petersen graph is not 3-edge-colorable, we prove that every 2-factor is isomorphic to $2C_5$.

Since a 2-factor is a union of disjoint cycles, a 2-factor H of the Petersen graph has an even number m of cross edges. If $m = 0$, then $H = 2C_5$. If $m = 2$, then the two cross edges have nonadjacent endpoints on the inner cycle or the outer cycle. On the cycle where their endpoints are nonadjacent, the remaining vertices force all five edges of that cycle into H (see illustration), which is impossible. Finally, if $m = 4$, then the cycle edges forced into H by the unused cross edge yield $2P_5$ in H in such a way that the only completion to a 2-factor is again $2C_5$.



Now we consider all simple graphs. We make $\Delta(G) + 1$ colors available and build a proper edge-coloring, incorporating edges one by one until we have a proper $\Delta(G) + 1$-edge-coloring of G. The algorithm runs surprisingly quickly, considering that it is hard to tell whether a graph has a proper $\Delta(G)$-edge-coloring.

Theorem: (Vizing [1964, 1965], Gupta [1966]) Every simple graph with maximum degree Δ has a proper Δ+1-edge-coloring.

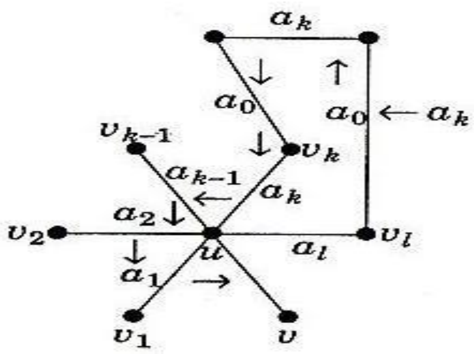
Proof: Suppose u,v is an edge left uncolored by a proper $\Delta(G)$ +1-edge coloring f of a proper subgraph G' of G. After possibly recoloring some edges, we extend the coloring to include uv ; call this an augmentations, we obtain a proper $\Delta(G)$ +1-coloring of G.

Since the number of colors exceeds $\Delta(G)$, every vertex has some color not appearing on its incident edges. Let a_0 be a color missing at u, and let a_1 be a color missing at v. We may assume that a_1 appears at *u*, or we could use a_1 on *uv*. Suppose v_1 is the neighbour of *u* along the edge of color a_1. At v_1 some color a_2 is missing. We may assume that a_2 appears at *u*, or we could recolor $uv_1$from a_1 to a_2 and then use a_1 on uv to extend the coloring.

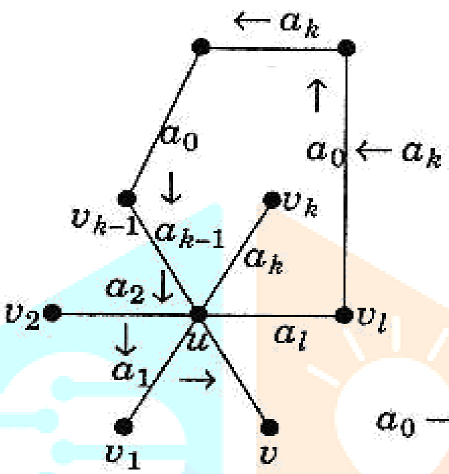
For $i \geq 2$, we continue this process. Having selected a new color a_i that appears at u, let $v_i$be the neighbor of u along the edge of color a_i . Let a_{i+1}be a color missing at v_i. If a_{i+1} is missing at u, then we shift color a_j from uv_j to uv_{j-1} for $1 \leq j \leq i$ (where $v_o = v$) to complete the augmentation. We call shifting of colors *down shifting fromi* . We are finished a_{i+1} appears at u, in which case the process continues.

Since we have only $\Delta(G) + 1$ colors to choose from, the iterative selection of a_{i+1} eventually repeats a color. Let l be the smallest index such that a color

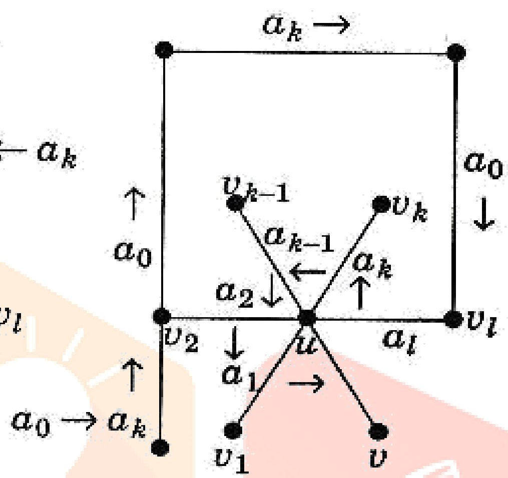
a_{l+1} missing at v_1 is in the list a_1 ,……. , a_l . Suppose $a_{l+1} = a_{k;}$ this color is missing at v_{k-1} and appears on uv_k . If a_o does not appear at v_l ,then we downshift from v_l and use color a_o on $u\, v_l$ to complete augmentation.



dia 1



dia 2



dia 3

Hence we may assume that a_o appears at v_l and that a_k does not. Let P be the maximum alternating path of edges color a_o and a_k that begins at v_l. There is only one such path, because each vertex has at most one incident edge in each color (we ignore edges not yet colored). Switching on P means interchanging colors a_o and a_k on the edges of P. Depending on the location of the other end of P, we describe a recoloring that completes the augmentation.

