Interval Valued Pythagorean Fuzzy Continuous Functions

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Abstract: In the present paper, we introduce the concepts of interval valued Pythagorean fuzzy almost continuous mapping and interval valued Pythagorean fuzzy weakly continuous mapping in interval valued Pythagorean fuzzy topological space and we study some of their properties. We will also introduce and investigate interval valued Pythagorean fuzzy $\alpha$-continuous function between interval valued Pythagorean fuzzy topological spaces and establish their corresponding characterizations.

I. INTRODUCTION

As a generalization of fuzzy set theory was first introduced by Zadeh[14]. Further Chang [3] introduced the fuzzy topological spaces with few results as continuity, closed and open sets. These spaces and their generalizations are later studied by several authors, one of which, developed by Sostak [12,8], used the idea of degree of openness. This type of generalization of fuzzy topological spaces was later rephrased by Chattopadhyay [4] and by Ramadan [10].

In 1983, Atanassov introduced the concept of Intuitionistic fuzzy set with elements comprising membership and non-membership degree [1,11]. Using this type of generalized fuzzy set, Coker [5,9] introduced the concept of Intuitionistic fuzzy topological spaces and studied some notions such as continuity and compactness. The different notion and definition for fuzzy topological spaces were given by Lowen [6].

In 1996, Coker and Demirci [2] introduced the basic definitions and properties of Intuitionistic fuzzy topological spaces in Sostak’s sense, which is a generalized form of fuzzy topological spaces developed by Sostak [12,8]. The concepts as fuzzy alpha open and closed sets and open map and continuous functions was developed by Rajvanshi and Singal. Yager [13] presented the concept of pythagorean fuzzy subset which is a typical fuzzy set. After the introduction of pythagorean fuzzy sets, it was widely used in the field of decision making and was applied for the real life applications. Olgun [7] introduced pythagorean fuzzy topology along with the definition of continuity in pythagorean fuzzy topological spaces and their characterizations. Thus it was further developed with connected, compactness of pythagorean topological spaces.

In this study, we introduce the following concepts: interval valued Pythagorean fuzzy almost continuous mapping, interval valued Pythagorean fuzzy weakly continuous mapping and interval valued Pythagorean alpha open sets and closed sets and interval valued Pythagorean alpha continuity in interval valued Pythagorean fuzzy topological space.

II. PRELIMINARIES

Definition 2.1

Let X be a nonempty fixed set and I the closed unit interval [0,1]. An Intuitionistic fuzzy set (IFS) A is an object having the form

\[ A = \{ < x, \mu_A(x), \nu_A(x) > : x \in X \} \],

where the mapping $\mu_A : X \rightarrow I$ and $\nu_A : X \rightarrow I$ denote the degree of membership and the degree of nonmembership of each element $x \in X$ to the set A and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. The complement of the IFS A is $\bar{A} = \{ < x, \nu_A(x), \mu_A(x) > : x \in X \}$. Obviously, every fuzzy set A on nonempty set X is an IFS having the form

\[ A = \{ < x, \mu_A(x), 1-\mu_A(x) > : x \in X \} \],

For a given nonempty set X, denote the family of all IFSs in X by the symbol $\mathcal{I}(X)$. 
Definition 2.2
An intuitionistic fuzzy topology (IFT) in Chang’s sense on a nonempty set $X$ is a family $\tau$ of IFSs in $X$ satisfying the following axioms:

$(T_1)$ $\tau(\emptyset) = \tau(\bigcap) = 1$;

$(T_2)$ $\tau(\bigcup A) \geq \tau(A)$ for any $A \subseteq X$;

$(T_3)$ $\tau(U) \geq \tau(A)$ for any $\{A_i : i \in I\} \subseteq X$.

In this case, the pair $(X, \tau)$ is called an Intuitionistic fuzzy topological spaces in Sostak’s sense (IFTS). For any $A \in \tau$, the number $\mu_\tau(A)$ is called the openness degree of $A$, while $\nu_\tau(A)$ is called the nonopenness degree of $A$.

Definition 2.3
An IFT in sostak’s sense on a nonempty set $X$ is an IFF $\tau$ on $X$ satisfying the following axioms:

$(T_1)$ $\tau(\emptyset) = \tau(\bigcap) = 1$;

$(T_2)$ $\tau(\bigcap A) \geq \tau(A)$ for any $A \subseteq X$;

$(T_3)$ $\tau(U) \geq \tau(A)$ for any $\{A_i : i \in I\} \subseteq X$.

