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Structural Characterizations of Bi-Ideals in Ternary Semirings: Interplay with One-Sided and Quasi-Ideal Structures

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ABSTRACT

Bi-ideals constitute an important class of ideal-like structures that extend the study of one-sided and quasi-ideals in algebraic systems. In this work, we investigate the structure and characterizations of bi-ideals in ternary semirings. Beginning with the fundamental notions of sub ternary semirings, left, lateral and right ideals, regular elements, and quasi-ideals, we examine the absorption condition that determines a bi-ideal. The study establishes several structural properties of bi-ideals in ternary semirings. In particular, it is shown that every one-sided ideal and every quasi-ideal is a bi-ideal, while suitable intersections of left, lateral and right ideals also produce bi-ideals. The stability of bi-ideals under arbitrary intersections, intersections with sub ternary semirings, and suitable ternary products is established. Further, for a bi-ideal (B) and a sub ternary semiring (T), the sets (BTT) and (TTB) are shown to retain the bi-ideal structure. The study also gives structural criteria for bi-ideals arising from products (ABC) of sub ternary semirings when one of the factors is a left, lateral, or right ideal. A central characterization demonstrates that a non-empty subset (B) is a bi-ideal if and only if it can be represented as a left ideal of an appropriate right ideal, equivalently as a right ideal of an appropriate left ideal. Finally, a necessary and sufficient characterization is obtained in terms of a left ideal, a lateral ideal, and a right ideal satisfying (RILBRIL). These results provide a unified structural view of bi-ideals and clarify their relationship with the various one-sided and quasi-ideal structures in ternary semirings.

Keywords

Ternary semiring; bi-ideal; quasi-ideal; left ideal; lateral ideal; right ideal; sub ternary semiring.

1.INTRODUCTION

The theory of ideals plays a fundamental role in the structural investigation of algebraic systems. In semigroups, rings, semirings and their generalizations, different forms of ideals provide effective tools for describing internal algebraic behaviour. In particular, left ideals, lateral ideals and right ideals capture the action of the underlying algebraic structure from different positions. Quasi-ideals provide a further generalization, while bi-ideals offer another important class of ideal-like subsets characterized by a suitable absorption condition.

Another important structure considered is the quasi-ideal. A non-empty subset (Q) of a ternary semiring (S) is called a quasi-ideal when it is a sub ternary semigroup under addition and satisfies the corresponding intersection condition [(SSQ)∩(SQS)∩(QSS)⊆Q] Thus, the quasi-ideal condition combines the three possible positions occupied by the subset within a ternary product. The document records that every quasi-ideal of a ternary semiring is a sub ternary semiring. An illustrative example is provided by the ternary semiring of natural numbers, where (3N) is a quasi-ideal.

The principal object of investigation is the **bi-ideal**. A non-empty subset (B) of a ternary semiring (S) is considered a bi-ideal when it is a sub ternary semiring and satisfies the corresponding ternary absorption condition. In the notation used throughout the document, this condition has the form [BSBSBB.] This condition expresses the ability of (B) to absorb alternating products involving elements of (S) while retaining the resulting elements inside (B). Every bi-ideal is therefore a sub ternary semiring, although the converse need not hold.

Thus, the purpose of the present study is to organize and establish the structural behaviour of bi-ideals in ternary semirings, emphasizing their connections with quasi-ideals and one-sided ideals.

2.PRELIMINARIES

2.1 Definition: A non-empty subset T of S is a sub ternary semiring of S if $(T, +)$ is a sub ternary semigroup of $(S, +)$ and abc ∈ T for all a, b,c ∈ T.

2.2 Definition: A non-empty subset T of S is a left (respectively lateral, right) ideal of S if T is a sub ternary semigroup of $(S, +)$ and $xya \in T$ (respectively $xay \in T, axy \in T$) for all $a \in T$, $x,y \in S$.

2.3 Definition: An element a of a ternary semiring is a regular if $a \in aMaMa$. If all elements of ternary semiring S are regular, then S is a regular ternary semiring.

2.4 Definition: A non empty subset Q of a ternary semiring S is a quasi-ideal if Q is a sub ternary semigroup of $(S, +)$ and $(SSQ) \cap (SQS) \cap (QSS) \subseteq Q$.

obviously, every quasi ideal of S is a sub ternary semiring of S.

