Algebra of $\alpha$ - Fuzzy Subgroup and Lagrange’s Theorem

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Abstract
In this paper, we use the concept of $\alpha$ — fuzzy subgroup and introduces Lagrange’s theorem for the $\alpha$ — fuzzy subgroup, order of an element in the $\alpha$ — fuzzy subgroup, and order of the $\alpha$ — fuzzy subgroup. Defined the intersection of $\alpha$ — fuzzy subgroups on different domain.

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1 Introduction
L.A. Zadeh used the term fuzzy set in 1965[8]. He introduced basic set operations, namely the union of fuzzy sets, the intersection of fuzzy sets, and the complement of a fuzzy set. In 1971[6], Rosenfeld used the concept of the fuzzy set given by Zadeh to develop the theory of fuzzy group and basic properties of a fuzzy group. In 1979[2], Anthony and Sherwood redefined the concept given by Rosenfeld. In 1981[4], P.S. Das proposed level fuzzy subset and level fuzzy subgroup. In 1992[3], Bhakat and Das used the concept of fuzzy subgroup and defined fuzzy cosets. In 1994[5], J. Kim and D. Kim introduced the notion of fuzzy $p$* subgroups and defined the order of an element in a fuzzy subgroup. In 2009[1], Abraham and Sebastian fuzzify the famous theorems of Cayley and Lagrange in group theory differently. In 2013[7], Sharma defined the $\alpha$ — fuzzy subgroup.

We defined the order of the $\alpha$ — fuzzy subgroup by using the order of fuzzy subgroup given by J.kim in [5]. Lagrange’s theorem for $\alpha$ — fuzzy subgroup is defined using concept of Abraham and Sebastian in [1].

2 Preliminaries

Definition 2.1 (fuzzy set) [8] Let $X$ be any set, then a fuzzy set $\tilde{A}$ of $X$ is a set of ordered pairs:

$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$

here $\mu_{\tilde{A}} : X \rightarrow [0,1]$ is called membership function.

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy subsets of a set $X$. Then the following expression are defined in

1. $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x), \forall \ x \in X$
2. $\tilde{A} = \tilde{B}$ if and only if $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$
3. The complement of the fuzzy set $\tilde{A}$ is $\tilde{A}^c$ and is defined as $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)$
4. $\mu_{\tilde{A} \cap \tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \forall \ x \in X$
5. $\mu_{\tilde{A} \cup \tilde{B}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \forall \ x \in X$
**Definition 2.2 (fuzzy subgroup)** [6] Let $G$ be any group. $\tilde{A}$ is a fuzzy set of $G$, then $\tilde{A}$ is called fuzzy subgroup (FSG) of $G$ if

1. $\mu_{\tilde{A}}(xy) \geq \min(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)) \quad \forall \ x, y \in G$
2. $\mu_{\tilde{A}}(x^{-1}) \geq \mu_{\tilde{A}}(x), \forall \ x \in G$

**Proposition 2.3** A fuzzy set $\tilde{A}$ of a group $G$ is a FSG if and only if $\mu_{\tilde{A}}(xy^{-1}) \geq \min(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)) \forall \ x, y \in G$

**Proposition 2.4** If $\tilde{A}$ is a FSG of group $G$, then

1. $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{A}}(e), \forall \ x \in G$, where $e$ is the identity element of group $G$
2. $\mu_{\tilde{A}}(xy^{-1}) = \mu_{\tilde{A}}(e) \Rightarrow \mu_{\tilde{A}}(x) = \mu_{\tilde{A}}(y), \forall \ x, y \in G$

**Definition 2.5 (left fuzzy coset)** [3] Let $\tilde{A}$ be a FSG of group $G$. For any $x \in G$, the fuzzy set $x\tilde{A}$ defined by $\mu_{x\tilde{A}}(y) = \mu_{\tilde{A}}(x^{-1}y) \quad \forall \ y \in G$ is called a left fuzzy coset of $\tilde{A}$.

**Definition 2.6 (right fuzzy coset)** [3] Let $\tilde{A}$ be a FSG of group $G$. For any $x \in G$, the fuzzy set $\tilde{A}x$ defined by $\mu_{\tilde{A}x}(y) = \mu_{\tilde{A}}(yx^{-1}) \quad \forall \ y \in G$ is called a right fuzzy coset of $\tilde{A}$.

