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BOUNDEDNESS OF SETS

L. N. MALLIKA SURIBHATLA

M.Sc., B.Ed.

VSM COLLEGE(AUTONOMOUS), RAMACHANDRAPURAM

Dr.B.R AMBEDKAR KONASEEMA DISTRICT

MANIKANTA SATYALA,

M.Sc., B.Ed.

VSM COLLEGE(AUTONOMOUS), RAMACHANDRAPURAM

Dr.B.R AMBEDKAR KONASEEMA DISTRICT

N.S.V. KIRAN KUMAR

M.Sc., M.Phil., B.Ed.

GOVERNMENT DEGREE COLLEGE MANDAPETA,

Dr.B.R AMBEDKAR KONASEEMA DISTRICT

ABSTRACT

Boundedness refers to the property of a mathematical object or function that is limited within a certain range or set of values. This concept is fundamental in many areas of mathematics, including calculus, analysis, and topology, as well as in other fields like physics, engineering, and economics. It allows mathematicians to study the behavior of mathematical objects and systems under various conditions and constraints, and to make predictions about their properties and behavior. Understanding boundedness is crucial for solving many mathematical problems and for developing new mathematical theories and models. Boundedness is often used to describe the behavior of a system or function, and it plays an important role in determining the convergence or divergence of a sequence or series.

In general, boundedness provides a useful way to understand and analyze mathematical objects and their properties, and it has numerous applications in both theoretical and practical contexts. In analysis, for example, a function is said to be bounded if its values are confined within a finite range, while a sequence is bounded if

its terms do not diverge to infinity. In topology, a set is bounded if it is contained within a finite radius, and in algebra, a group is said to be bounded if all its elements are confined within a finite range. Understanding boundedness is essential for many areas of mathematics, and is a key concept in the development of rigorous mathematical proofs and theorems.

KEYWORDS:

Boundedness, bounded above, bounded below, lower bound, upper bound, infimum, supremum, locally bounded and uniformly bounded.

INTRODUCTION

Boundedness on sets is a concept from mathematical analysis that describes the extent to which a set is limited in size or scope. An object is said to be bounded if it is constrained within a finite range or region. More specifically, a set is said to be bounded if there exists a finite number, called a bound, such that all the elements of the set are within a certain distance from this number. For example, a function is said to be bounded if its range is a bounded set, meaning that its values do not "escape" to infinity. Similarly, a sequence is said to be bounded if its terms are contained within a finite range.

In particular, a set is said to be bounded if it can be enclosed within a finite region of space. For example, a circle in a plane is a bounded set because it can be enclosed within a finite radius. On the other hand, a line in a plane is an unbounded set because it cannot be enclosed within any finite radius.

In a more general sense, a set is bounded if it has a finite diameter or radius. This means that all elements in the set are within a certain distance from some fixed point or center.

Basic concept:

In mathematics, boundedness refers to a property of a set, function, or sequence that is restricted to a finite range of values.

A **Set** is said to be bounded if it is contained within some finite region of space. More specifically, a set is bounded if there exists a finite number M such that the absolute value of each element in the set is less than or equal to M.

i.e., A set S is sad to be bounded if there exists a finite number M such that $|x| \leq M, \forall x \in S$

For example, the interval [-1, 1] is a bounded set, because all its elements lie between -1 and 1. On the other hand, the set of all real numbers is not bounded, because it extends infinitely in both directions.

A **Function** is said to be bounded if its values are limited within a certain range, regardless of the domain of the function. In other words, a function is bounded if there exists a finite number M such that the absolute value of the function is less than or equal to M for all possible inputs.

i.e., A function $f(x)$ defined on some set X with real or complex values is said to be bounded if there exists a finite number M such that $|f(x)| \leq M, \ \forall x \in X$.

For example, the function $f(x) = sin(x)$ is bounded since the absolute value of $sin(x)$ is always less than or equal to 1 for all values of x.

A **Sequence** is said to be bounded if its values do not exceed a certain limit or range. In other words, a sequence is said to be bounded if there exists a finite number K such that the absolute value of each element of sequence is less than or equal to K.

i.e., a Sequence $\{S_n\}$ is said to be bounded iff there exists K$\in R^+$such that $|S_n| \leq K$ or $-K \leq S_n \leq K$ for all n.