2.5 Example: Let N be the set of natural numbers. Then N is a ternary semiring with respect to usual addition and multiplication. Then $Q = 3N$ is a quasi ideal of a ternary semiring N.

2.6 Definition: A non empty subset B of ternary semiring S is said to be a bi-ideal of S if B is a sub ternary semiring of S and $BSBSB \subseteq B$. Clearly every bi-ideal is a sub ternary semiring but converse is not true.

2.7 Example: Consider the ternary semiring $S = M_{2x2}(N_0)$, where N denotes the set of all natural numbers and $N_0 = N \cup \{0\}$. If S forms a ternary semiring with ABC = usual matrix product of A, B, C for all A, B, C $\in S$.

B = $\left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} / x, y \in N_0 \right\}$ is a sub ternary semiring of S. But B is not a bi-ideal of S.

2.8 Result: Let S be a ternary semiring then the following statements holds

(i) Any one sided (three sided) ideal of a ternary semiring S is a bi-ideal of S.

(ii) Intersection of a right ideal, a lateral ideal and a left ideal of a ternary semiring S is a bi-ideal of S.

(iii) Every quasi-ideal of S is a bi-ideal of S.

(iv) Arbitrary intersection of bi-ideals of S is also a bi-ideal of S and hence the set of all bi-ideals of S forms a complete lattice.

(v) If B is a bi-ideal of S, then Bss and ssB are bi-ideals of S, for any s ∈ S.

(vi) If B is a bi-ideal of S, then bBc is a bi-ideal of S, for b, c ∈ S.

(vii) If B is a bi-ideal of S and if T is a sub semiring, then $B \cap T$ is a bi-ideal of T.

(viii) If A, B and C are bi-ideals of S, then ABC and CBA are bi-ideals of S.

(ix) For any $a \in S$, SSa is a left ideal and aSS is a right ideal of S.

3.MAIN RESULTS

3.1 Theorem: If B is a bi-ideal and T is a sub ternary semiring of ternary semiring S, then BTT and TTB are bi-ideals of S.

Proof: Let $a, b \in BTT$. Hence $a = \sum_{i=1}^{n} a_i s_i t_i$ and $b = \sum_{j=1}^{n} b_j u_j v_j$ where $a_i, s_i, t_i, b_j, u_j, v_j \in BTT$ being a finite sum $a + b \in BTT$.

Hence BTT is a sub semigroup of (S , +).

Consider $(BTT)(BTT)(BTT) = (BTT\ BTT\ B)TT \subseteq ((BSBSB)\ TT \subseteq BTT$, But BTT is a sub ternary semiring of S.

Now $BTTSBTTSBTT = B(TTS)B(TTS)BTT$

$\subseteq BSBSBTT$

$\subseteq BTT$.

That is $BTTSBTTSBTT \subseteq BTT$

Therefore BTT is a bi-ideal of S.

Similarly, we can prove that TTB is a bi-ideal of S.

3.2 Theorem: If B is a bi-ideal of a ternary semiring S and C is a bi-ideal of B such that $C^3 = CCC = C$, then C is a bi-ideal of S.

Proof: Let B be a bi-ideal of a ternary semiring S. Then we get $BSBSB \subseteq B$.

Let C be a bi-ideal of B such that $C^3 = CCC = C$.

Hence $CBCBC \subseteq C$.

Since C is a sub ternary semiring of B, it is a sub ternary semiring of S.

Further $CSCSC = (CCC)SCS(CCC)$

$= CC(CSCSC)CC$

$\subseteq CC(BSBSB)CC$

$\subseteq CCBCC$

$\subseteq C.$

Thus $CSCSC \subseteq C.$

Hence C is a bi-ideal of S.

3.3 Theorem: Let A,B and C are three sub ternary semirings of a ternary semiring S and $M = ABC$. If A is a left ideal of S, then M is a bi-ideal of S.

Proof: Let AB and C be any two sub ternary semirings of a ternary semiring S.

Suppose that A is a left ideal of ternary semiring S.

Hence A is a bi-ideal of S.

