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Geometrical Control Theory with Applications

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Abstract

This paper we studied dynamical systems and control theory. We deal with specific characteristics of a dynamical system namely stability and optimal control. In industry economics and engineering both characteristics are common. We picked out some examples and geometrical solutions which are significant such as portfolio selection and related economics problem. We described the corresponding differential equations and used optimal control theory to treat the problem, and we find some geometrical solutions in dynamical systems and control theory.

1.1 Introduction [5]

Control theory is the area of application-oriented mathematics that deals with the basic principle underlying the analysis and design of control systems. It is a branch of mathematics that studies the properties of control systems i.e. dynamical systems whose behavior can be modified by a command.

Goal: to control an object, to influence its behavior to achieve a desired goal.

Examples: Watts's steam engine governor, CD players and automobiles, home temperature controlling system or industrial robots and airplanes autopilots. Control mechanisms are widespread in nature and are used by living organisms to maintain essential variables such as body temperature and blood sugar levels for instance. In engineering, feedback control has a long history: As far back as the early Romans, one finds water levels in aqueducts being kept constant through the use of various combinations of valves. In more moderns' days, it is now also found in the medical field as well with the use of optimal control chemotherapy protocols and surgery robotics.

Two Main Lines of Work in Control Theory:

A good model of the object to be controlled is available and one wants somehow to optimize its behavior, this is known as optimal control.

Uncertainty about the model or about the environment in which the object operates.

It is feedback control.

The best work in systems and control theory makes it clear that the creative modeling of phenomena and the conceptualization of systems using mathematics and the subsequent mathematical analysis of these models to get a deeper understanding of the phenomena are what is central to the field.

1.1.1 Control systems [9]

Differential equations effectively model mathematically the real world from physics such as Newton's law to produce a computational model for understanding biological complex processes for instance. Let $x \in \mathbb{R}^n$ be the state variable, the evolution of a system whose behavior is determined by its state only is expressed as:

$$\dot{x}(t) = f(x)$$

If an initial value is known x_0 which is equivalent to knowing the state at an initial time t_0: $x_0 = x(t_0)$, the future behavior of the system is completely determined by solving the corresponding Cauchy problem: $\dot{x}(t) = f(x)$, $x(t_0) = x_0$. The motion of the planets in the solar system is a solution to a Cauchy problem, it is a typical example of evolution that cannot be altered. However, as will be seen later, when analyzing the motion of a spacecraft an external force can be applied using thrusters. This falls into the category of control theory where we can actively influence the evolution of the system. A new parameter, called the control, appears in the differential equation:

$$\dot{x}(t) = f(x(t), u(t))$$

where f is a C^1-mapping, $u(.)$ belongs to an admissible family of functions whose regularity varies depending on the application and that we assume here to be the set of bounded measurable functions and takes its value in a subset of $\mathbb{R}^m$. For control systems, the rate of change $\dot{x}(t)$ and therefore the evolution of the system depends not only on the state itself but also on the control u that varies in time. The control is chosen depending on the application, the goal might be to steer the system from one state to another, to maximize the terminal value of one of the parameters, or to minimize a certain cost function for instance.

1.2 Controllability: [12]

Is it possible to bring the state variable from any initial condition to any final condition in a finite time?

1.2.1 linear systems:

$$\begin{cases} x^{\cdot}(t) = A(t)x(t) + B(t)u(t) + r(t) \\ x(t_0) = x_0 \end{cases}$$

where $x(t)\ \mathbb{R}^n, u(t)\mathbb{R}^n$, $A(t) \in M_n(\mathbb{R})$, $B(t) \in M_{n,m}(\mathbb{R})$ and $r(t) \in M_{n,1}(\mathbb{R})$ for all $t \in [t_0, t_f]$.

→ Kalman condition

1.2.2 nonlinear systems:[7]

→ much more complex

→ Poincare's Recurrence theorem, Poisson-stability, linearization, local controllability.

