



AN INTRODUCTION: KNOT THEORY

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ABSTRACT:

The study of the Knot Proposition involves interesting artefacts that are found in real-world physical space. Even though the topic of the knot proposition is well known to all and its problems are clearly stated, they can also be found in fields such as biology, chemistry, and medicine. It is frequently difficult to apply sophisticated solutions to even the most basic problems. This essay covers a variety of subjects connected to various facets of contemporary life.

Keywords: Knots, Biology, Chemistry, Physics, Mathematics.

INTRODUCTION

J.W. Alexander, an American mathematician who lived from 1888 to 1971, was the first to demonstrate the significance of the knot premise in the study of three-dimensional topology. The work of the German mathematician H. Seifert, among others, from the late 1920s to the 1930s served to emphasise this even more. Additionally, there was a lot of effort put into studying the connection between algebraic figures and knot proposal around this period in Germany. In the 1950s, following World War II, research into knot proposition advanced rapidly in the United States. This exploration had a significant impact on the development of knot theory in Japan, which has persisted to the present day.

By virtue of the outcome of Smith's hypothesis on periodic mappings, it was demonstrated in the 1970s that the knot proposition is connected to the algebraic number proposition, among other effects. The epochal knot steady discovered by V.F.R. Jones in the early 1980s was the reason why the knot proposition transitioned from the study of topology to the study of high-quality pharmaceuticals. When it was demonstrated that the knot proposition is almost related to the soluble models of statistical mechanics, this was further underlined. The bounds of the knot proposition are always changing as it expands and changes. Now, in addition, they excel in a few specific fields of fine chemistry and biology. To briefly elaborate, it

has been observed experimentally that specific DNA notes in biology can take the shape of particular knots. We intend to introduce the contribution of various mathematicians to knot proposition as well as the relationship and operation of knot proposition to other domains in the chapters that follow.

HISTORICAL WAY

Theophil Vandermonde, a French mathematician, may have made the first mention of knots from a fine perspective in a paper titled "Remarques sur les problems de situation" in 1771. The study of fine properties of knots, often known as knot theory, dates back to the 19th century and the system for tabulating knots developed by the German mathematician Carl Friedrich Gauss. Gauss used the sophisticated understanding of knots in his electrodynamics research. He was curious about the significance of the work being done on a glamorous pole in an unfettered wind in a current circle. He gave two non-cutting circles some thought. He was qualified to receive a response. He made the discovery that is now known as the "Gauss linking number" during this operation. Under ambient isotopy, this quantity remains constant. It was the first technique to discriminate between two links that were not original. Johann Benedict Listing, a student of Gauss, developed an interest in knots after reading his work. He initially used the term "Topology" in his essay "VorstudienzurTopologie," which also featured a discussion of little knots and their bracket. His focus was on the parity between a knot and its glass image, or knot chirality.

The discovery of the abecedarian group, which served as the basis for algebraic topology, by Julius Henry Poincare in 1900 marked a significant advancement in the study of knots. Henreich Tietze distinguished between the unknot and the trefoil knot in 1908 using the abecedarian group of the surface of the knot in R^3 known as the knot group. In 1905, Wilhelm Wirtinger presented a mechanism for randomising a group donation. The idea of a knot group was improved by Max Dehn, who also created an algorithm to build the abecedarian group of a link's complement and demonstrated that the trefoil knot is not amphichiral. Mathematicians slowly began their investigation into knots.

Mathematicians were more intrigued by the knot proposition in the 1920s. This was an area of inquiry that was later applied to merely fine generalisations. Emil Artin created the Braid Theory in the early 1920s. This claim could be operated on using everything from number mechanics to combinatory to fabric assiduity. Through the plat proposition, the knot proposition was able to regain its importance. Mathematicians were more intrigued by the knot proposition in the 1920s. This was an area of inquiry that was later applied to merely fine generalisations. Emil Artin created the Braid Theory in the early 1920s. This claim could be operated on using everything from number mechanics to combinatory to fabric assiduity. Through the plat proposition, the knot proposition was able to regain its importance.

