GRAPH EQUATIONS FOR LINE GRAPHS, MIDDLE GRAPHS, SEMI-TOTAL CLOSED NEIGHBORHOOD BLOCK GRAPHS AND SEMI-TOTAL CLOSED EDGE NEIGHBORHOOD BLOCK GRAPH

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ABSTRACT: Graph equations are those in which the unknowns are graphs. During the past fifty years, the development in these areas has emerged as one of the significant branches in Graph Theory. Let $L(G)$ and $M(G)$ denote the line graph and middle graph respectively. In this paper, we solve the graph equations:

$$L(G) \cong \operatorname{BN}_n(H), \quad \bar{L}(G) \cong \operatorname{BN}_n(H), \quad M(G) \cong \operatorname{BN}_n(H), \quad \bar{M}(G) \cong \operatorname{BN}_n(H),$$

The symbol $\cong$ stands for isomorphism between two graphs.

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I. INTRODUCTION

Kulli and Patil [7] solved the following equations:

$$L(G) \cong e(H), \quad M(G) \cong e(H),$$

$$\bar{L}(G) \cong e(H), \quad \bar{M}(G) \cong e(H)$$

Basavangoud and Mathad [2] solved graph equations:

$$L(G) \cong B_m(H), \quad M(G) \cong B_m(H),$$

$$\bar{L}(G) \cong B_m(H), \quad \bar{M}(G) \cong B_m(H),$$

$$J(G) \cong B_m(H), \quad \bar{J}(G) \cong B_m(H).$$

Also Basavanagoud and Mathad (see [3]) obtained solutions for graph equations:

$$L(G) \cong m(H), \quad M(G) \cong m(H),$$

$$\bar{L}(G) \cong n(H), \quad \bar{M}(G) \cong n(H),$$

$$J(G) \cong n(H), \quad \bar{J}(G) \cong n(H).$$
\[
J(G) \equiv m(H), \quad \overline{M(G)} \equiv m(H).
\]

Cvetković and Simić [5] solved graph equations \(L(G) \equiv T(H)\), \(\overline{L(G)} \equiv T(H)\). Akiyama et.al. [1] solved graph equations \(L(G) \equiv M(H); M(G) \equiv T(H)\) : \(\overline{M(G)} \equiv T(H)\) and \(\overline{L(G)} \equiv M(H)\).

In this chapter, we solve the following graph equations:

\[
\begin{align*}
L(G) & \equiv \text{BN}_{nc}(H) \\
\overline{L(G)} & \equiv \text{BN}_{nc}(H) \\
M(G) & \equiv \text{BN}_{nc}(H) \\
\overline{M(G)} & \equiv \text{BN}_{nc}(H) \\
L(G) & \equiv \text{BEN}_{nc}(H) \\
\overline{L(G)} & \equiv \text{BEN}_{nc}(H) \\
M(G) & \equiv \text{BEN}_{nc}(H) \\
\overline{M(G)} & \equiv \text{BEN}_{nc}(H).
\end{align*}
\]

That is, we shall find all pairs \((G, H)\) of graphs satisfying equations. Both \(G\) and \(H\) are called solution of graphs.

For a graph \(G\), let \(V(G)\), \(E(G)\) and \(b(G)\) denote its vertex set, edge set and the set of blocks of \(G\) respectively.

Hamada and Yoshimura [6] defined a graph \(M(G)\) as an intersection graph \(\Omega(F)\) on the vertex set \(V(G)\) of any graph \(G\). Let \(E(G)\) be the edge set of \(G\) and \(F = \bigvee V(G) \cup E(G)\) where \(\bigvee V(G)\) indicates the family of one-point subsets of the set \(V(G)\). Let \(M(G) \equiv \Omega(F)\). \(M(G)\) is called the middle graph of \(G\).

