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CONCEPT OF INVERSE OF A SINGULAR MATRIX

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ABSTRACT – In the chapter on Matrices, we have studied that a matrix is invertible if and only if it is a non-singular matrix. That is, matrices having determinant value as zero are non-invertible. In this paper, I will explain the concept that inverse of a matrix depends upon the identity element and using it I will show that we can obtain inverse of a singular matrix also.

KEYWORDS – Identity element, singular and non-singular matrices, invertible matrix Binary operation.

1. INTRODUCTION

The theory which characterizes invertible matrices as non-singular matrices was developed using $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as the identity element.

It is important to learn that the definition of inverse depends upon the identity element. If we change the identity element then the definition of inverse also changes. So one must take a caution while computing inverse of a matrix.

2. ***DEFINITION – 1 Binary operation:*** An operation say * in the set M is said to be a binary operation if for a, b ∈ **M**. We have a * b ∈ **M** (always).

FOR EXAMPLE: (a) On the set of natural numbers (N), '+' is a binary operation as for a, b ∈ **N**. we have a + b ∈ **N**.

(b) But '–' is not a binary operation on **N** as for a, b ∈ **N**, a – b ∉ **N** (always).

DEFINITION – 2 Identity element: An element say 'e' in the set **M** w.r.t. a binary operation say * is said to be the identity element in **M** if:

(i) e ∈ M (ii) a * e = a = e * a for all a ∈ M.

Example 1: (a) On the set of Natural numbers (N) w.r.t multiplication as a binary operation, 1 is the identity element as 1 ∈ N and 1 . a = a = a . 1 for all a ∈ **N**.

(b) **Note:** On the set of Natural numbers (**N**) w.r.t addition as a binary operation, '0' is not the identity element as 0 ∉ **N**.

(although a + 0 = a = a + a for all a ∈ N)

DEFINITION – 2 Invertible element: An element a in the set M is s.t.b. invertible w.r.t. a binary operation * if there exist an other element b ∈ M such that a * b = e = b * a.

Example 2: (a) On the set of integer (**Z**) w.r.t. addition as a binary operation for any a ∈ **Z**, we have – a ∈ **Z** such that a + (–a) = 0 = (–a) + a.

(b) Note: On the set of integers (**Z**) w.r.t multiplication, for any $0 \neq a \in Z, \frac{1}{a} \notin Z$, (always). So we observe that in **Z** w.r.t multiplication only 1 and -1 are inventible elements.

Example 3: Let $M = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} : a \neq 0, a \in R \right\}$ and * be the operation on M defined as usual matrix multiplication.

Then we shall show that:

(i) * is a binary operation on M.
(ii) Identity element exists in M w.r.t. *
(iii) Each element in M is invertible w.r.t. *

Note: Any element in M is a 2 × 2 singular matrix.

Solution (i) Let $\begin{bmatrix} a & a \\ a & a \end{bmatrix}$ and $\begin{bmatrix} b & b \\ b & b \end{bmatrix}$ be any two elements in M, then we have $\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} 2ab & 2ab \\ 2ab & 2ab \end{bmatrix} \in M$

(∵ 2ab ≠ 0, 2ab ∈ R)

∴ * is a binary operation on M.

(ii) Let $\begin{bmatrix} e & e \\ e & e \end{bmatrix}$ be the required identity element in M, then $\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} e & e \\ e & e \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$

⇨ $\begin{bmatrix} 2ae & 2ae \\ 2ae & 2ae \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$

⇨ 2ae = a

⇨ $e = \frac{1}{2} \quad (\because a \neq 0)$

∴ $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is the identity element in M.

Note: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ also satisfies the equation

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

But since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin M$, so we can't take it as the identity element.

Hence, we have consider $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ as the identity element because $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \in M$.

(iii) Let $\begin{bmatrix} a & a \\ a & a \end{bmatrix} \in M$ be any arbitrary element. Suppose $\begin{bmatrix} b & b \\ b & b \end{bmatrix} (b \neq 0)$ be its inverse in M.

$$\therefore \begin{bmatrix} a & a \\ a & a \end{bmatrix}\begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2ab & 2ab \\ 2ab & 2ab \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow 2ab = \frac{1}{2}$$

$$\Rightarrow b = \frac{1}{4a}$$

$\therefore \begin{bmatrix} 1/4a & 1/4a \\ 1/4a & 1/4a \end{bmatrix} \in M$ is the inverse of $\begin{bmatrix} a & a \\ a & a \end{bmatrix}$ the singular. So, we have shown that the singular matrix $\begin{bmatrix} a & a \\ a & a \end{bmatrix} (a \neq 0, a \in R)$ is invertible.