**Definition 2.7 (fuzzy normal subgroup)** [6] If $\tilde{A}$ is a FSG of group $G$, then $\tilde{A}$ is called a fuzzy normal subgroup (FNSG) of $G$ if $\mu_{\tilde{A}}(xyx^{-1}) \geq \mu_{\tilde{A}}(y) \quad \forall \ x, y \in G$

**Definition 2.8 (t-level subset)** [4] Let $\tilde{A}$ be a fuzzy set of a group $G$. For $t \in [0,1]$, the $t$-level subset of $\tilde{A}$ is the set $t_{\tilde{A}} = \{x \in G : \mu_{\tilde{A}}(x) \geq t\}$.

**Definition 2.9 (t-level subgroup)** [4] Let $\tilde{A}$ be a FSG of group $G$. For $t \in [0,1]$ with $t \leq \mu_{\tilde{A}}(e)$ the subgroup $t_{\tilde{A}}$ of $G$ is called $t$-level subgroup of $\tilde{A}$.

**Definition 2.10 ($\alpha$-Fuzzy Subset)** Let $A$ be a fuzzy subset of a group $G$. Let $\alpha \in [0,1]$. Then the fuzzy set $\tilde{A}^\alpha$ of $G$ is called the $\alpha$-fuzzy subset of $G$ (with respect to fuzzy set $\tilde{A}$) and is defined as $\mu_{\tilde{A}^\alpha}(x) = \min(\mu_{\tilde{A}}(x), \alpha), \forall \ x \in G$

**Definition 2.11 ($\alpha$-Fuzzy Subgroup)** If $\tilde{A}$ is an $\alpha$-fuzzy subset of group $G$. $\tilde{A}$ is called $\alpha$-fuzzy subgroup (FSG) of $G$ if $\tilde{A}^\alpha$ is a fuzzy subgroup of $G$. i.e. if the following conditions hold

1. $\mu_{\tilde{A}^\alpha}(xy) \geq \min(\mu_{\tilde{A}^\alpha}(x), \mu_{\tilde{A}^\alpha}(y)), \forall \ x, y \in G$
2. $\mu_{\tilde{A}^\alpha}(x^{-1}) = \mu_{\tilde{A}^\alpha}(x), \forall \ x \in G$.

**Definition 2.12 (fuzzy order of an element)** [5] Let $\tilde{A}$ be a fuzzy subgroup of a group $G$. Given $x \in G$, the smallest positive integer $n$ such that $\mu_{\tilde{A}}(x^n) = \mu_{\tilde{A}}(e)$ is called the fuzzy order of $x$ with respect to $\tilde{A}$. If no such $n$ exists, $x$ is said to have infinite fuzzy order with respect to $\tilde{A}$. The fuzzy order of $x$ with respect to $\tilde{A}$ is denoted by $FO_{\tilde{A}}(x)$.

**Definition 2.13 (order of a fuzzy subgroup)** [5] Let $\tilde{A}$ be a fuzzy subgroup of a group $G$. The least positive integer $n$ such that $\mu_{\tilde{A}}(x^n) = \mu_{\tilde{A}}(e), \forall \ x \in G$, is called the order of $\tilde{A}$ and denoted by $O(\tilde{A})$. If no such $n$ exists, $\tilde{A}$ is said to have an infinite order.

**Theorem 2.14 (Lagrange's theorem for fuzzy subgroups)** [1] Let $H$ be a subgroup of a group $G$ and let $n$ be the order of a fuzzy subgroup $\tilde{A}$ of $G$. then $O(\tilde{A}|_H) \mid O(\tilde{A})$. 
Proof. $O(\tilde{A}) = n$. Then $\mu_{\tilde{A}}(x^n) = \mu_{\tilde{A}}(e)$, $\forall \ x \in G$.
Now $\mu_{\tilde{A}|H}(x) = \mu_{\tilde{A}}(x)$, $\forall \ x \in H$
$\Rightarrow O(\tilde{A}|H) \leq O(\tilde{A})$.
If $O(\tilde{A}|H) = n$, then $O(\tilde{A}|H)|O(\tilde{A})$.
If $O(\tilde{A}|H) < n$, let $O(\tilde{A}|H) = m$. Then $\mu_{\tilde{A}|H}(x^m) = \mu_{\tilde{A}|H}(e)$, $\forall \ x \in H$
$\Rightarrow m|n$, i.e. $O(\mu_{\tilde{A}|H})|O(\mu_{\tilde{A}})$.