The open-neighborhood \(N(u)\) of a vertex \(u\) in \(V(G)\) is the set of all vertices adjacent to \(u\).

\[
N(u) = \{ v/ uv \in E(G) \}
\]

The closed-neighborhood \(N[u]\) of a vertex \(u\) in \(V(G)\) is given by

\[
N[u] = \{ u \} \cup N(u)
\]

For each vertex \(u_i\) of \(G\), a new vertex \(u_i'\) is taken and the resulting set of vertices is denoted by \(V_i(G)\).

For a graph \(G\), we define the semi-total closed neighborhood block graph \(\text{BN}_{nc}(G)\) of a graph \(G\) is defined as the graph having vertex set \(V(G) \cup V_1(G) \cup b(G)\) with two vertices are adjacent if they correspond to adjacent vertices of \(G\) or one corresponds to a vertex \(u_i'\) of \(V_1(G)\) and the other to a vertex \(v_i\) or \(w_i\) in \(N[u_i]\) or one corresponds to a vertex \(u_i\) of \(V(G)\) and the other to a vertex \(b_i\) of \(b(G)\) and \(u_i\) lies on \(b_i\).

The open-neighborhood \(N(e_i)\) of an edge \(e_i\) in \(E(G)\) is the set of edges adjacent to \(e_i\).

\[
N(e_i) = \{ e_j / e_i \text{ and } e_j \text{ are adjacent in } G \}
\]

The closed-neighborhood \(N[e_i]\) of an edge \(e_i\) in \(E(G)\) is given by

\[
N[e_i] = \{ e_i \} \cup N(e_i)
\]

For each edge \(e_i\) of \(G\), a new vertex \(e_i'\) is taken and the resulting set of vertices is denoted by \(E_i(G)\).

For a graph \(G\), we define the semi-total closed edge neighborhood block graph \(\text{BEN}_{nc}(G)\) of a graph \(G\) as the graph having vertex set \(E(G) \cup E_1(G) \cup b(G)\) with two vertices are adjacent if they correspond to adjacent edges of \(G\) or one corresponds to an element \(e_i'\) of \(E_1(G)\) and the other to an element \(e_j\) of \(E(G)\) where \(e_j\) is in \(N[e_i]\) and the other to a vertex \(b_i\) of \(b(G)\) and \(e_i\) lies on \(b_i\).

In Figure 1, a graph \(G\) and its \(\text{BN}_{nc}(G)\) and \(\text{BEN}_{nc}(G)\) are shown.

Beineke has shown in [4] that a graph \(G\) is a line graph if and only if \(G\) has none of the nine specified graphs \(F_i, i = 1, 2, \ldots, 9\) as an induced subgraph as stated in the Theorem 6.A and \(\overline{F_i}\) is the complement of \(F_i\) as in Theorem 6.B. We depict here some of the nine graphs which are useful to extract our later results. These are \(F_1 = K_{1,3}\), \(F_2\) and \(F_3 = K_{5} - x\), where \(x\) is any edge of \(K_{5}\).

Before investigating the solution of the above graph equations, we construct a class of graphs which are useful in our later discussion.

Let \(K_{1,n}, n \geq 4\) be a star with endedges \([u_{1}, u_{2}, \ldots, u_{n}]\). Take \(1 \leq m \leq n\) copies of the star \(K_{1,2}\) with endedges \([v_{1}w_{1}, v_{1}w_{2}], [v_{2}w_{2}, v_{2}w_{4}], \ldots, [v_{m}w_{2m-1}, v_{m}w_{2m}]\). Consider the graph \(K_{1,n} \cup mK_{1,2}\) and denote this by \(K_{n,2}^{0}\). Next identify \(u_1\) with \(w_1\) and \(u_2\)
with \( w_2 \). The resulting graph is denoted by \( K_{n,2}^1 \). Further identifying \( u_1 \) with \( w_3 \) and \( u_4 \) with \( w_4 \), we obtain the new graph which is denoted by \( K_{n,2}^2 \) and so on. In this way, we obtain the graph \( K_{n,2}^m \).