For example, the sequence {1/n} is bounded because its values approach zero as n approaches infinity and never exceed a certain value. Also, the sequence $\{(-1)^n\}$ is bounded since its terms alternate between -1 and 1, which are both within a finite range. On the other hand, the sequence {n} is not bounded since its terms increase without limit.

Boundedness of a set:

An aggregate S is said to be bounded if it is both bounded below and bounded above.

- An Aggregate S is bounded ⟺ there exists $u, v \in R$ such that $v \leq x \leq u$ for all $x \in S$.
- An Aggregate S is bounded ⟺ there exists $k \in R$ such that $|x\ | \leq K$ for all $x \in S$.

Ex: 1. A finite set is bounded.
2. The aggregates Z, Q and R are not bounded.
3. If $S = \{\frac{1}{n}/n \in N\}$ then S is bounded and inf S = 0, Sup S = 1

Upper bound:

The upper bound of a set is a value that is greater than or equal to every element in the set. More formally, let S be a set of real numbers. A real number M_1 is an upper bound of S if and only if $M_1 \geq x$ for all $x \in$ S. Also, any number greater than M_1 is also an upper bound of S.

- For example, if we have the set S = {1, 2, 3, 4, 5}, then 6 is an upper bound of S, since 6 is greater than every element in S. Similarly, 7, 8, 9, ... are also upper bounds of S.
- Multiple upper bounds may exist for a set.

Bounded above:

An Aggregate S is said to be **bounded above** if there exists $M_1 \in R$ such that $x \in S \Longrightarrow x \leq M_1$.The number M_1 is called an upper bound of S.

- A number $M \in$ R is not an upper bound of an aggregate S if there exists at least one y ∈ S such that $y > M$ or $y \nleq M$.
- R^- is bounded above set and '0' is an upper bound.
- R^+ is not bounded above.
- Every finite set is bounded above.

Least upper bound (l.u.b) or Supremum(sup):

The smallest value among the upper bounds of the set S is known as the least upper bound (l.u.b). Which also called as the supremum of a set S. In other words, if M is an upper bound of S and there is no real number in S less than M, then we say that M is called least upper bound or supremum of set S. The least upper bound is denoted by sup(S) or sometimes by l.u.b.(S).

- For example, Let S = { $x \in Z/x \leq 2$ }. In this case, 2 is an upper bound of S, also 3,4, …. are upper bounds. But 2 is the smallest value among them. So, the least upper bound of S is exactly 2, which is denoted as sup(S) = 2 or l.u.b.(S) = 2.
- M $\in R$ is supremum of S $\Longrightarrow$ for each $\epsilon > 0, M - \epsilon$ is not an upper bound of S $\Longrightarrow$ there exists at least one $x \in S$ such that $M - \epsilon < x < M$.

Lower bound:

The Lower bound of a set is a value that is less than or equal to every element in the set. More formally, let S be a set of real numbers. A real number M_2 is a lower bound of S if and only if $M_2 \leq x$ for all $x \in$ S. Also, any number less than M_2 is also a lower bound of S.

- For example, if we have the set S = {1, 2, 3, 4, 5}, then 0 is a lower bound of S, since 0 is less than every element in S. Similarly, -1, -2, -3, …. are also lower bounds of S.
- Multiple lower bounds may exist for a set.

Bounded below:

An Aggregate S is said to be **bounded below** if there exists $M_2 \in R$ such that $x \in S \Longrightarrow x \geq M_2$.The number M_2 is called a lower bound of S.

- A number $M \in$ R is not a lower bound of an aggregate S if there exists at least one y ∈ S such that $y < M$ or $y \ngeq M$.
- N is bounded below and 1 is a lower bound.
- R^- is not bounded below.
- Every finite set is bounded below.

Greatest Lower bound (g.l.b) or Infimum(inf):

The greatest value among the lower bounds of the set S is known as the greatest lower bound (g.l.b). Which also called as the infimum of a set S. In other words, if 'm' is a lower bound of S and there is no real number in S greater than 'm', then we say that 'm' is called greatest lower bound or infimum of set S. The greatest lower bound is denoted by inf(S) or sometimes by g.l.b.(S).

