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# COMMON FIXED POINT THEOREM IN B-METRIC SPACE EMPLOYING COMMON (E.A) PROPERTY

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***Abstract***

*The aim of the paper is to prove a fixed point theorem for four mappings in b- metric space by utilising the concept of common (E.A) property. In this paper if we have prove that if E,H,L and M are mappings on a b- metric space, then the pairs (E,H) and (L,M) have a point of coincidence in γ. Also if these pairs are weakly compatible then they have a unique common fixed point. This is also explains with the help of example.*

*Keywords:-Fixed point, common (E.A) property, b- metric space, weakly compatible mappings.*

## 1. Introduction

The famous Banach contraction principle introduced by Banach [1], secure the existence and uniqueness of fixed points for a contraction mapping in metric space. One of the significant generalizations of metric space is called b- metric space. The concept of b- metric space was introduced by Bakhtin [2]. The E.A property was introduced by Aamri and Moutawakil [3]. Roshan [4] *et.al* used the notion of almost observe contractive mappings in ordered complete b- metric spaces and established some fixed and common fixed point results. Czerwik [5] first produce a generalization of Banach fixed point theorem in b- metric space. Berinde [6] explained the concept of weak contraction from the case of single-valued mappings is extended to multi-valued mappings and then corresponding convergence theorems for the Picard iteration associated to a multi-valued weak contraction. Aydi *et al*. [7] proved a common fixed point results for single valued and multi-valued mapping satisfying a weak φ- contraction in b- metric spaces. After that, several interesting results about the existence of a fixed point for single- valued and multi- valued operators in b- metric spaces have been obtained ([8], [9], [10], [11], [12], [13].

**Definition 1.1 [2]**:- let $\Gamma$ be a non- empty set and s ≥ 1 be a given real number. A function $d:\Gamma\times\Gamma\rightarrow R$ is called a b- metric space provided that for all $\alpha,\beta,\delta\in\Gamma$,

I. $d(\alpha,\beta)=0,$ if and only if $\alpha=\beta$, (Identification)

II. $\mathrm{d}(\alpha,\beta)=\mathrm{d}(\beta,\alpha)$, (Symmetry)

III. $\mathrm{d}(\alpha,\delta)\leq s[\mathrm{d}(\alpha,\beta)+\mathrm{d}(\beta,\delta)]$. (Triangular Inequality)

A pair ( $\Gamma$ , d) is called a b- metric space.

**Example 1.2 [11]:-** Let ( $\Gamma$ , d) be a metric space and $\rho(\alpha,\beta)=(\mathrm{d}(\alpha,\beta))^{p}$, where $p>1$ is a real number. Then $\rho$ is a b- metric with $s=2^{p-1}$.

**Definitions 1.3 [11]:-** Let ( $\Gamma$ , d) be a b- metric space. Then a sequence $\{\alpha_n\}$ in $\Gamma$ is called a Cauchy sequence if and only if for all $\varepsilon>0$ there exist $n(\varepsilon)\in\mathrm{N}$ such that for each $n,m\geq n(\varepsilon)$ we have $d(\alpha_n,\alpha_m)<\varepsilon.$

**Definition 1.4 [11]**:- Let ( $\Gamma$ , d) be a b- metric space then a sequence $\{\alpha_n\}$ in $\Gamma$ is called convergent sequence if and only if there exist $\alpha\in\Gamma$ such that for all there exists $n(\varepsilon)\in\mathrm{N}$ such that for all $n\geq n(\varepsilon)$ we have $d(\alpha_n,\alpha)<\varepsilon$. In this case we write $\lim_{n\to\infty}\alpha_n=\alpha.$