3 Lagrange’s theorem for $\alpha$ – fuzzy subgroups

Definition 3.1 ($\alpha$ – fuzzy order of an element) Let $\tilde{A}$ be a fuzzy subset of a group $G$. Let $\alpha \in [0,1]$ such that $\tilde{A}^\alpha$ is a $\alpha$ – FSG of $G$ with respect to $\tilde{A}$. Given $x \in G$, the smallest positive integer $n$ such that $\mu_{\tilde{A}^\alpha}(x^n) = \mu_{\tilde{A}^\alpha}(e)$ is called the $\alpha$ – fuzzy order of $x$ with respect to $\tilde{A}$. If no such $n$ exists, $x$ is said to have infinite $\alpha$ – fuzzy order with respect to $\tilde{A}$. The $\alpha$ – fuzzy order of $x$ with respect to $\tilde{A}$ is denoted by $FO_{\tilde{A}}(x)$.

Definition 3.2 (order of a $\alpha$ – fuzzy subgroup) Let $\tilde{A}$ be a fuzzy subset of a group $G$. Let $\alpha \in [0,1]$ such that $\tilde{A}^\alpha$ is a $\alpha$ – FSG of $G$ with respect to $\tilde{A}$. The least positive integer $n$ such that $\mu_{\tilde{A}^\alpha}(x^n) = \mu_{\tilde{A}^\alpha}(e)$, $\forall \ x \in G$, is called the order of $\tilde{A}^\alpha$ with respect to $\tilde{A}$ and denoted by $O(\tilde{A}^\alpha)$. If no such $n$ exists, $\tilde{A}^\alpha$ is said to have an infinite order.

Some Results:
1. If $\tilde{A}$ be a $\alpha$ – FSG of a group $G$ and $H$ be a subgroup of $G$, then $\tilde{A}|H$ is a $\alpha$ – FSG of $H$.
2. If $\tilde{A}$ is a FSG of a group $G$, then $\tilde{A}$ is also a $\alpha$ – FSG of $G$.
3. Intersection of two $\alpha$ – FSG’s of a group $G$ is also $\alpha$ – FSG of $G$.
4. If $\tilde{A}$ and $\tilde{B}$ be two fuzzy subset of $X$. Then $(\tilde{A} \cap \tilde{B})^\alpha = \tilde{A}^\alpha \cap \tilde{B}^\alpha$

Theorem 3.3 Let $H$ be a subgroup of a group $G$ and let $n$ be the order of a $\alpha$ – fuzzy subgroup $\tilde{A}$ of $G$. then $O(\tilde{A}^\alpha|H)|O(\tilde{A}^\alpha)$. 

Proof. $O(\tilde{A}^\alpha) = n$. Then $\mu_{\tilde{A}^\alpha}(x^n) = \mu_{\tilde{A}^\alpha}(e)$, $\forall \ x \in G$.
Now $\mu_{\tilde{A}^\alpha|H}(x) = \mu_{\tilde{A}^\alpha}(x)$, $\forall \ x \in H$
$\Rightarrow O(\tilde{A}^\alpha|H) \leq O(\tilde{A}^\alpha)$.
If $O(\tilde{A}^\alpha|H) = n$, then $O(\tilde{A}^\alpha|H)|O(\tilde{A})$.
If $O(\tilde{A}^\alpha|H) < n$, let $O(\tilde{A}^\alpha|H) = m$. Then $\mu_{\tilde{A}^\alpha|H}(x^m) = \mu_{\tilde{A}^\alpha|H}(e)$, $\forall \ x \in H$
$\Rightarrow m|n$, i.e. $O(\mu_{\tilde{A}^\alpha|H})|O(\mu_{\tilde{A}^\alpha})$.