- For example, Let S = { $x \in Z/x \geq 3$ }. In this case, 3 is a lower bound of S, also 2, 1, 0, -1, -2, ... are lower bounds. But 3 is the greatest value among them. So, the greatest lower bound of S is exactly 3, which is denoted as inf(S) = 3 or l.u.b.(S) = 3.
- m $\in R$ is infimum of S $\Longrightarrow$ for each $\epsilon > 0, m + \epsilon$ is not a lower bound of S $\Longrightarrow$ there exists at least one $x \in S$ such that $m < x < m + \epsilon$.

Boundedness of a function:

A function $f(x)$ defined on some set X with real or complex values is said to be bounded if set of all its values lies in limited range.

- A function $f(x)$ defined on some set X with real or complex values is said to be bounded if there exists a finite number K$\in R$ such that $|f(x)| \leq K, \ \forall x \in$ X
- A function $f(x)$ defined on X is said to be bounded if $a \leq f(x) \leq b, \forall x \in X$.
- If $f(x)$ is real-valued defined on X is said to be **bounded above** by A if $f(x) \leq$ A for all $x \in$ X.
- If $f(x)$ is real-valued defined on X is said to be **bounded below** by B if $f(x) \geq$ B for all $x \in$ X.

Ex: 1. For example, the function $f(x) = 1/x$ is bounded on the interval $[1, \infty)$ because its values are always greater than 0 and less than or equal to 1.

2. The function $g(x) = x^2 + x$ is not bounded since its range is not restricted.

3. The function $f(x) = sin(x)$ is bounded since its range lies between -1 and 1.

Sequence:

A function $\boldsymbol{s}: Z^+ \rightarrow R$ is called a sequence of real numbers or a sequence. A sequence S is a specified rule which associates to each $n \in Z^+$exactly one $S(n)$ or $S_n \in R$. A Sequence is thus the set $\{(n, S_n)/\ n \in Z^+\}$.

- We denote a sequence by $\{S_n\}$ or $< S_n >$ or $S_1, S_2, S_3, \dots\dots S_n, \dots.$ or $S(1), S(2), S(3), \dots\dots S(n), \dots..$ or $\{S_n\}_{n=1}^{\infty}$.
- The set of all terms of a sequence is called range set or range of the sequence. Range set of the sequence $\{S_n\} = \{S_n/\ n \in Z^+\}$.
- The range of a sequence may be finite or infinite.
- The range sequence is a non-empty subset of R.

Boundedness of a Sequence:

A Sequence $\{S_n\}$ is bounded if it is both bounded below and bounded above.

A Sequence $\{S_n\}$ is **bounded below** if the range of the sequence is bounded below. i.e., if there exists a real number K_1 such that $K_1 \leq S_n, \forall\ n \in Z^+$.The number $K_1$is called **lower bound** of the sequence $\{S_n\}$.

If $K_1$is a lower bound of the sequence $\{S_n\}$, then any number less than K_1 is also lower bound of $\{S_n\}$.If 'l' is lower bound of $\{S_n\}$ and any real number less than 'l' is not a lower bound of $\{S_n\}$ then 'l' is called **greatest lower bound** or **infimum** of $\{S_n\}$.

A Sequence $\{S_n\}$ is **bounded above** if the range of the sequence is bounded above. i.e., if there exists a real number K_2 such that $K_2 \geq S_n, \forall\ n \in Z^+$.The number $K_2$is called **upper bound** of the sequence $\{S_n\}$.

If $K_2$is an upper bound of the sequence $\{S_n\}$, then any number greater than K_2 is also upper bound of $\{S_n\}$.If 'u' is an upper bound of $\{S_n\}$ and any real number greater than 'u' is not an upper bound of $\{S_n\}$ then 'u' is called **least upper bound** or **supremum** of $\{S_n\}$.