Problem 1:

We assume to have a cart moving horizontally in a frictionless environment. For simplicity, assume that it has unit mass $m = 1$: We denote its initial position by $x(0)$ and $v(0)$ is its initial velocity. If no extremal forces are applied, the evolution of the cart is simply given by $x(t) = x(0) + v(0)_t$ where t represents the time. If now we assume there is an extremal force pushing on the cart, denoted by the control function $u(.)$, the system becomes $x^{\cdot\cdot}(t) = u(t)$ which is equivalent to the first-order system:

$$x^{\cdot}(t) = v(t), \qquad v^{\cdot}(t) = u(t)$$

That can be integrated to obtain

$$x(t) = x(0) + v(0)t + \int_0^t (t-s)u(s)ds$$

$$v(t) = v(0) + \int_0^t u(s)ds$$

Assume that the force satisfies the constraint $|u(t)| \leq 1$ and consider the problem of steering the system to the origin with zero speed. Assuming $(x(0), v(0)) = (2,2)$, this goal is achieved by open-loop control.

$$u(t) = \begin{cases} -1 \quad if \quad 0 \leq t < 4 \\ 1 \quad \text{if } 4 \leq t < 6 \\ 0 \quad if \quad 6 \leq t, \end{cases}$$

Note that because of the uniqueness of solutions, this control would not accomplish the same task in connection with any other initial data different from (2, 2).

1.3 Stabilization:[17]

How can we make a control system insensitive to perturbations?

Problem 2:

if (xe, ue) is an equilibrium point of the autonomous control system

$$x^{\cdot}(t) = f(x(t), u(t))$$

i.e

$$f(xe, ue) = 0.$$

Does exist a control u such that, for all € ≥ 0, there exists μ ≥ 0 such that, for all $x_0 \in B(x_e, \eta)$ and $t \geq 0$ all solution to the system

$$\begin{cases} x^{\cdot}(t) = f(t, x(t), u(t)) \\ x(t_0) = x(0) \end{cases}$$

$$satisfies\ ||x(t) - xe|| \leq \epsilon?$$

Linear system controllability

Nonlinear systems Lyapunov functions

1.4 Optimal Control [16]:

Given a control system, can we determine the optimal solutions for a given optimization criterion? When the initial and final states are {fixed, it is equivalent to finding the solution to the following boundary value problem:

$$\begin{cases} x^{\cdot}(t) = f(t, x(t), u(t)) \\ x(t_0) = x_0 \in M_0, x(t_f) = x_f \in M_1 \end{cases}$$

Which minimizes the cost?

$$min_{u(.)\in U} = \int_{t_0}^{t_f} f_0\big(t, x(t), u(t)\big)dt + g(t_f, x_f)$$

Where $f_0 : R \times M \times U \to R$ is smooth and $g : R \times M \to R$ is continuous.

- $\int_{t_0}^{t_f} f_0\big(t, x(t), u(t)\big)dt$: Lagrange cost.
- $g(t_f, x_f)$: Mayer cost.

In 1696, Johan Bernoulli challenged his colleagues with the following problem: Consider two points A and B such that A is above B. Assume that an object is located the point A with no initial velocity and is only subjected to gravity. What is the curve between A and B so that the object travels from A to B in minimal time? We know that the straight line is the shortest way between two points. Is it the fastest way? NO! The fastest way is a cycloid arc.

Optimal control theory is at the crossroad of:

1- Theory of Differential Equations/Dynamical Systems (Finding solutions to differential systems, the problem of existence and unicity of optimal solutions).
2- Differential geometry (optimal synthesis strongly depends on the geometric properties of the problem, the modern theory of optimal control).
3- Optimization.
4- Modeling (relevance of the way that an optimal control problem is set up).
5- Numerical Analysis (numerical methods to approximate optimal solutions).
6- Applications (solving real-world problems).