4 some results on $\alpha$ fuzzy subgroup

Theorem 4.1 Let $\tilde{A}$ be a FSG of a group $G$. For $\alpha \in [0,1]$ with $\mu_{\tilde{A}}(e) \leq \alpha$, $\tilde{A}^\alpha = \tilde{A}$

Proof. For $\alpha \in [0,1]$ and $\mu_{\tilde{A}}(e) \leq \alpha$. Since $\tilde{A}$ is a FSG, therefore $\mu_{\tilde{A}}(xy) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$ and
$\mu_{\tilde{A}}(x^{-1}) \geq \mu_{\tilde{A}}(x)$ for all $x, y \in G$.
Now for any $x \in G$
$\mu_{\tilde{A}}(xx^{-1}) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(x^{-1})\}$
$\Rightarrow \mu_{\tilde{A}}(e) \geq \mu_{\tilde{A}}(x)$
Thus
\[
\mu_{\tilde{A}}(x) \leq \mu_{\tilde{A}}(e) \quad \forall \ x \in G
\]

(1)

\[
\Rightarrow \mu_{\tilde{A}}(x) \leq \alpha \quad \forall \ x \in G
\]

(2)

Now for any \( x \in G \)

\[
\mu_{\tilde{A}^\alpha}(x) = \min\{\mu_{\tilde{A}}(x), \alpha\}
\]

(3)

\[
\Rightarrow \mu_{\tilde{A}^\alpha}(x) = \mu_{\tilde{A}}(x)
\]

therefore \( \mu_{\tilde{A}^\alpha}(x) = \mu_{\tilde{A}}(x) \quad \forall \ x \in G \). Thus

\[
\tilde{A}^\alpha = \tilde{A}
\]

Corollary 4.2 Let \( \tilde{A} \) be a FSG of a group \( G \). For \( \alpha \in [0,1] \) with \( \mu_{\tilde{A}}(e) \leq \alpha \), \( O(\tilde{A}^\alpha) = O(\tilde{A}) \).

Corollary 4.3 Let \( \tilde{A} \) be a FSG of a group \( G \). For any \( x \in G \) and \( \alpha \in [0,1] \) with \( \mu_{\tilde{A}}(e) \leq \alpha \), \( FO_{\tilde{A}^\alpha}(x) = FO_{\tilde{A}}(x) \).

5 \( \alpha \) — fuzzy subgroup on different domain

Definition 5.1 Let \( \tilde{A} \) and \( \tilde{B} \) are fuzzy set of any set \( X \) and \( Y \) respectively. If \( X \cap Y \neq \phi \), then \( \tilde{A} \cap \tilde{B} \) is a fuzzy set of \( X \cap Y \).

Here \( \mu_{\tilde{A} \cap \tilde{B}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \forall \ x \in X \cap Y \)

Theorem 5.2 Let \( G \) be any group. \( H \) and \( K \) are subgroups of \( G \) such that \( H \cap K \neq 0 \). If for \( \alpha \in [0,1] \), \( \tilde{A} \) and \( \tilde{B} \) are the \( \alpha \) — fuzzy subgroup of \( H \) and \( K \) respectively, then \( \tilde{A} \cap \tilde{B} \) is a \( \alpha \) — fuzzy subgroup of \( X \cap Y \).

Proof. For any \( x, y \in X \cap Y \)

\[
\mu_{(\tilde{A} \cap \tilde{B})^\alpha}(xy) \geq \min\{\mu_{(\tilde{A} \cap \tilde{B})^\alpha}(x), \mu_{(\tilde{A} \cap \tilde{B})^\alpha}(y)\}
\]

(4)

and

\[
\mu_{(\tilde{A} \cap \tilde{B})^\alpha}(x^{-1}) \geq \mu_{(\tilde{A} \cap \tilde{B})^\alpha}(x)
\]

(5)

Hence \( \tilde{A} \cap \tilde{B} \) is a \( \alpha \) — FSG of \( H \cap K \).

6 Conclusion

In this paper, we have introduced the concept of order of the \( \alpha \)-fuzzy subgroup, and Lagrange’s theorem for \( \alpha \) — fuzzy subgroup. Concept of the intersection of two \( \alpha \) — fuzzy subgroup defined on different domain is discussed.

Further work is based on the intersection of \( \alpha \) — fuzzy set in different domain.

References