If P reaches v_k, then it reaches v_k along an edge with color a_o, continues along $v_k u$ in color a_k, and stops at u, which lacks color a_o. In this case, we downshift from v_k and switch on P(see the dia 1 above). Similarly, if P reaches v_{k-1}, give color a_o to uv_{k-1}, and switch on P(see the dia 2 above). Finally, suppose P does not reach v_k or v_{k-1}, so it ends at some vertex outside{u, v_l, v_k, v_{k-1}). In this case, we downshift from v_l, give a color a_o to uv_l and switch on P(see the dia 3 above).

Vizing's Adjacency Lemma:

A graph G with at least two edges is minimal with respect to chromatic index if $\chi 0(G - e) = \chi 0(G) - 1$ for every edge e of G. Since isolated vertices have no efect on edge colorings, it is natural to rule out isolated vertices when considering such minimal graphs. Therefore, the added hypothesis is that a minimal graph G is connected is equivalent to the assumption that G has no isolated vertices.

Two of the most useful results dealing with these minimal graphs are also results of Vizing [22], which are presented without proof.

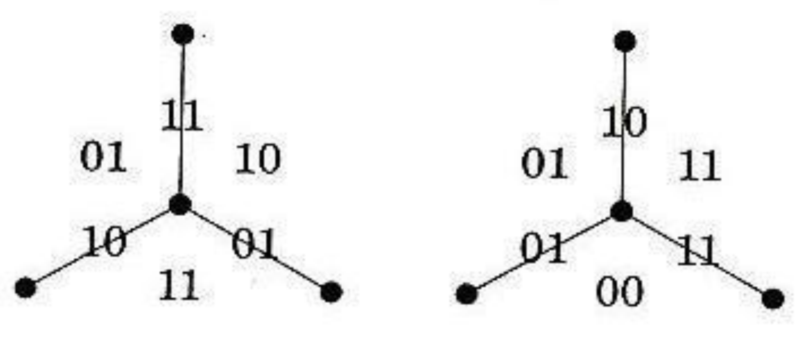
Theorem: Let G be a connected graph of Class 2 that is minimal with respect to chromatic index. Then every vertex of G is adjacent to at least two vertices of degree $\Delta(G)$. In particular, G contains at least three vertices of degree $\Delta(G)$.

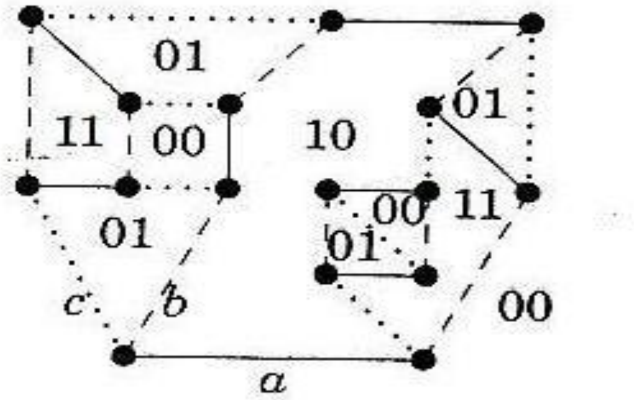
Theorem: Let G be a connected graph of Class 2 that is minimal with respect to chromatic index.If u and v are adjacent vertices with $\deg(u) = k$, then v is adjacent to at least $(G) - k + 1$ vertices of degree $\Delta(G)$.

A **Tait coloring** is a 3-edge coloring of a cubic graph.

Theorem [Tait(1878)] : A simple 2-edge-connected 3-regular plane graph is 3-edge-colorable if and only if it is 4-face-colorable.

Proof: Let G be such a graph. Suppose first that G is 4-face-colorable; we obtain a 3-edge-coloring. Let the four colors be denoted by binary ordered pairs: c_o=00, c_1 =01, c_2 =10, c_3 = 11. We obtain a proper 3-edge-coloring of G by assigning to the edge between faces with colors c_i and c_j the color obtainted by adding c_i and c_j as vectors of length 2, using coordinate wise addition modulo 2. Because G is 2-edge-connected, each edge bounds two distinct faces, and hence the color 00 never occurs as a sum. It suffices to prove that the 3 edges at a vertex receive distinct colors. At vertex *v* the faces bordering the three incident edges are pairwise adjacent, so these three faces must have three distinct colors $\{c_i , c_j , c_k\}$, as illustrated below. If color 00 is not in this set, then sum of any two of these is the third, and hence $\{c_i , c_j , c_k\}$, is the set of colors on the three edges. If c_k =00, then c_i and c_j appear on two of the edges, and third receives the color $c_i + c_j$, which is the color not in $\{c_i , c_j , c_k\}$.





Now suppose G has a proper 3-edge-coloring using colors *a, b, c* on the subgraphs E_a , E_b , E_c ;we construct a 4-face-coloring using four colors defined above. Since G is 3-regular, each color appears at every vertex, and the union of any two of E_a, E_b , E_c is 2-regular, which makes it a union of disjoint cycles. Each face of this subgraph is a union of faces of the original graph. Let $H_1 = E_a \cup E_b$ and $H_2 = E_b \cup E_c$. To each face of G, assign the color whose *i*th coordinate ($i \in \{1,2\}$) is the parity of the number of cycles in H_i that contain it

(0 for even, 1 for odd). We claim this is a proper 4-face-coloring, as illustrated above. If two faces F, F' are separated by an edge *e*, they are distinct faces, since G is 2-edge-connected. This edge belongs to a cycle C in atleast one of H_1 , H_2 (in both if *e* has color *b*). By the Jordan Curve Theorem, one of F, F' is inside C and the other is outside. However, forever other cycle in H_1 or H_2 , F, F' are on the same side. Hence if *e*has color *a, c, or b*, then the parity of the number of cycles containing F and F' is different in H_1 , in H_2 or in both, respectively. This means that F and F' receive different color in the face-coloring we have constructed.

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