\[
G:
\]

\[
\begin{align*}
1 & \quad b_1 \\
2 & \quad e_1 \\
3 & \quad e_2 \\
4 & \quad b_2 \\
& \quad e_4
\end{align*}
\]

\[
\begin{align*}
& \quad e_3
\end{align*}
\]

Further identifying \( u_3 \) with \( w_3 \) and \( u_4 \) with \( w_4 \), we obtain the new graph which is denoted by \( K_{n,2}^2 \) and so on. In this way, we obtain the graph \( K_{n,2}^m \).

\[
\begin{align*}
& \quad b_1 \\
& \quad e_1 \\
& \quad e_2 \\
& \quad b_2 \\
& \quad e_4
\end{align*}
\]

\[
\begin{align*}
& \quad e_3
\end{align*}
\]

A graph \( G^* \) is the endedge graph of a graph \( G \) if \( G^* \) is obtained from \( G \) by adjoining an endedge \( u_i u'_i \) at each vertex \( u_i \) of \( G \) (see [1]). Hamada and Yoshimura have proved in [6] that \( M(G) \cong L(G^*) \).

2. THE SOLUTION OF \( L(G) \cong BN_{tc}(H) \)

In this case, we observe that if \( H \) has an edge then \( F_1 \) is an induced subgraph of \( BN_{tc}(H) \). Hence \( H \) is \( nK_1, n \geq 1 \). The corresponding \( G \) is \( nK_{1,2} \).

Thus we obtain the following result.

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Figure 1.
THEOREM 2.1. The following pairs (G, H) are all pairs of graphs satisfying the graph equation \( L(G) \cong BN_e(H) \):

\[(nK_{1,2}, nK_1), n \geq 1\]

3. THE SOLUTION OF \( \overline{L(G)} \cong BN_e(H) \)

In this case H satisfies the following properties:

i) If H has at least one edge, then it is connected, since otherwise, \( F_1 \) is an induced subgraph of \( BN_e(H) \).

ii) H does not contain cut-vertices, since otherwise, \( F_2 \) is an induced subgraph of \( BN_e(H) \).

iii) H does not contain \( C_n, n \geq 5 \) as an induced subgraph, since otherwise, \( F_3 \) is an induced subgraph of \( BN_e(H) \).

iv) H does not contain a vertex which is not adjacent to two mutually adjacent vertices, since otherwise, \( F_4 \) is an induced subgraph of \( BN_e(H) \).

v) H is not complete bipartite graph \( K_{m,n} \), m \geq 4 or n \geq 4, since otherwise, \( F_5 \) is an induced subgraph of \( BN_e(H) \).

We consider the following cases:

Case 1. Assume H is disconnected. Then from observation (i), it follows that H is totally disconnected graph.

For \( n = 1 \), \( H = K_1 \) and G = 2K.

For n = 2, H = 2K and G = C_4.

For n = 3, H = 3K and G = K_4.

Case 2. Assume H is connected. We consider the following subcases:

Subcase 2.1. \( H = K_n \). It follows that \( (K_{1,n+1} \cup nK_2, K_n) \), n \geq 2 and \( (K_3 \cup 2K_2, K_2) \) are the solutions.

Subcase 2.2. \( H = K_{m,n} \). From observation (v), it follows that H is \( K_{2,2} \) or \( K_{2,3} \) or \( K_{3,3} \).

For \( H = K_{2,2} \), G is the graph shown in Fig. 2 (a).

For \( H = K_{2,3} \), G is the graph shown in Fig. 2 (b).

For \( H = K_{3,3} \), G is the graph shown in Fig. 2 (c).