Now $MSMSM = (ABC)S(ABC)S(ABC)$

$= A(BCS)A(BCS)ABC$

$\subseteq (ASASA)BC$ since $BCS \subseteq S$.

$\subseteq (ABC)$ since A is a bi-ideal of S.

$= M.$

That is $MSMSM \subseteq M.$

Hence M is a bi-ideal of S.

3.4 Theorem: Let A, B and C are three sub ternary semirings of a ternary semiring S and $M = ABC$. If B is a lateral ideal of S, then M is a bi-ideal of S.

Proof: follows the above **Theorem 3.3**.

3.5 Theorem: Let A B and C be three sub ternary semirings of a ternary semiring S and $M= ABC$. If C is a right ideal of S, then M is a bi-ideal of S.

Proof: follows the above **Theorem 3.3**.

3.6 Theorem: For any non-empty subset B of S, following statements are equivalent:

(1) B is a bi-ideal of S.

(2) B is a left ideal of some right ideal of S.

(3) B is a right ideal of some left ideal of S.

Proof: (1)$\Leftrightarrow$(2). Suppose that B is a bi-ideal of S.

Therefore $BSBSB \subseteq B$.

Since BSS is a right ideal of S, $(BSS)B \subseteq B$.

Therefore B is a left ideal of a right ideal BSS of S.

Conversely, suppose that B is a left ideal of a right ideal R of S.

Then $RRB \subseteq B$, $RSS \subseteq R$ and B is a sub semiring of S.

Hence $BSBSB \subseteq RSRSB$
$\subseteq RRRSB$ (since $B \subseteq R$)
$\subseteq R(RRS)B$
$\subseteq RRB$ (since $RRS \subseteq R$)
$\subseteq B$.
$BSBSB \subseteq B$.
Therefore B is a bi-ideal of S.

(1)⇔(3):

Suppose that B is a bi-ideal of S. Then $BSBSB \subseteq B$.

Since SSB is a left ideal of S, $B(SSB) \subseteq B$.

Therefore B is a right ideal of a left ideal SSB of S.

Conversely,

suppose B is a right ideal of a left ideal L of S.

Then $BLL \subseteq B$, $SSL \subseteq L$ and B is a sub ternary semiring of S.

We get $BSBSB \subseteq BSLSL$

$\subseteq BS(LLL) \subseteq B(SLL)L$

$\subseteq BLL \subseteq B$.

Hence B is a bi-ideal of S.

Similarly we can prove (1) ⇔(2)⇔ (3).

3.7 Theorem: A sub ternary semiring B of a ternary semiring S is a bi-ideal of S if and only if there exist a left ideal L, lateral ideal I and a right ideal R of S such that $RIL \subseteq B \subseteq R \cap I \cap L$.

Proof: Suppose that B is a bi-ideal of a ternary semiring S.
Then $BSBSB \subseteq B$.
Hence $R = BSS$ is a right ideal of S, $I=SBS$ is a lateral ideal and $L = SSB$ is a left ideal of S.
$RIL = (BSS)SBS(SSB)$
$= B(SSS)B(SSS)B$
$\subseteq BSBSB$
$\subseteq B$.
Therefore $RIL \subseteq B$.
As B is a bi-ideal of S, B is a right ideal of a left ideal L and also a left(lateral) ideal of a right ideal R.
Therefore $B \subseteq R \cap I \cap L$.

Thus $RIL \subseteq B \subseteq R \cap I \cap L$.

Hence $RIL \subseteq R \cap I \cap L$.

Conversely, suppose that $RIL \subseteq B \subseteq R \cap I \cap L$.

Consider $BSBSB \subseteq (R \cap I \cap L)S(R \cap I \cap L)S(R \cap I \cap L)$

$\subseteq RSISL$

$\subseteq RIL$

$\subseteq B$.

Hence $BSBSB \subseteq B$.

This shows B is a bi-ideal of S.

CONCLUSION

The results establish that bi-ideals occupy a central position in the ideal theory of ternary semirings. They connect quasi-ideals and one-sided ideals through the ternary absorption condition and provide several equivalent and constructive ways of identifying bi-ideals. The framework developed in the document can serve as a foundation for further investigations of specialized classes of ternary semirings and for the study of additional ideal structures arising from ternary operations.

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