KNOTS:

This notion leads naturally to the definition of a knot as a continuous simple closed curve in R^3 . Such a curve consists of a continuous function $f : [0,1] \rightarrow R^3$ with $f(0) = f(1)$ and with $f(x) = f(y)$ implying one of three possibilities:

$$x = y$$

$$x = 0 \text{ and } y = 1$$

$$x = 1 \text{ and } y = 0$$

$$C = \{x, y\} \text{ belongs to } R^3: x^2 + y^2 = 1$$

The remedies are either to introduce the concept of differentiability or to use polygonal curves instead of differentiable ones in the definition.

KNOT:

In the mathematical field of topology, **knot theory** is the study of mathematical knots. While inspired by knots which appear in daily life, such as those in shoelaces and rope, a mathematical knot differs in that the ends are joined so it cannot be undone, the simplest knot being a ring (or "unknot"). In mathematical language, a knot is an embedding of a circle in 3-dimensional Euclidean space, R^3 .

A Knot is a simple closed polygonal curve R^3 . The ordered set (p_1, p_2, \dots, p_n) defines a knot.

The knot being the union of the line segments $[p_1, p_2], [p_2, p_3], \dots, [p_{n-1}, p_n]$ and $[p_n, p_1]$.

A knot is a closed loop of "string," where the string has no thickness at all. It must not intersect itself, since that would cause branches in the "string," but may cross over itself.

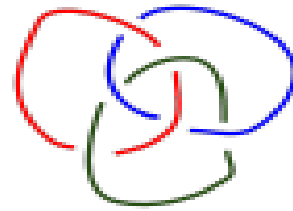
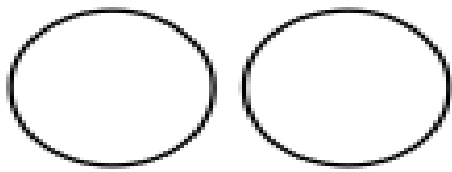


Figure represents three common points

- From left to right: the unknot
- Middle: the trefoil knot
- And final one eight knot.

LINK:

A link is a group of one or more knots. Those knots do not necessarily need to be connected to each other as the name suggests. The only requirement is that the link consist of more than one discrete piece of string.

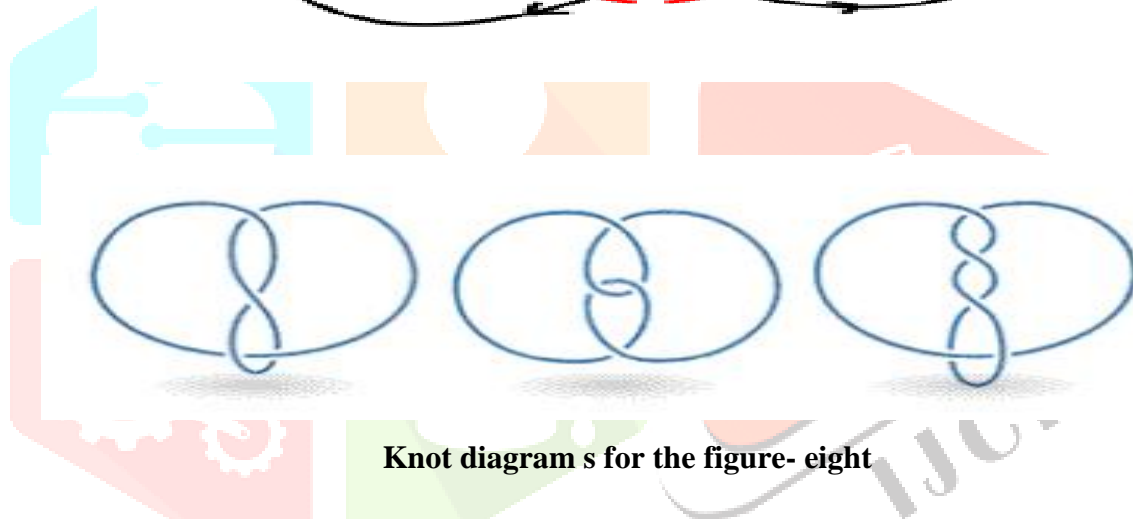
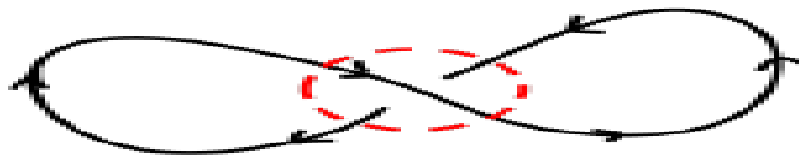