- A sequence $\{S_n\}$ is bounded $\Leftrightarrow$ there exists $K_1, K_2 \in R$ such that $S_n \in [K_1\ K_2]$. i.e., $K_1 \le S_n \le K_2,, \forall\ n \in Z^+$.
- A sequence $\{S_n\}$ is bounded $\Leftrightarrow$ there exists $K \in R^+$ such that $|\,S_n| \le K,\ \forall\ n \in Z^+$ or $-K \le S_n \le K,, \forall\ n \in Z^+$.
- A Sequence $\{S_n\}$ is said to be **unbounded** if $\{S_n\}$ is neither bounded below nor bounded above.
- A real number l is **infimum** of the sequence $\{S_n\}$ if (i) $S_n \ge l, \forall\ n \in Z^+$ and (ii) for each $\epsilon > 0$ there exists a positive integer m such that $S_m < l + \epsilon$.
- A real number u is **supremum** of the sequence $\{S_n\}$ if (i) $S_n \le u, \forall\ n \in Z^+$ and (ii) for each $\epsilon > 0$ there exists a positive integer m such that $S_m > u - \epsilon$.
- Every subsequence of a bounded sequence is bounded.

Ex: 1. If $\{S_n\}$ defined by $S_n = 1 + (-1)^n, \forall\ n \in Z^+$.Then the range set of $\{S_n\} = \{0,2\}$.So $\{S_n\}$ is bounded and $l.u.b$ of $\{S_n\} = 2, g.l.b$ of $\{S_n\} = 0$.

2. If $\{S_n\} = \{\frac{1}{n} /\ n \in Z^+\}$ then $\{S_n\}$ is bounded as for each $n \in Z^+, \frac{1}{n} > 0$ and $\frac{1}{n} \le 1 \Rightarrow 0 < \frac{1}{n} \le 1$.Here $l.u.b$ of $\{S_n\} = 1, g.l.b$ of $\{S_n\} = 0$.

3. If $\{S_n\} = (-1)^n n, \forall\ n \in Z^+$ then $S_1 = -1, S_2 = 2, S_3 = -3, \dots \dots$ i.e., $S_{2m} > 0$ and $S_{2m+1} < 0,\ \forall\ m \in Z^+$ $\Rightarrow$for any $K \in R^+ \exists\ m \in Z^+ \ni S_{2m} > K$

$\Rightarrow \{S_n\}$ is not bounded above. Similarly, for any $K \in R^+ \exists\ m \in Z^+ \ni\ S_{2m+1} < K \Rightarrow \{S_n\}$ is not bounded below. Thus $\{S_n\}$ is not bounded.

Boundedness in a metric space:

A metric space is a set equipped with a distance function called a metric that measures the distance between any two points in the set. Boundedness is a property of a metric space that describes how "spread out" the points in the space are.

A metric space is said to be bounded if there exists a real number M such that the distance between any two points in the space is less than or equal to M. In other words, the diameter of the space (i.e., the maximum distance between any two points in the space) is finite.

- A metric space is an ordered pair (M,d) where M is a set and d is a metric (a distance between the points) on M.
- let (X, d) be a metric space, and let A be a subset of X. A is said to be bounded if there exists a real number M > 0 such that for any two points x, y in A, we have $d(x, y)\ \le\ M$.
- For example, consider the metric space (R, d), where R is the set of real numbers and $d(x, y)\ =\ |x\ -\ y|$ is the standard metric. This metric space is unbounded because the distance between any two points in R can be arbitrarily large. However, the metric space [0,1] with the same metric is bounded because the distance between any two points in [0,1] is at most 1.
- If there is an integer $r\ >\ 0$ such that for all $s, t \in S$, we have $d\ (s, t)\ <\ r$, then a subset S of a metric space (M, d) is said to be bounded. If M is bounded as a subset of itself, the metric space (M, d) is said to be a bounded metric space (or d is a bounded metric).
- A metric space is complete and totally bounded if it is compact.
- A subset of Euclidean space R^n is compact if and only if it is closed and bounded.

Boundedness in topological vector space:

In a topological vector space, a set is said to be bounded if it is contained in a bounded set. A set is bounded if it is contained within a ball of finite radius centered at the origin.

let X be a topological vector space and A be a subset of X. Then A is said to **Neumann bounded** or **bounded** in X if any of the following condition is true.

- for every neighborhood U of the origin, there exists a scalar λ such that A ⊆ λU.
- Each neighborhood of the origin absorbs A.
- For every neighborhood U of the origin there exists a scalar $r > 0$ such that $sA \subseteq U$ for all scalars satisfying $|s| \leq r$.
- For every neighborhood U of the origin there exists a scalar $r > 0$ such that $tA \subseteq U$, for all real $0 < t \leq r$.