Subcase 2.3. H is neither a complete graph nor a complete bipartite graph. It follows from observations (ii), (iii), and (iv) that H is a block not containing a vertex which is not adjacent to two mutually adjacent vertices. Therefore, in this case, H = \( K_n = \{e_1, e_2, \ldots, e_m\} \), where 1 \leq m < n. The corresponding G is the graph \( K_{n+1,2} \cup (n-2m)K_2 \).

Hence the graph equation 3 is solved and we obtain the following:

THEOREM 3.1. The following pairs (G, H) are all pairs of graphs satisfying the graph equation \( \overline{L(G)} \cong BN_e(H) \):

\[(2K_2, K_1); (C_4, 2K_1); (K_1, 3K_2); (K_{1,n+1} \cup nK_2, K_n), n \geq 2; \]

\[(K_{m+1,2} \cup (n-2m)K_2, K_n - \{e_1, e_2, \ldots, e_m\}), 1 \leq m < n; (G_1, K_{3,2});\]

\[(G_2, K_{2,3}) \text{ and } (G_3, K_{3,3}), \text{ where } G_1, G_2 \text{ and } G_3 \text{ are the graphs shown in Fig. 2(a), 2(b) and 2(c) respectively}.\]
4. THE SOLUTION OF \( M(G) \cong BN_\ell(H) \)

Theorem 2.1 gives solutions of the graph equation \( L(G) \cong BN_\ell(H) \). But there is no solution which is of the form \( (G^*, H) \). Hence, there is no solution of the graph equation \( M(G) \cong BN_\ell(H) \).

Thus, we have the following result.

**THEOREM 4.1.** There is no solution for the graph equation \( M(G) \cong BN_\ell(H) \).

5. THE SOLUTION OF \( \overline{M(G)} \cong BN_\ell(H) \)

Theorem 3.1 gives the solution of the graph equation \( \overline{L(G)} \cong BN_\ell(H) \). Among these solutions, \( (2K_2, K_1) \) is of the form \( (G^*, H) \). Therefore the solution of \( \overline{M(G)} \cong BN_\ell(H) \) is \( (2K_1, K_1) \).

Thus, we have the following result.

**THEOREM 5.1** There is only solution \( (2K_1, K_1) \) of the graph equation \( \overline{M(G)} \cong BN_\ell(H) \).

6. THE SOLUTION OF \( L(G) \cong BEN_\ell(H) \)

In this case, we observe that if \( H \) has adjacent edges then \( F_1 \) is an induced subgraph of \( BEN_\ell(H) \). Therefore, \( (nK_2^+, nK_2) \), \( n \geq 1 \) is the solution. Hence we have the following result.

**THEOREM 6.1.** The following pairs \((G, H)\) are all pairs of graphs satisfying the graph equation \( L(G) \cong BEN_\ell(H) \):

\[
(nK_2^+, nK_2), n \geq 1.
\]

7. THE SOLUTION OF \( \overline{L(G)} \cong BEN_\ell(H) \)

Any graph \( H \) which is a solution of the above equation satisfies the following properties:

i) If \( H \) has at least one edge, then it is connected, since otherwise, \( \overline{F_2} \) is an induced subgraph of \( BEN_\ell(H) \).

ii) \( H \) does not contain more than one cut-vertex, since otherwise, \( \overline{F_1} \) is an induced subgraph of \( BEN_\ell(H) \).

iii) \( H \) does not contain cut-vertex which lies on more than two blocks, since otherwise, \( \overline{F_1} \) is an induced subgraph of \( BEN_\ell(H) \).

iv) \( H \) does not contain cut-vertex which lies on blocks other than \( K_2 \), since otherwise, \( \overline{F_1} \) is an induced subgraph of \( BEN_\ell(H) \).

v) \( H \) does not contain \( C_n \), \( n \geq 5 \), as a subgraph, since otherwise, \( \overline{F_2} \) is an induced subgraph of \( BEN_\ell(H) \).

vi) \( H \) does not contain an edge which is not adjacent to two mutually adjacent edges, since otherwise, \( \overline{F_2} \) is an induced subgraph of \( BEN_\ell(H) \).

vii) \( H \) is not a complete graph \( K_n \), \( n \geq 5 \), since otherwise, \( \overline{F_2} \) is an induced subgraph of \( BEN_\ell(H) \).

viii) \( H \) is not a complete bipartite graph \( K_{m,n} \), \( m \geq 3 \) or \( n \geq 3 \), since otherwise, \( \overline{F_2} \) is an induced subgraph of \( BEN_\ell(H) \).