Two common links: Unlink (left)

Borromean link(Right)

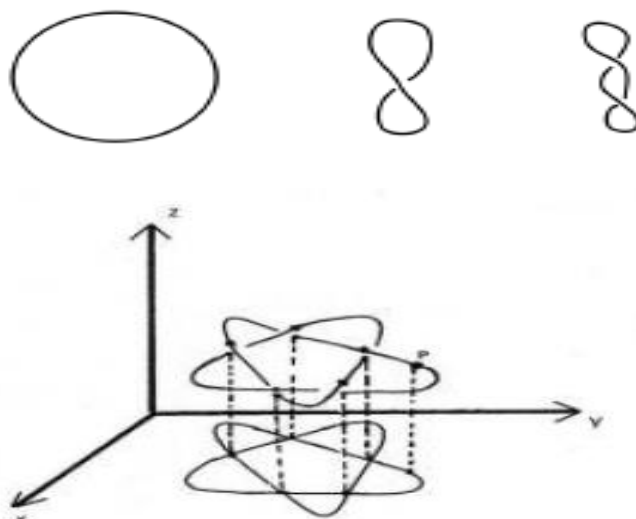
REGULAR PROJECTION (OR) CROSSING:

A knot projection is called a regular projection if no three points on the knot project to the same point, and no vertex projects to the same point as any other point on the knot.

**Knot diagrams for the figure-eight**

A knot diagram is the regular projection of a knot to the plane with broken lines indicating where one part of the knot under crosses the other part. Informally, an orientation of a knot can be thought of as a direction of travel around the knot.

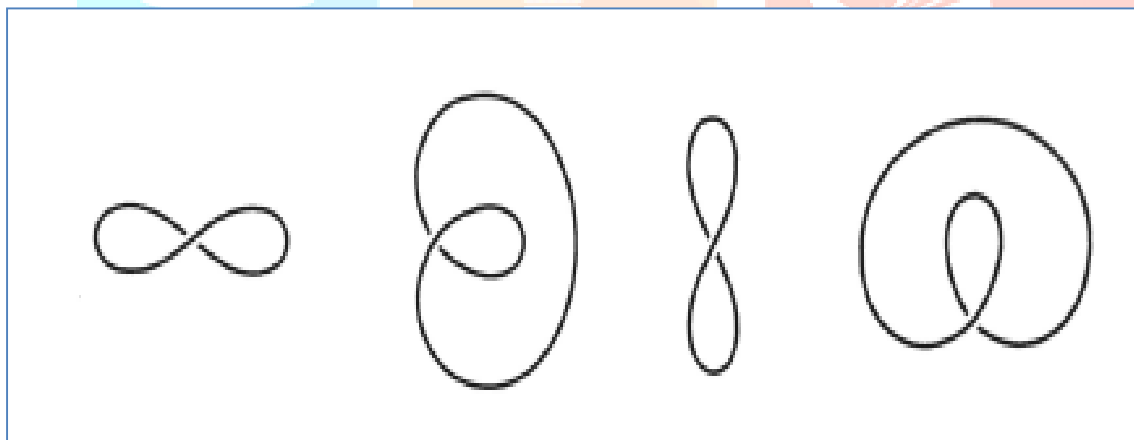
Projection: A knot has different projections, which are different expressions of the same knot. They appear different but can actually be twisted and pulled to become the same knot.