Locally bounded:

A function or sequence is said to be locally bounded if it is bounded in some small neighborhood around each point. More precisely, a function or sequence f is said to be locally bounded at a point p if there exists a neighborhood U of p such that $|f(x)| \leq M$ for all x in U, where M is a fixed positive real number.

For a function $f: X \rightarrow Y$, where X and Y are metric spaces, f is locally bounded if for each point p in X, there exists a positive real number r such that the set $\{f(x): x \in B(p,r)\}$ is bounded, where $B(p,r)$ denotes the open ball of radius r centered at p.

- a set S in a metric space X is locally bounded if for each point p in X, there exists a positive real number r such that the set S intersected with $B(p,r)$ is bounded.
- Consider the $f(x) = sin(x)/x$ function as an example. Although this function is locally bounded everywhere except at $x = 0$, it is not bounded over the entire real line. To see this, consider any point $p \neq 0$, and let U be a small neighborhood around p. Then, since $sin(x)$ is bounded between -1 and 1 for all x, we have $|f(x)| \leq 1/|x| \leq M$ for all x in U, where M is any positive real number greater than $1/|p|$.
- The function $f: R \rightarrow R$ defined by $f(x) = \frac{1}{x^2+1}$ is bounded, because $0 \leq f(x) \leq 1, \forall x$. Thus f is also locally bounded.
- The function $f: R \rightarrow R$ defined by $f(x) = 2x + 3$ is not bounded, as it becomes arbitrarily large. However, it is locally bounded because for each a, $|f(x)| \leq M$ in the neighborhood $(a-1, a+1)$, where $M = 2|a| + 5$.
- The function $f: R \rightarrow R$ defined by $f(x) = \begin{cases} \frac{1}{x}, & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$ is neither bounded nor locally bounded. This function accepts values of any magnitude in any neighborhood of 0.
- Any continuous function is locally bounded.

Uniformly bounded:

A family of bounded functions is said to be uniformly bounded if all of its members have a common constant as a boundary. The absolute value of any function within the family is greater than or equal to this constant.

Let $F = \{f_i: X \rightarrow K, i \in I\}$ be a set of family of functions defined on X and K is the set of real or complex numbers, then F is said to be uniformly bounded if there exists a real number M such that $|f_i(x)| \leq M, \forall i \in I, \forall x \in X$.

Let $F = \{f_i : X \to Y, i \in I\}$ be a set of family of functions defined on X and Y is a metric space with metric d, then F is said to be uniformly bounded if there exists an element a in Y and real number M such that $d(f_i(x), a) \le M, \forall i \in I, \forall x \in X$.

Ex:

1. Every sequence of bounded functions that converges uniformly is uniformly bounded.
2. The family of functions $f_n(x) = \sin nx,\ n \in Z$ is uniformly bounded by 1.
3. The family of functions $f_n(x) = n \cos nx,\ n \in Z$ is not uniformly bounded. Since each $f_n(x)$ is bounded by $|n|$, but there does not exist a real number M such that $|n| \le M, \forall n \in Z$.

Properties of Boundedness:

1. If a set or a function or a sequence is bounded, then any subset of that set or a function or a sequence is also bounded.
2. The sum or difference of two bounded functions is also a bounded function. Similarly, the product of a bounded function and a bounded constant is a bounded function.
3. A function is bounded and uniformly continuous on an interval if it is defined on a closed, bounded interval.
4. A set is closed and bounded if and only if it is compact. The Heine-Borel theorem describes this.
5. Boundedness is relative: A set or function may be bounded with respect to one metric or norm but not with respect to another. For example, a function may be bounded with respect to the Euclidean norm but unbounded with respect to the L^1 norm.
6. Boundedness and Convergence: A sequence that is bounded is not necessarily convergent, but if a sequence is convergent, then it must be bounded.
7. Boundedness and Continuity: A continuous function on a closed and bounded interval is bounded.
8. Boundedness and Compactness: A set is compact if and only if it is closed and bounded
9. Boundedness implies convergence: A sequence must converge to a limit if it is bounded. The Bolzano-Weierstrass theorem refers to this.
10. A bounded set is always closed. Conversely, a closed set need not be bounded.
11. Boundedness is preserved under continuous functions: If $f(x)$ is a continuous function and S is a bounded set, then $f(S)$ is also bounded.
12. Boundedness is preserved under compact sets: If S is a bounded set and K is a compact set, then S ∩ K is also bounded.