III. INTERVAL VALUED PYTHAGOREAN FUZZY ALMOST CONTINUOUS AND INTERVAL VALUED PYTHAGOREAN FUZZY WEAKLY CONTINUOUS MAPPING:

Definition 3.1
Let $A$ be an IVPFTS $(X, \tau)$. For $a \in \mathbb{I}_0, b \in \mathbb{I}_1$ with $[a^2, b^2] \leq 1$.

A is called,

i) interval valued Pythagorean fuzzy regular open (IVPFO) set of $X$ if $\text{int}_{a,b}(cl_{a,b}A) = A$

ii) interval valued Pythagorean fuzzy regular closed (IVPFC) set of $X$ if $cl_{a,b}(\text{int}_{a,b}A) = A$

THEOREM 3.2:
Let $A$ be an IVPFS in an IVPTS $(X, \tau)$. Then, for $a^2 \in \mathbb{I}_0, b^2 \in \mathbb{I}_1$ with $[a^2, b^2] \leq 1$.

i) If $A$ is IVPFO (resp., IVPFC), set then $t(A) \geq <a, b>$ (resp., $t^*(A) \geq <a, b>$)

ii) A is IVPFO set if and only if $\bar{A}$ is IVPFC set.

Proof:
We will prove (ii) only.

A is IVPFO $\iff \text{int}_{a,b}(cl_{a,b}A) = A$

$\iff cl_{a,b}(\text{int}_{a,b}A) = \bar{A}$

$\iff \bar{A}$ is IVPFC

Theorem 3.3
Let $(X, \tau)$ be an IVPFTS. Then,

i) the union of two is IVPFC sets is IVPFC set.

ii) the intersection of two IVPFC sets is IVPFC set.
i) Let A, B be any two IVPFRC sets. By theorem 3.2, we have $\tau^*(A) \leq <\alpha, \beta>$, $\tau^*(B) \leq <\alpha, \beta>$ then $\tau^*(A) \wedge \tau^*(B) \leq <\alpha, \beta>$, but $\text{int}_{a,b}(A) \cup \text{int}_{a,b}(B) \subseteq A \cap B$, this implies that $\text{cl}_{a,b}(\text{int}_{a,b}(A)) \subseteq \text{cl}_{a,b}(A) = A$ and hence $\text{cl}_{a,b}(\text{int}_{a,b}(B)) \subseteq \text{cl}_{a,b}(B) = B$. Now, $A = \text{cl}_{a,b}(\text{int}_{a,b}(A)) \subseteq \text{cl}_{a,b}(\text{int}_{a,b}(A)) = A$ and hence $A \cap B$ is IVPFRC set.

ii) It can be proved by the same manner.

**THEOREM 3.4:**
Let $(X, \tau)$ be an IVPFTS. Then,

1. If $A \in \zeta^Y$ such that $\tau^*(A) \geq <\alpha, \beta>$ then $\text{int}_{a,b}(A)$ is $(\alpha, \beta)$- IVPFRO set,

2. If $B \in \zeta^X$ such that $\tau^*(B) \geq <\alpha, \beta>$ then $\text{cl}_{a,b}(B)$ is IVPFRC set.

**Proof:**

i) Let $A \in \zeta^X$ such that $\tau^*(A) \geq <\alpha, \beta>$ clearly,

$$\text{int}_{a,b}(A) \subseteq \text{int}_{a,b}(\text{cl}_{a,b}(A)),$$

which implies that

$$\text{int}_{a,b}(A) \subseteq \text{int}_{a,b}(\text{cl}_{a,b}(\text{int}_{a,b}(A))).$$

Now, since

$$\tau^*(A) \geq <\alpha, \beta>,$$

then $\text{cl}_{a,b}(\text{int}_{a,b}(A)) \subseteq A$;

This implies that

$$\text{int}_{a,b}(\text{cl}_{a,b}(\text{int}_{a,b}(A))) \subseteq \text{int}_{a,b}(A).$$

Thus, $\text{int}_{a,b}(\text{cl}_{a,b}(\text{int}_{a,b}(A))) = \text{int}_{a,b}(A)$

Hence, $\text{int}_{a,b}(A)$ is IVPFRO set.

ii) It can be proved by the same manner.

**DEFINITION 3.5:**
A mapping $f: (X, \tau_1) \rightarrow (Y, \tau_2)$ from an IVPFTS $(X, \tau_1)$ to another IVPFTS $(Y, \tau_2)$ is called

(i) Interval valued Pythagorean fuzzy strong continuous if and only if $\tau_1(f^{-1}(\alpha)) = \tau_2(A)$, for each $A \in \zeta^Y$.