It follows from observation (ii) that \( H \) contains at least one cut vertex. We consider the following cases:

**Case 1.** If \( H \) has the cut-vertex, then from observations (iii) and (iv), \( H = K_{1,2} \). The corresponding \( G \) is the graph \( K_{1,2}^* \) together with an endedge adjacent to a vertex of maximum degree.

**Case 2.** If \( H \) has no cut-vertices, then we consider the following subcases:

**Subcase 2.1.** \( H = K_n \). In this case, it follows from observation (vii) that \( (K_{1,2} \cup K_2, K_1) \), \( (K_{1,4} \cup 3K_2, K_3) \) and \( (K_{3,2,7,2}, K_4) \) are the solutions.

**Subcase 2.2.** \( H = K_{m,n} \). Then from observation (viii), it follows that \( (K_{1,2}^+, K_{2,2}) \) and \( (G', K_{1,2}) \) where \( G' \) is the graph \( K_{1,2}^* \) together with an endedge adjacent to a vertex of maximum degree are the solutions.

**Subcase 2.3.** \( H \) is neither a complete graph nor a complete bipartite graph. It follows from observation (vi) that \( H \) is a graph which does not contain an edge which is not adjacent to two mutually adjacent edges. In this case \( (K_{6,2}^2, K_4 - x) \), where \( x \) is any edge of \( K_4 \) and \( (K_{5,2}^2, C_4) \) are the solutions.
Thus, we have the following result.

**THEOREM 7.1.** The following pairs \((G, H)\) are all pairs of graphs satisfying the graph equation \(\overline{L}(G) \equiv \text{BEN}_n(H)\):

\((G', K_{1,2})\), where \(G'\) is the graph \(K^{1,2}_{1,2}\) together with an endedge adjoined to a vertex of maximum degree; \((K_{1,2} \cup K_2, K_2)\);

\((K_{1,4} \cup 3K_2, K_4)\); \((K^3_{7,2}, K_4)\); \((K^2_{6,2}, K_4 - x)\), where \(x\) is any edge of \(K_4\); and \((K^2_{5,2}, C_4)\).

8. **THE SOLUTION OF** \(M(G) \equiv \text{BEN}_n(H)\)

Theorem 6.1 gives solution of the equation \(L(G) \equiv \text{BEN}_n(H)\) as \((nK^2_2, nK_2\)), \(n \geq 1\). This is in the form \((G^*, H)\). Therefore, the solution of \(M(G) \equiv \text{BEN}_n(H)\) is given by \((nK_2, nK_2\)), \(n \geq 1\). Thus we have the following theorem.

**THEOREM 8.1.** The following pairs \((G, H)\) are all pairs of graphs satisfying the graph equation \(M(G) \equiv \text{BEN}_n(H)\):

\((nK_2, nK_2\)), \(n \geq 1\).

9. **THE SOLUTION OF** \(\overline{M}(G) \equiv \text{BEN}_n(H)\)

Theorem 7.1 gives the solution of the graph equation \(\overline{L}(G) \equiv \text{BEN}_n(H)\). None of these solutions is of the form \((G^*, H)\). Therefore, there is no solution of the equation \(\overline{M}(G) \equiv \text{BEN}_n(H)\).

Thus we obtain the following result.

**THEOREM 9.1.** There is no solution of the graph equation \(\overline{M}(G) \equiv \text{BEN}_n(H)\).

**REFERENCES**

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