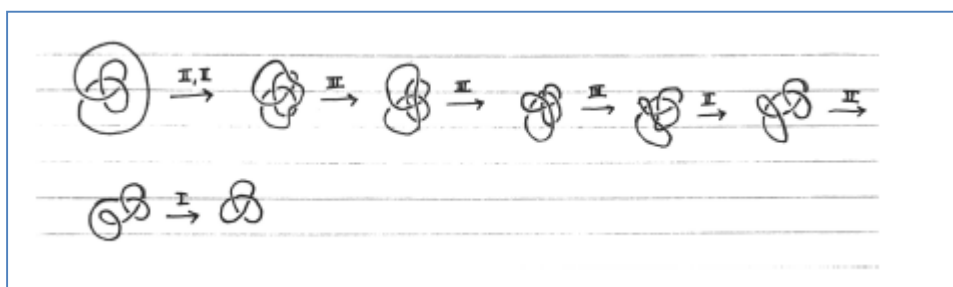
PROJECTION OF A KNOT :

Twisting and pulling is a very vague term, however. There are a few specifications we can take on one projection of a knot to transform it into another called Reidemeister moves.

Nontrivial knots (i.e. not the unknot) must have more than one crossing in a projection. This makes sense, because a knot with no crossings is simply a circular loop, the simplest projection of the unknot. But we can also easily show that knots with exactly one crossing are trivial as well. All one-crossing knots can be simplified to one of the four shown in below



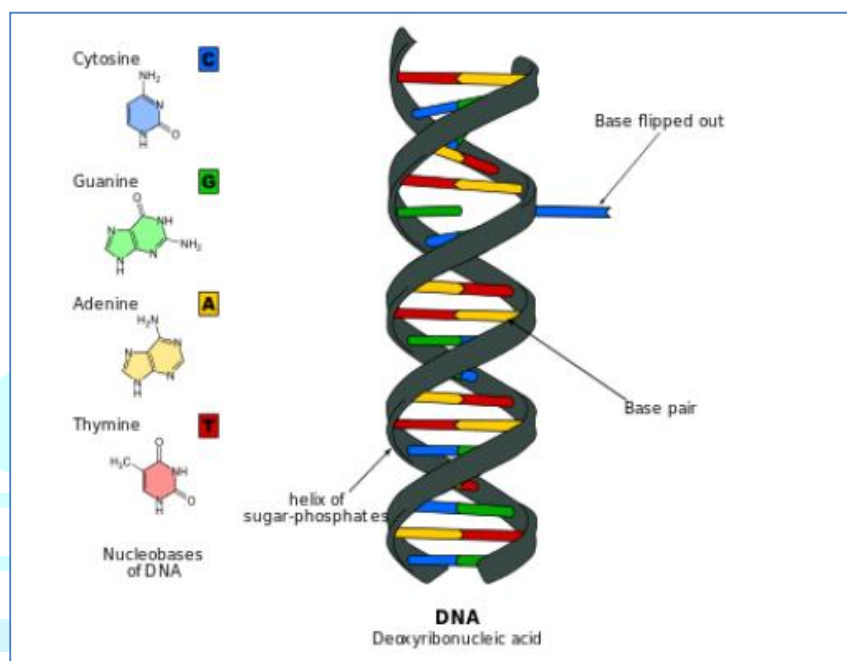
When moving between different projections of the unknot as shown in above figures, we used only Type I Reidemeister moves by adding and removing twists. More complex knot projections, however, will require more complex series of Reidemeister moves to be transformed.



A more complicated series of Reidemeister moves to simplify a projection of the trefoil knot.

IN BIOLOGY:**REPRESENTATION OF DNA:**

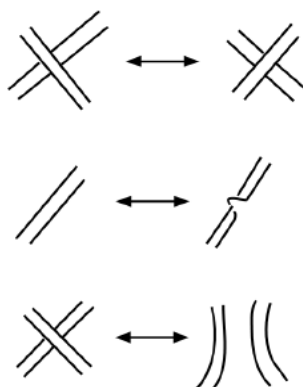
DNA has a very particular structure illustrated in figure, a DNA molecule is composed of two molecular strands twisted together in a double helix held together by pairs of nitrogenous bases bonded together in between them. There are four possible bases: adenine (A), thymine (T), guanine (G), or cytosine (C). However, there are only four specific pairings - AT, TA, CG, and GC – as adenine only bonds with thymine and guanine only with cytosine.