(ii) Interval valued Pythagorean fuzzy almost continuous if and only if $\tau_1(f^{-1}(\alpha)) \geq <\alpha, \beta>$, for each $(\alpha, \beta)$- IVPFRO set $A$ of $Y$.

(iii) $(\alpha, \beta)$- Interval valued Pythagorean fuzzy weekly continuous if and only if $\tau_2(A) \geq <\alpha, \beta>$ implies $\tau_1(f^{-1}(\alpha)) \geq <\alpha, \beta>$, for each $A \in \zeta^X$.

**REMARK 3.6:**
From the above definition, it is clear that the following implications are true for $\alpha \in I_l, \beta \in I_l$, with $[\alpha^b]^2 + [\beta^b]^2 \leq 1$

Interval valued Pythagorean fuzzy almost continuous mapping

Interval valued Pythagorean fuzzy strong continuous mapping

Interval valued Pythagorean fuzzy weakly continuous mapping

But, reciprocal implications are not true in general.

**Theorem 3.7:**
Let $f: (X, \tau_1) \rightarrow (Y, \tau_2)$ be a mapping from an IVPFTS $(X, \tau_1)$ to another IVPFTS $(Y, \tau_2)$. Then the following statements are equivalent.

(i) $f$ is $(\alpha, \beta)$- Interval valued Pythagorean fuzzy almost continuous;

(ii) $\tau^*_2(f^{-1}(\beta)) \geq <\alpha, \beta>$, for each IVPFRC set $B$ of $Y$;

(iii) $f^{-1}(B) \subseteq \text{int}_{a,b}(\text{cl}_{a,b}(\text{int}_{a,b}(B)))$ for each $B \in \zeta^Y$ such that $\tau_2(B) \geq <\alpha, \beta>$;

(iv) $\text{cl}_{a,b}(f^{-1}(\text{int}_{a,b}(B))) \subseteq f^{-1}(B)$, for each $B \in \zeta^Y$ such that $\tau_2(B) \geq <\alpha, \beta>$ where $\alpha \in I_l, \beta \in I_l$ with $[\alpha^b]^2 + [\beta^b]^2 \leq 1$

**PROOF:**

(i) $\Rightarrow$ (ii)

Let $B$ be IVPFRC set of $Y$.

Then by theorem 3.2, $\bar{B}$ is IVPFRO set.

By (i), we have $\tau_1(f^{-1}(\bar{B})) = \tau_2(f^{-1}(\beta)) \geq <\alpha, \beta>$

(ii) $\Rightarrow$ (i)

It is analogous to the proof of (ii) $\Rightarrow$ (i)

(i) $\Rightarrow$ (iii)

Since that $\tau_2(B) \geq <\alpha, \beta>$, then $B = \text{int}_{a,b}(B) \subseteq \text{int}_{a,b}(\text{cl}_{a,b}(B))$ and hence,

$f^{-1}(B) \subseteq f^{-1}(\text{int}_{a,b}(\text{cl}_{a,b}(B)))$.

Since, $\tau_2(\text{cl}_{a,b}(B)) \geq <\alpha, \beta>$ then by theorem 3.4, $\text{int}_{a,b}(\text{cl}_{a,b}(B))$ is $(\alpha, \beta)$- IVPFRO set.

So, $\tau_1(f^{-1}(\text{int}_{a,b}(\text{cl}_{a,b}(B)))) \geq <\alpha, \beta>$. Then,
Let $f: (X, \tau_1) \to (Y, \tau_2)$ be a mapping from an IVPFTS $(X, \tau_1)$ to another IVPFTS $(Y, \tau_2)$. Then the following statements are equivalent.

i) $f$ is Interval valued Pythagorean fuzzy weakly continuous;

ii) $f(c_l\alpha, (A)) \subseteq c_l\alpha, (f(A))$ for each $A \epsilon \zeta^X$.

**Proof:** (i) $\rightarrow$ (ii)

Let $A \epsilon \zeta^X$. Then,

$$f^{-1}(c_l\alpha, (f(A))) = f^{-1}[\{K \epsilon \zeta^Y: \tau_2(K) \geq \alpha, \beta\}, K \supseteq f(A)]$$

$$= f^{-1}\left[\{K \epsilon \zeta^Y: \tau_2(\bar{K}) \geq \alpha, \beta\}, K \supseteq f(A)\right]\nonumber$$

$$\subseteq f^{-1}\left[\{K \epsilon \zeta^Y: \tau_2(\bar{K}) \geq \alpha, \beta\}, K \supseteq f(A)\right]\nonumber$$

$$\supseteq f^{-1}\left[\{K \epsilon \zeta^Y: \tau_2(\bar{K}) \geq \alpha, \beta\}, K \supseteq f(A)\right]\nonumber$$

Therefore $f(c_l\alpha, (A)) \subseteq f(f^{-1}(c_l\alpha, (f(A)))) \subseteq c_l\alpha, (f(A))$.