**Definition 1.5 [11]:-** The b- metrics space is complete if every Cauchy sequence is convergent.

**Definition 1.6 [12]:** A pair of maps E and H is called weakly compatible pair if they commute at coincidence points.

## 2. Main Result

**Definition 2.1:** Let E, H, L, M: $\Gamma\rightarrow\Gamma$ be mappings on b- metric space ( $\Gamma$ , d). The pairs (E, L) and (H, M) satisfy the common (E.A) property, if there exist sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that

$$\lim_{n\to\infty}E\alpha_n=\lim_{n\to\infty}L\alpha_n=\lim_{n\to\infty}H\beta_n=\lim_{n\to\infty}M\beta_n=k,\ \text{for some } \mathrm{k}\in\Gamma.$$

In the following result, the notion the control functions $\psi,\varphi:[0,\infty)\rightarrow[0,\infty)$ is used which are continuous, non decreasing function with $\psi(\mathrm{t})=0$ and $\varphi(\mathrm{t})=0$ if and only $t=0$.

**Theorem 2.2:-** Let E, H, L, M: $\Gamma \to \Gamma$ be mappings on b- metric space ($\Gamma$, d) with s ≥ 1 and satisfying the following conditions:-

I. $\psi(s^2 d(E\alpha, H\beta)) \leq \psi(N_S(\alpha,\beta)) - \varphi(N_S(\alpha,\beta))$

Where

$$N_S(\alpha,\beta) = \max\left\{d(L\alpha, M\beta), d(E\alpha, L\alpha), d(H\beta, M\beta), \frac{d(E\alpha, M\beta)+d(L\alpha, H\beta)}{2s}\right\},$$

II. The pairs (E, L) and (H, M) share the common (E.A) property.

III. If L($\Gamma$) and M($\Gamma$) are closed subsets of $\Gamma$.

Then, the pairs (E, L) and (H, M) have a point of coincidence in $\Gamma$. Further, E, H, L and M have a unique common fixed point provided that both the pairs (E, L) and (H, M) are weakly compatible.

Proof:- The pairs (E, L) and (H, M) share the common (E. A) property, there exist sequences

$\{\alpha_n\}$ and $\{\beta_n\}$ in $\Gamma$ such that

$$\lim_{n\to\infty} E\alpha_n = \lim_{n\to\infty} L\alpha_n = \lim_{n\to\infty} H\beta_n = \lim_{n\to\infty} M\beta_n = k, \text{ for some } k \in \Gamma. \tag{1.1}$$

Since $L(\Gamma)$ is a closed subset of $\Gamma$ therefore, there exists a point $m_1 \in \Gamma$

such that $Lm_1 = k.$ (1.2)

We put $\alpha = m_1$ and $\beta = \beta_n$ in equation ( I ), one have

$$\psi(s^2 d(E m_1, H\beta_n)) \leq \psi(N_s(m_1,\beta_n)) - \varphi(N_S(m_1,\beta_n)), \tag{1.3}$$

where, $N_S(m_1,\beta_n) = \max\left\{d(L m_1, M\beta_n), d(E m_1, L m_1), d(M\beta_n, H\beta_n), \frac{d(E m_1, M\beta_n)+d(L m_1, H\beta_n)}{2s}\right\}.$

Taking the limits as n→∞ and by using (1.3), we have

$$\psi(s^2 d(k, Em_1)) \leq \psi(d(k, Em_1)) - \varphi(d(k, Em_1)).$$

Using the definition of ψ, one can easily get

$$s^2 d(k, Em_1) \leq d(k, Em_1).$$

This implies, $d(k, Em_1) = 0$, hence $Em_1 = k$. (1.4)

From equation (1.2) and (1.4), we get $Em_1 = Lm_1 = k.$

Therefore, $m_1$ is a coincidence point of the pair (E, L).

If M($\Gamma$) is a closed subset of $\Gamma$. Therefore, there exists a point $w_1 \in \Gamma$ such that

$\mathrm{Mw}_1 = k.$ (1.5)

Now, we assert that, $Hw_1 = Mw_1$.