Example: (4) Let $M = \left\{ \begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix} : 0 \neq a, a \in R \right\}$ and suppose * is defined as usual matrix multiplication on M.

Then, we shall show that:

(i) * is a binary operation on M.

(ii) Identity element exists in M w.r.t. *.

(iii) Each element in M is invertible w.r.t. *.

Solution (i) Let $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} b & b \\ 0 & 0 \end{bmatrix}$ are any two elements in M.

Since $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ab & ab \\ 0 & 0 \end{bmatrix} \in M$

($\because$ $ab \neq 0$)

(ii) Let $\begin{bmatrix} e_1 & e_1 \\ 0 & 0 \end{bmatrix} \in M$ then we have

$$\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e_1 & e_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} ae_1 & ae_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad ae_1 = a \Rightarrow e_1 = 1$

$\therefore \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is the identity element in M.

Note: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ also satisfies $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$.

But we can't take $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as the identity element as $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin M$.

Example: (5) Let $M = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a \neq 0,\ b \neq 0,\ a, b \in R \right\}$ and suppose * is defined as usual matrix multiplication on M.

Then, we observe that:

(i) * is a binary operation on M.

(ii) Identity element does not exists in M w.r.t. *.

Solution (i) Since $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ac & ad \\ 0 & 0 \end{bmatrix}$ (which belongs to M).

So * is a binary operation on M

(ii) Let $\begin{bmatrix} e_1 & e_2 \\ 0 & 0 \end{bmatrix} \in M$. Then we have $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e_1 & e_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} ae_1 & ae_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \quad e_1 = 1,\ e_2 = \frac{b}{a}$

$\therefore$ We have $\begin{bmatrix} e_1 & e_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & b/a \\ 0 & 0 \end{bmatrix}$.

But $\begin{bmatrix} e_1 & e_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} e_1a & e_1b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \Rightarrow e_1 = 1$

$\therefore \quad \begin{bmatrix} e_1 & e_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & e_2 \\ 0 & 0 \end{bmatrix}$.

Here e_2 can take any value so uniqueness of identity gets disturbed.

So, identity element does not exists in M w.r.t. *.

Example: (6) Let $M = \left\{ \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix} : a \neq 0 \right\}$ and * be an operation on M defined as usual matrix multiplication.

Then, we shall show that:

(i) * is a binary operation on M.

(ii) Identity element exists in M w.r.t. *.

(iii) Each element is invertible in M w.r.t. *.

Solution (i) Left to the readers.

(ii) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in \mathrm{M}$ is the identity elements (verify).

(iii) For ay element $\begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix}$ in M, $\begin{bmatrix} 1/a & 0 \\ 1/a & 0 \end{bmatrix} \in \mathrm{M}$ be its inverse (verify).

Food For Thought

Let $\mathrm{M} = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} : a \neq 0, b \neq 0, \ a, b \in \mathrm{R} \right\}$ and * is an operation on M defined as usual matrix multiplication.

Then, think about these questions:

(i) Is * is a binary operation on M?

(ii) Is identity element exists in M w.r.t. * ?

(iii) Is each element in M is invertible w.r.t. * ?

CONCLUSION: It is important to learn that the definition of inverse of a matrix depends upon the identity element. If we change the identity element then the definition of inverse also changes. So, one must take a caution while computing inverse of a matrix.

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REFERENCES

Binary operation [1]

Invertible matrices [2]

Singular and non singular matrices [3], [4]

- Textbook titled “Binary Arithmetic and Boolean Algebra” by Angelo C. Gillie. [1]
- Textbook titled “Abstract Algebra” by Dr. S. K. Bhambri and DR. V. K. Khanna. [2]
- Textbook titled “Linear Algebra and its Applications” by David C lay. [3]
- Research paper titled “Invertible Matrix” by S. Jose Published by “Journal of Applied Mathematics and its Applications”. [4]