(ii) $\rightarrow$ (i)

Let $B \epsilon \zeta^Y$ such that $\tau_2(B) \geq \alpha, \beta$. Then $\tau_2(\bar{B}) = \tau_2(B) \geq \alpha, \beta$. So, we have $c_l\alpha, (\bar{B}) = \bar{B}$.

Further, since $f(c_l\alpha, (f^{-1}(B))) \subseteq c_l\alpha, (f(f^{-1}(B))) \subseteq c_l\alpha, (B) = \bar{B}$.

We have $c_l\alpha, (f^{-1}(\bar{B})) \subseteq f^{-1}(\bar{B})$, then $c_l\alpha, (f^{-1}(\bar{B})) = f^{-1}(\bar{B})$.

This implies that $\tau_1(f^{-1}(\bar{B})) = \alpha, \beta$.

Therefore $\tau_1(f^{-1}(\bar{B})) = \tau_1(f^{-1}(B)) \geq \alpha, \beta$ . Hence $f$ is Interval valued Pythagorean fuzzy weakly continuous.

**Theorem 3.9:**

Let $f: X \to Y$ be an Interval valued Pythagorean fuzzy continuous mapping with respect to the IVPFTS $\tau_1$ and $\tau_2$ respectively.

Then for every IVPFS $A$ in $X$, $f(c_l\alpha, (A)) \subseteq c_l\alpha, (f(A))$ where $\alpha \epsilon I_0$, $\beta I_1$ with $[a^2] + [b^2] \leq 1$.

**Proof:**

Let $f: X \to Y$ be an Interval valued Pythagorean fuzzy continuous mapping with respect to $\tau_1$ and $\tau_2$, and let $A \epsilon \zeta^X$. Then,

$$f^{-1}(c_l\alpha, (f(A))) = f^{-1}\left[\{K \epsilon \zeta^Y: \tau_2(K) \geq \alpha, \beta\}, f(A) \subseteq K\right]\nonumber$$

$$\supseteq f^{-1}\left[\{K \epsilon \zeta^Y: \tau_2(K) \geq \alpha, \beta\}, f(A) \subseteq K\right]\nonumber$$

$$\supseteq f^{-1}\left[\{K \epsilon \zeta^Y: \tau_2(K) \geq \alpha, \beta\}, f(A) \subseteq K\right]\nonumber$$

This implies that $f(c_l\alpha, (A)) \subseteq c_l\alpha, (f(A))$.

**Theorem 3.10:**

Let $f: X \to Y$ be an Interval valued Pythagorean fuzzy continuous mapping with respect to the IVPFTS $\tau_1$ and $\tau_2$ respectively. Then for every IVPFS $A$ in $Y$, $c_l\alpha, f^{-1}(f(A)) \subseteq f^{-1}(c_l\alpha, (A))$ where $\alpha \epsilon I_0$, $\beta I_1$ with $[a^2] + [b^2] \leq 1$.

**Proof:**

Let $A \epsilon \zeta^Y$ we get theorem 3.9

$$c_l\alpha, f^{-1}(f(A)) \subseteq f^{-1}(f(c_l\alpha, (f^{-1}(f(A))))) \subseteq f^{-1}(c_l\alpha, (A)) \quad \ldots \quad (2.18)$$

Hence $c_l\alpha, f^{-1}(f(A)) \subseteq f^{-1}(c_l\alpha, (A))$ for every $A \epsilon \zeta^Y$.

**Definition 3.11:**

A IVPFS $P = < x, \mu_P, \gamma_P >$ of a IVPFTS $(X, \tau)$ is called an interval valued Pythagorean fuzzy $\alpha$ open set if $P \subseteq (c_l(int(P)))$. A IVPFS whose complement is a interval valued Pythagorean fuzzy $\alpha$ open set (IVPFOs) is called a interval valued Pythagorean fuzzy $\alpha$ closed set (IVPFOCS).