**DNA Structure**

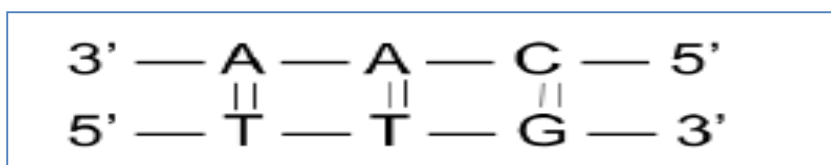
Because of the small number of possible pairings, each DNA molecule must contain millions of pairs to hold all the genetic information necessary for life. To make matters worse, this genetic material is usually very tangled, which makes it very difficult to replicate, copy, or modify the DNA when the cell's biological mechanisms require it.

DNA MANIPULATION AND TOPOISOMERASES:

In order to make these disorganized tangles of DNA easier to work with, cells use enzymes called topoisomerases to manipulate the DNA topologically. Specific enzymes can perform very sophisticated manipulations on DNA, but topoisomerases can also perform several more general actions, as pictured in the following figure.

**Three actions of DNA performed by topoisomerases**

Scientists use circular (cyclic) DNA, or DNA molecules whose ends are joined, to examine the effects of these enzymes. On linear DNA, any knots caused by the enzyme would slip off the end, while they would be captured in a circular molecule. Both single-stranded and duplex (double-stranded) forms of cyclic DNA are common not only in bacteria and viruses but also in human mitochondria. However, although they appear naturally, scientists often apply enzymes to artificially created, synthetic cyclic DNA to determine their effects. To create this cyclic DNA, scientists attach the tail ends of a strand of linear duplex DNA to the head ends. However, a wrinkle appears when we consider that DNA has a set orientation. Each strand contains sugar molecules, and depending on the orientation of the sugar molecules at either end, one end is defined as 3' and the other as 5'. Further, the structure of linear duplex DNA is anti parallel, so one strand's 3' end will correspond to the other's 5' end, and vice versa.



PHYSICS APPLICATIONS:

Statistical mechanics is an topic in physics where the overall state of a system is studied rather than that of the individual particles. A simple example might be how temperature describes average kinetic energy of a set of particles, rather than the kinetic energy of a specific particle. Statistical mechanics is useful in describing things like phase transition (which does not occur for a single molecule), or the effects of magnetization on a conductor. It can be difficult to model a large system. One simple model for that purpose is the Ising model, which models situations in which only particles that are near to each other interact. To visually describe an Ising model, we can use a graph in which the vertices represent particles and the edges represent interactions. Because metals are often shaped in regular, repeating patterns, this allows Ising models to be particularly simple when modelling them. Ising models of 3D states have been very difficult to solve mathematically, but the study of 2D lattices provides a foundation for later study of them. In Ising models, vertices receive a positive or negative 1 value. In magnetization, this value would denote spin.

Each edge of the graph has energy based on the states of the vertices on either side of the edge. We can say that in general the energy of an edge that has vertices of the same value is E_{\pm} , and if the vertices have different values the energy of the edge between them is E_{\neq} . The energy of the system, $E(s)$, is then the sum of energies of all the edges. We can define the following:

$$\omega(s_i, s_j) = \exp\left(\frac{-E(s_i, s_j)}{kT}\right)$$

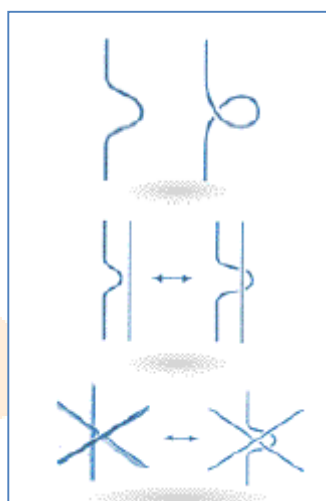
$$P = \sum_s \exp\left(\frac{-E(s)}{kT}\right) = \sum_s \pi \omega(s_i, s_j).$$

Essentially, what this equation states is that the partition function simply equals to the product of the ω function for all edges in a state, summed across all possible states.

Equivalence of Knots:

A knot K is said to be *equivalent* (or *equal*) to a knot K' if we can obtain K' from K by applying the elementary knot moves a finite number of times.

The notion of equivalence satisfies the definition of an equivalence relation; it is reflexive, symmetric, and transitive. Knot theory consists of the study of equivalence classes of knots. In general it is a difficult problem to decide whether or not two knots are equivalent or lie in the same equivalence class, and much of knot theory is devoted to the development of techniques to aid in this decision. A Reidemeister move is an operation that can be performed on the diagram of a knot without altering the corresponding knot.