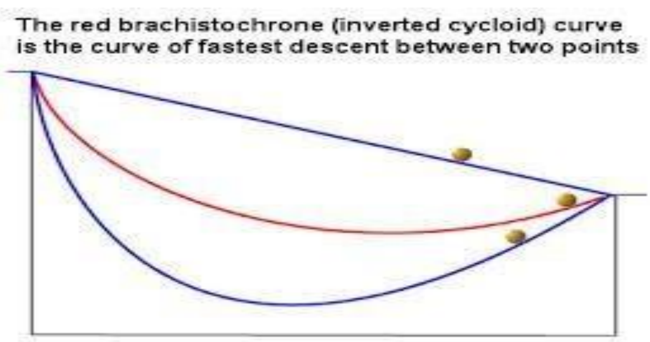

Figure 1: Cycloid is the solution to the Brachistochrone problem.

Example of Applications: The controlled restricted 3-Body problem. [8]

The motion of the spacecraft in the Earth-Moon system is not fully determined by the motion of the two planets. Indeed, the thrust provided by the engines of the spacecraft can be used to alter the acceleration of the spacecraft. In other words, it is the control that we have on spacecraft dynamics. The influence of this control must be added to the equations of motion? Thus, we get the equations of the controlled restricted 3-body problem.

$$\begin{cases} \ddot{x} - 2\dot{y} - x = \dfrac{\dot{\partial} v}{\partial x} + u_1 \\ \ddot{y} - 2\dot{x} - x = \dfrac{\dot{\partial} v}{\partial y} + u_2 \end{cases}$$

where $u = (u_1;\ u_2)$ is the control term which represents the impact of the engine thrust on the spacecraft motion? We introduce the coordinates $q = (x, y, x^{\cdot}, y^{\cdot})$. As a result, the controlled equations of motion are written as the bi-input control system

$$\dot{q} = f_0(q) + f_1(q)u_1 + f_2(q)u_2$$

Where

$$f_0(0) = \begin{cases} q_3 \\ q_4 \\ 2q_4 + q_1 - (1-\mu)\dfrac{q_1 + \mu}{((q_1+\mu)^2 + q_2{}^2)^{\frac{3}{2}}} - \mu - \dfrac{q_1 - 1 + \mu}{((q_1 - 1 + \mu)^2 q_2{}^2)^{\frac{3}{2}}} \\ -q_3 + q_2 - (1-\mu)\dfrac{q_2}{((q_1+\mu)^2 + q_2{}^2)^{\frac{3}{2}}} - \mu\dfrac{q_2}{((q_1 - 1 + \mu)^2 q_2{}^2)^{\frac{3}{2}}} \\ F_1(q) = \dfrac{\partial}{\partial q_3}, F_2(q)\dfrac{\partial}{\partial q_4} \end{cases}$$

Time-minimal problem

Our objective is to minimize the transfer time between the geostationary orbit OG and a circular parking orbit OL around the Moon when low thrust is applied. This problem consists of solving the optimal control problem

$$\begin{cases} \dot{q} = F_0(q) + \in (F_1(q)u_1 + (F_2(q)u_2, \in > 0 \\ min_{u(0) \in B_{R^2}(0,1)} \int_{t_0}^{t_f} dt \\ q(0) \in O_G, q(t_f) \in O_{L,} \end{cases}$$

In other words, we want to find the solutions to the controlled equations of motion of the spacecraft, starting from the geostationary orbit and ending at the circular parking orbit around the Moon, which minimize the transfer time $\int_{t_0}^{t_f} dt$. The control bound ϵ represents the maximum thrust allowed, which can be arbitrarily set to any sufficiently small value. Unfortunately, this optimal control problem cannot be solved analytically. It is highly non-linear and has singularities. However, necessary optimality conditions and numerical calculations can produce simulations of locally optimal solutions. The methodology is the following:

1- Apply Pontryagin's Maximum Principle which provides necessary optimality conditions.
2- Turn the Pseudo-Hamiltonian system deduced from the Application of Pontryagin's Maximum Principle into a true Hamiltonian system. This can be done by using the Implicit function Theorem.
3- Use the shooting method to compute candidate solutions to the problem.
4- Check the local optimality of these extremals by using the second-order optimality condition. To do so, we need to compute the first conjugate time along each extremal and verify that it is greater than the transfer time

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