Let us put $\alpha = m_1$ and $\beta = w_1$ in equation (I), we have

$$\psi(\mathrm{s}^2 d(\mathrm{E}\,m_1, \mathrm{Hw}_1)) \le \psi(\mathrm{N}_S(\mathrm{m}_1, \mathrm{w}_1)) - \varphi(\mathrm{N}_S(\mathrm{m}_1, \mathrm{w}_1)), \quad (1.6)$$

where,

$N_S(\mathrm{m}_1, \mathrm{w}_1) = \max\left\{ d(\mathrm{L}\,m_1, \mathrm{Mw}_1), d(\mathrm{E}\,m_1, \mathrm{L}\,m_1), d(\mathrm{Mw}_1, \mathrm{Hw}_1), \dfrac{d(\mathrm{E}\,m_1, \mathrm{Mw}_1) + d(\mathrm{L}\,m_1, \mathrm{Hw}_1)}{2s} \right\}$. By using equation (1.6), we have

$$\psi(\mathrm{s}^2 d(\mathrm{k}, \mathrm{Hw}_1)) \le \psi(\mathrm{d}(\mathrm{k}, \mathrm{Hw}_1)) - \varphi(\mathrm{d}(\mathrm{k}, \mathrm{Hw}_1)).$$

Using the definition of $\psi$, one can have

$$\mathrm{s}^2 d(\mathrm{k}, \mathrm{Hw}_1) \le \mathrm{d}(\mathrm{k}, \mathrm{Hw}_1).$$

Hence $Hw_1 = k.$ (1.7)

From equation (1.5) and (1.7), we get $Hw_1 = Mw_1 = k.$

Therefore, $w_1$ is a coincidence point of the pair (H, M).

Thus, $Em_1 = Lm_1 = Hw_1 = Mw_1 = k$.

The weak compatibility of the pairs (E, L) and (H, M) implies $Ek = Lk$ and $Hk = Mk$. We will show that k is a common fixed point of E, H, L and M from equation (1)

We put $\alpha = k$ and $\beta = w_1$ in equation (I)

$$\psi(\mathrm{s}^2 d(\mathrm{Ek}, \mathrm{Hw}_1)) \le \psi(\mathrm{N}_s(\mathrm{k}, \mathrm{w}_1)) - \varphi(N_S(\mathrm{k}, \mathrm{w}_1)), \quad (1.8)$$

where,

$$N_S(\mathrm{k}, \mathrm{w}_1) = \max\left\{ d(\mathrm{Lk}, \mathrm{Mw}_1), d(\mathrm{Ek}, \mathrm{Lk}), d(\mathrm{Mw}_1, \mathrm{Hw}_1), \frac{d(\mathrm{Ek}, \mathrm{Mw}_1) + d(\mathrm{Lk}, \mathrm{Hw}_1)}{2s} \right\}$$

$$= \max\left\{ d(\mathrm{Ek}, \mathrm{k}), \mathrm{d}(\mathrm{Ek}, \mathrm{Ek}), \mathrm{d}(\mathrm{k}, \mathrm{k}), \frac{d(\mathrm{Ek}, \mathrm{k}) + \mathrm{d}(\mathrm{Ek}, \mathrm{k})}{2s} \right\}$$

$$= d(\mathrm{Ek}, \mathrm{k})$$

From equation (1.8), we get

$\psi(d(\mathrm{Ek},\mathrm{k})) \le \psi(\mathrm{d}(\mathrm{Ek},\mathrm{k})) - \varphi(d(\mathrm{Ek},\mathrm{k}))$.

So $Ek = Lk = k$. Similarly it can be shown that $Hk = Mk = k$.

To prove the uniqueness of fixed point, suppose r is another fixed point of E, H, L and M.