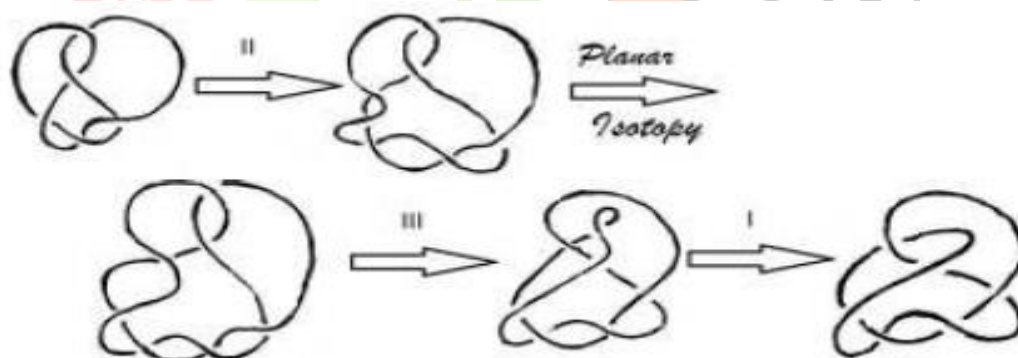


Type I,II,III Reidemeister moves

Theorem: If two knots are equivalent, their diagrams are related by a sequence of Reidemeister moves.

The German mathematician Kurt Reidemeister proved that if there are two projections of the same knot, then one can be obtained from the other by a series of **Reidemeister moves** and planar isotopies.

The following examples demonstrate this.



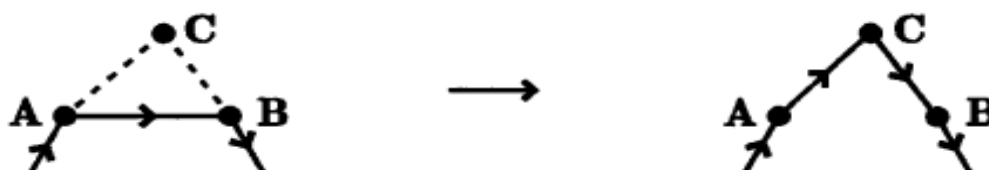
The equivalence of two projections to show the Reidemeister moves.

Two knots K_1 and K_2 are said to be *equivalent* if there exists a homeomorphism of R^3 onto itself which maps K_1 onto K_2 . If two knots are equivalent then they are said to be of the *same type*. Any knot equivalent to an unknot is of *trivial type*. If a knot is equivalent to a polygonal knot then it is said to be a *tame*; otherwise a *wild knot*.

A knot has no starting point and no endpoint, i.e., it is a simple closed curve. Therefore, we can assign an orientation to the curve. As is the custom, we shall denote the orientation of a knot by an arrow on the curve. It is immediately obvious that any knot has two possible orientations.



If two oriented knots K and K' can be altered with respect to each other by means of *oriented* elementary knot moves, Figure then we say K and K' are equivalent with *orientation preserved* and we write $K \cong K'$.



Two knots that are equivalent without an orientation assigned are not *necessarily* equivalent (with orientation) when we assign an orientation to the knots.. The two knots in the above figure are certainly equivalent without an orientation assigned; it is not, however, immediately obvious whether they are equivalent with orientation.

Theorem:

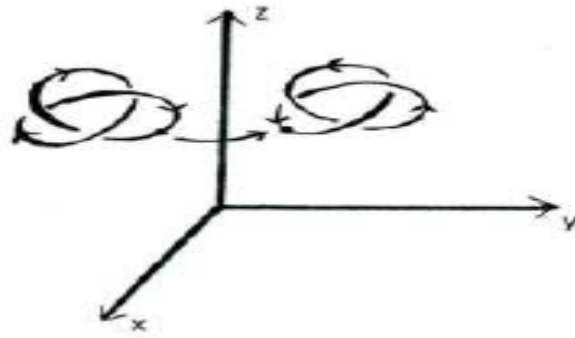
If two knots K_1 and K_2 that lie in S^3 are equivalent, then their complements $S^3 - K_1$ and $S^3 - K_2$ are homeomorphic

Show first that the knot K , shown in below figure, is the mirror image of K, K^* , and second K and K^* are equivalent with orientation..