**Proposition 3.12:**

Let $(X, \tau)$ be a IVPFS. Then arbitrary union of IVPFOs is an IVPFOs and arbitrary intersection of IVPFO sets is an IVPFOCS.
Thus $P$ is interval valued Pythagorean fuzzy $\alpha$ open set and every IVPFCS is a IVPF$\alpha$C but the converse is not true.

**Definition 3.13:**
The interval valued Pythagorean fuzzy $\alpha$ closure of a IVPFS $P$ in a IVPFS $(X, \tau)$ represented as $cl_{\alpha}(P)$ and defined by $cl_{\alpha}(P) = \cap \{C_i | C_i$ is IVPF$\alpha$C set and $P \subseteq C_i \}.$

**Proposition 3.14:**
In a IVPFTS $X, \tau$ a IVPFS is IVPF$\alpha$C if and only if $P = cl_{\alpha}(P)$.

**Proof:**
Assume that $P$ is a IVPF$\alpha$C set. Then $P \subseteq \{C_i | C_i$ is IVPF$\alpha$C set and $P \subseteq C_i \}$

$= cl_{\alpha}(P)$

Conversely, consider $P = cl_{\alpha}(P)$.

$P \subseteq \{C_i | C_i$ is IVPF$\alpha$C set and $P \subseteq C_i \}$

Thus $P$ is interval valued Pythagorean fuzzy $\alpha$- closed set.

**Proposition 3.15:**
In a IVPFTS $X, \tau$ the following hold for interval valued Pythagorean fuzzy $\alpha$- closure.

1. $cl_{\alpha}(\emptyset) = \emptyset$
2. $cl_{\alpha}(P)$ is a IVPF$\alpha$C in $(X, \tau)$ for every IVPFS $P$ in $X$.
3. $cl_{\alpha}(P) \subseteq cl_{\alpha}(R)$ whenever $P \subseteq R$ for every $P$ and $R$ in $X$.
4. $cl_{\alpha}(cl_{\alpha}(P)) = cl_{\alpha}(P)$ for every IVPFS $P$ in $X$.

**Proof:**
(1) The proof is obvious.

(2) By preposition, $P$ is IVPF$\alpha$C if and only if $P = cl_{\alpha}(P)$ we get $cl_{\alpha}(P)$ is a IVPF$\alpha$C for every $P$ in $X$.

(3) By same preposition, we get $P = cl_{\alpha}(P)$ and $R = cl_{\alpha}(R)$, whenever $P \subseteq R$, we have $cl_{\alpha}(P) \subseteq cl_{\alpha}(R)$.

(4) Let $P$ be a IVPFS in $X$. we know that $P = cl_{\alpha}(P)$

$cl_{\alpha}(P) = cl_{\alpha}(cl_{\alpha}(P))$

Thus $cl_{\alpha}(cl_{\alpha}(P)) = cl_{\alpha}(P)$ for every $P$ in $X$.

**Definition 3.16:**
Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be IVPFTS. A mapping $f : X \rightarrow Y$ is named as interval valued Pythagorean fuzzy $\alpha$ - continuous (IVPF$\alpha$CN) if the inverse image of each IVPFOS of $Y$ is a IVPF$\alpha$O set in $X$.

**Theorem 3.17:**
Let $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ be a mapping from a IVPFTS $(X, \tau_X)$ to a IVPFTS $(Y, \tau_Y)$. If $f$ is interval valued Pythagorean fuzzy $\alpha$ - continuous, then

1. $F(cl(int(cl(P)))) \subseteq cl(f(P))$ for all IVPFS $P$ in $X$.
2. $cl(int(cl(f^{-1}(B)))) \subseteq f^{-1}(cl(B))$ for all $B$ in $Y$.

**Proof:**
Assume that $f$ is an IVPF$\alpha$CN mapping. Let $P$ be a IVPFS in $X$, then $cl(f(P))$ is a IVPFCS in $Y$ and this $f^{-1}(cl(f(P)))$ is a IVPF$\alpha$C set in $X$. thus

$cl(int(cl(P))) = cl(int(cl(f(P))))$

$\subseteq cl(int(cl(f^{-1}(cl(f(P))))))$

$\subseteq f^{-1}(cl(f(P)))$

So $f(cl(int(cl(P)))) \subseteq cl(f(P))$.

Now let $B$ be a IVPFS in $Y$. Then $f^{-1}(B)$ an IVPFS in $X$. by (i),

$F(cl(int(cl(f^{-1}(B))))) \subseteq cl(f(f^{-1}(B)))$

$\subseteq cl(B)$

Thus $cl(int(f^{-1}(B))) \subseteq f^{-1}(cl(B))$. 

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