Limitations of Boundedness:

1. **Boundedness does not guarantee convergence:** Even though a set or function is bounded, this does not imply that it will converge to a certain value or limit. For example, a function might alternate between two boundaries forever without ever converging to a limit. For example, the sequence $(-1)^n$ is bounded between -1 and 1, but it does not converge to a limit.
2. **Boundedness may not be unique:** There may be several boundaries for a set or function. For example, the function $f(x) = sin(x)/x$ is bounded between -1 and 1, but it also has bounds of -2 and 2.
3. **Boundedness may depend on the domain:** Boundedness doesn't provide any information about how a function or set will behave outside of the specified range. The domain of the function or set determines the boundedness of the attribute. For example, the function $f(x) = 1/x$ is unbounded on the interval (0,1), but it is bounded on the interval [1,2].
4. **Boundedness may not be preserved under operations:** Boundedness is not always preserved under operations such as addition, multiplication, or composition of functions. For example, the product of two bounded functions may not be bounded.
5. **Boundedness may not imply continuity:** A function can be continuous yet not bounded. For example, the function $f(x) = 1/x$ on the interval (0,1) is bounded but not continuous at $x = 0$.

6. **Boundedness does not always imply differentiability:** Even though a function is bounded, it cannot be differentiable. For instance, the function $f(x) = |x|$ is bounded but not differentiable at $x = 0$.
7. **Boundedness is not always easy to prove:** It might be challenging to prove boundedness, especially for complicated functions. It may refer to more advanced mathematical methods and an in-depth understanding of the behaviour of that particular function.
8. There are some mathematical theorems or qualities for which boundedness is not necessarily a necessary condition. A convergent subsequence can be found in any bounded sequence in Euclidean space, according to the Bolzano-Weierstrass theorem, however not all bounded sets possess this property.

Applications:

Numerous disciplines, including mathematics, economics, physics, and engineering, use boundedness extensively. Applications of boundedness include:

1. **Analysis of Functions:** An essential concept in the study of functions is boundedness. It helps in determining a function's range and proving the existence of limits. For instance, it is simpler to demonstrate that a function has a limit if it is bounded.
2. **Optimization:** Boundedness is an important factor in determining the viability of solutions in optimization problems. A feasible solution must fulfil all restrictions, including variable boundaries. Furthermore, boundedness helps in the prevention of numerical instability and the divergence of optimization methods.
3. **Control Theory:** In control theory, where the goal is to develop controllers that guarantee the system remains stable and bounded, boundedness is a key feature. Unbounded systems are susceptible to instability, oscillations, and even explosion.
4. **Probability Theory:** Boundedness is a concept used in probability theory to describe and determine the features of some probability distributions. For instance, if the support of a probability distribution fits within a finite range, the distribution is said to be bounded.
5. **Economics:** In economics, boundedness is a key notion that is utilized to explain resource, input, and output limitations. A production function, for instance, is said to be bounded if the output is constrained by the resources and inputs available.
6. **Physics:** In physics, the concept of boundedness is utilized to define concepts like energy, momentum, and angular momentum. In a closed system, these quantities are conserved, which means they don't change over time.

Overall, boundedness is a fundamental concept that is used in many fields to define properties, establish limits, and ensure stability.

Conclusion:

Numerous branches of mathematics, including calculus, analysis, topology, and number theory, all depend on the concept of boundedness. It is used to demonstrate basic mathematical concepts as well as to illustrate the convergence or divergence of sequences, series, and functions.

Although boundedness is a valuable notion, there are some restrictions that need to be taken into account when using it in various situations. To avoid assuming the wrong things or coming to the wrong conclusions, it is crucial to be aware of these limits.

In real-world applications, such as in engineering, physics, economics, and computer science, boundedness is crucial. It enables efficient system optimization, modelling and analysis of physical processes, and practical issue solving.

Although the concept of boundedness is valuable in mathematics, it is vital to be aware of its restrictions and and take into account additional characteristics of functions and sequences in order to completely comprehend their behavior.

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