Knot decomposition and the semi-group of a knot:

Consider $\phi(x, y, z) = (-x, -y, z)$, a 180° rotation about the z - axis. This is an orientation preserving auto- homeomorphism of R^3 . If K_1 is a trefoil knot oriented clockwise and K_2 a trefoil knot oriented anticlockwise, then ϕ maps K_1 onto K_2 . Thus K_1 is equivalent to K_2 which has a reverse orientation and hence the trefoilknot is invertible.



K_1 Mapped to K_2 which has reverse orientation.

The Knot shown in the figure with 8 crossings is not invertible.



Cross number: The crossing number of a knot K , denoted $C(K)$, is the least number of crossings that occur, ranging over all possible diagrams.

Prime knot: knot is called prime if for any decomposition as a connected sum, one of the factors is unknotted.

Prime decomposition theorem:

Every knot can be decomposed as the connected sum of nontrivial prime knots. If $K = K_1 \# K_2 \# \dots \# K_n$ and $K = J_1 \# J_2 \# \dots \# J_m$ with each K_i and J_i nontrivial prime knots, then $m = n$, and, after reordering each K_i is equivalent to J_i .

If this knot is composed with itself in two different ways, then two distinct knots which are not equivalent are obtained.



Knots are composed in two different ways

Theorem: The uniqueness and existence of a decomposition of knots.

- Any knot can be decomposed into a finite number of knots.
- This decomposition, excluding the order, is unique. That is, if a knot is decomposed

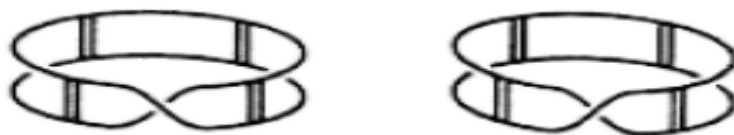
in two ways as $K_1 K_2 K_3 \dots K_m$ and $K'_1 K'_2 K'_3 \dots K'_n$ then $n=m$. Also, renumbering the subscript suitably of $K_1, K_2, K_3, \dots, K_m$ $K_1 \approx K'_1, K_2 \approx K'_2, K_3 \approx K'_3, \dots, K_m \approx K'_m$.

Proposition:

$K_1 \# K_2$ is equivalent to $K_2 \# K_1$ with orientation i.e commutative law holds for sum of two knots. Also Associative law holds.

Synthesis of knotted molecules:

A molecule of DNA contains many atoms and is very complicated. There are some simple molecules that can be knotted or linked. A chain of same atoms, bonded in the same way can form a knotted chain or an unknot. Thus two or more molecules can be formed. These molecules may be distinct with different properties. For example, consider the two molecules in the figure.



Same set of atoms and bonds are made by two molecules

The first one is a Mobius band ladder with four rungs and right hand twist while the second one is with left hand twist. The two molecules are made up of the same atoms and bonds but have different molecular graph. The first embedding of the graph cannot be deformed to the second embedding of the graph through three dimensional space, that is, the two molecules are homeomorphic but are not isotopic

A pair of molecules that are homeomorphic but not isotopic are called a pair of **topological stereoisomers**. same atoms are bonded in the same sequence to form three molecules. The first is an unknot, the second a left hand trefoil and the third a right hand trefoil. The three molecules are topological stereoisomers with each other.



Three topological stereoisomers.

Chemists were interested in topological stereoisomers because they could synthesize new molecules.

CONCLUSION:

Although knot theory is generally thought of as a quite specialised discipline, it is routinely used to model and explain key concepts in a variety of other fields of knowledge. In biology, knots can be used to investigate how topoisomerase enzymes modify DNA by adding or removing tangles, and in chemistry, knots can be used to describe the structure of topological stereoisomers. Ultimately, scientists have also made significant contributions to the depth of the area and its implications for further practical application, even though mathematicians have been crucial to advancing knot proposal as a theoretical field.

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