We put $\alpha = k$ and $\beta = r$ in equation ( I ) we have

$$\psi(\mathrm{s}^2\, d(\mathrm{Ek},\mathrm{Hr})) \le \psi(\mathrm{N}_s(\mathrm{k},\mathrm{r})) - \varphi(N_s(\mathrm{k},\mathrm{r})), \tag{1.9}$$

where,

$$\mathrm{N}_S(\mathrm{k},\mathrm{r}) = \max\left\{ d(\mathrm{Lk},\mathrm{Mr}), d(\mathrm{Ek},\mathrm{Lk}), d(\mathrm{Mr},\mathrm{Hr}), \frac{d(\mathrm{Ek},\mathrm{Mr}) + d(\mathrm{Lk},\mathrm{Hr})}{2s} \right\}$$

$$= \max\left\{ d(\mathrm{k},\mathrm{r}), \mathrm{d}(\mathrm{k},\mathrm{k}), \mathrm{d}(\mathrm{r},\mathrm{r}), \frac{d(\mathrm{k},\mathrm{r}) + \mathrm{d}(\mathrm{k},\mathrm{r})}{2s} \right\}$$
$$= d(\mathrm{k},\mathrm{r}).$$

Hence we have $\psi(d(\mathrm{k},\mathrm{r})) = \psi(d(\mathrm{k},\mathrm{r})) - \varphi(\mathrm{d}(\mathrm{k},\mathrm{r}))$,

Which implies that $\mathrm{d}(\mathrm{k},\mathrm{r}) = 0$, so $k = r$.

Hence, E, H, L and M have a common fixed point k.

**Example 2.3:-** Let $\Gamma = [-1,1]$ and define $\mathrm{d} : \Gamma \times \Gamma \to [0,\infty)$ as fallows

$$\mathrm{d}(\alpha,\beta) = \begin{cases} 0, & \alpha = \beta \\ (\alpha+\beta)^2, & \alpha \ne \beta \end{cases}.$$

The $(\Gamma,\mathrm{d})$ is a b- metric space with s= 2, $\psi(\mathrm{t}) = \sqrt{\mathrm{t}}$ and $\varphi(\mathrm{t}) = \frac{t}{10}$ for $\mathrm{t} \in [0,\infty)$.

Let $\mathrm{E},\mathrm{H},\mathrm{L},\mathrm{M} : \Gamma \to \Gamma$ be defined by $E\alpha = \frac{\alpha}{3}$, $H\alpha = -\frac{\alpha}{3}$, $L\alpha = \alpha$, $M\alpha = -\alpha$ for all $\alpha \in \Gamma$.

We take $\{\alpha_n\} = \left\{\frac{1}{n}\right\}$ and $\{\beta_n\} = \left\{-\frac{1}{n}\right\}$ then we have

$$\lim_{n\to\infty} E\alpha_n = \lim_{n\to\infty} L\alpha_n = \lim_{n\to\infty} H\beta_n = \lim_{n\to\infty} M\beta_n = 0.$$

Therefore, both pairs (E, L) and (H, M) satisfy the common (E.A) property.

Here, $L(\Gamma)$ and $M(\Gamma)$ are closed subsets of $\Gamma$. Thus all the conditions of the theorem (2.2) are satisfied. Here, 0 is a unique common fixed point of the (E, L) and (H, M)

The following results can easily prove with the help of this main theorem (2.2)

**Corollary 2.4:-** Let E, H, L, M: $\Gamma \to \Gamma$ be mappings on b- metric space ($\Gamma$, d) and satisfying the contractive condition ( I ) of the theorem (2.2).Suppose that the pairs (E, L) and (H, M) are weakly compatible and share the common (E.A) property. Then the maps E, H, L and M have unique common fixed point, provided either of the following two conditions hold good:

$L(\Gamma)$ is complete and $\mathrm{E}(\Gamma) \subset \mathrm{M}(\Gamma)$

$\mathrm{M}(\Gamma)$ is complete and $\mathrm{H}(\Gamma) \subset L(\Gamma)$.

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