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Differentiable Functions on Tensor Product of Locally Convex Spaces

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ABSTRACT

Due to Grothendieck, the linear topological dual of the spaces of bounded operators $L(X,Y)$, equipped with the topology τ of uniform convergence on compact subsets of the space X. The continuous linear functional on tensor product space $L((X,Y),\tau)$ consists of all maps θ of the form

$\theta(T) = \sum_{i=1}^{\infty}({y_i}^*, Tx_i)$, where $x_i \in X, {y_i}^* \in Y^*$ Satisfies the conditions $\sum_i^{\infty}\|x_i\|.\|{y_i}^*\| < \infty$.

On tensor product $E \otimes F$ of LCTVS X and Y; π-topology is the strongest locally convex topology for which the canonical bilinear mapping $(x,y) \rightsquigarrow x \otimes y$ of $E \times F$ into $E \otimes F$ is continuous. The space $E \otimes F$ with τ will be denoted by $E \otimes_{\pi} F$, which is Hausdorff iff E and F both are Hausdorff. For LCTVS E, the space $\mathcal{L}^{\mathrm{m}}(X\ ;\ E)$ of m-times differentiable functions $f: X \to E$; X is either a locally compact topological space($\mathrm{m} = 0$) or an open subset of R^{n}, and then m can be any integer. When X is an LCS, $\mathcal{L}^0(X\ ;\ E)$ is Banach space and then $\mathcal{L}^0(X) \otimes Y$ is dense in $\mathcal{L}^0(X\ ;\ E)$.

Keywords: Compact sets; Dense; Fréchet space ; Norms etc.

INTRODUCTION

Let f be a mapping of the open set X of R^n into the TVS E. Then f is **differentiable** at a point x^0 of X if there are n vectors $e_1, e_2, \dots \dots \dots e_n$ in E, such that

$$|x - x^0|^{-1}\{f(x) - f(x^0) - \sum_{j=1}^{n}(x_j - x_j^0)e_j\}$$

converges to zero in E as the number $|x - x^0| > 0$ converges to zero. The vectors e_j are then called the **first partial derivatives** of f at the point x^0.(Francois Treves; 1967). The collection of all m-times differentiable functions $f: X \to E$, is also a vector space which is denoted by $\mathcal{L}^{\mathrm{m}}(X\ ;\ E)$. Given an arbitrary compact subset K of X, $\mathcal{L}_{\mathrm{c}}^{\mathrm{m}}(X\ ;\ E)$ is the subspace of $\mathcal{L}^{\mathrm{m}}(X\ ;\ E)$ consisting of the functions with support contained in K, provided with the topology induced by $\mathcal{L}^{\mathrm{m}}(X\ ;\ E)$.

A **tensor product** of two vector spaces E and F , denoted by $E \otimes F$, is a pair (M, θ) where M is another vector space and θ is a bilinear mapping $\theta: E \times F \to M$ which satisfied the following conditions:

I. The θ −image of $E \times F$ spans the whole space M.
II. E and F are $\theta -$ linearly disjoint. i.e., For the finite subsets of same order, $(x_1, x_2, \dots \dots \dots x_r)$ and $(y_1, y_2, \dots \dots \dots y_r)$ of E and F respectively; $\sum_{i=1}^{r}\theta(x_i, y_i) = 0$.

Then if $x_1, x_2, \dots\dots\dots x_r$ are linearly independent implies $y_1 = y_2 = \cdots \dots\dots = y_r = 0$

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When E and F are locally convex spaces, the **projective tensor product topology** on $E \otimes F$ is the strongest locally convex topology π on $E \otimes F$ for which the bilinear map(the canonical map) $\beta: E \times F \to E \otimes F$ is continuous. The locally convex space $E \otimes F$, equipped with this topology, is called the projective tensor product of E and F and denoted by $E \otimes_\pi F$(C. Perez-Garcia and W. H. Schikhof; 2010).

The projective tensor product $E \otimes_\pi F$ is the completion of the tensor product space ($E \otimes F$,π). Every element $z \in E \otimes_\pi F$ admits a representation

$$z = \sum_{i=1}^{\infty} x_i \otimes y_i \text{such that } \sum_{i=1}^{\infty} \|x_i\| \|y_i\| < \infty$$

where, without any loss of generality, $\{\|x_i\|\}_{i=1}^{\infty} \in c_0$ and $\{\|y_i\|\}_{i=1}^{\infty} \in \ell_1$ and

$\pi(z) = \inf \{\sum_{i=1}^{\infty} \|x_i\| \|y_i\| : z = \sum_{i=1}^{\infty} x_i \oplus y_i\}$(P. H´ajek, R.J. Smith; 2012).

Tensor product of $\mathcal{L}^{\mathrm{m}}(X)$ and E is denote by $\mathcal{L}^{\mathrm{m}}(X) \otimes E$, which the subspace of $\mathcal{L}^{\mathrm{m}}(X\,;E)$, consisting of the functions whose image is contained in a finite dimensional subspace of E . Let $\theta \in \mathcal{L}^{\mathrm{m}}(X) \otimes E$, $e_1, e_2, \dots\dots, e_d$ be linearly independent vectors of E such that the image of θ is contained in the linear subspace of E they span. Thus we may write, for all $x \in X$, $\theta(x) = \theta_1(x)e_1 + \theta_2(x)e_2 + \cdots \dots\dots + \theta_d(x)e_d;\, where\ \theta_1, \theta_2, \dots\dots\dots, \theta_d$ are complex-valued functions, m times continuously differentiable.

Theorem-1 If E is complete, so is $\mathcal{L}^{\mathrm{m}}(X\,;E)$.

Proof. As E is complete, a Cauchy filter $\mathcal{F}$ on $\mathcal{L}^{\mathrm{m}}(X\,;E)$. converges pointwise to a function $f: X \to E$. Since X is locally compact therefore the convergence of the filter $\mathcal{F}$ is uniform on a neighborhood of every point because it is uniform on compact subsets of X. Thus the limit function $f: X \to E$ is continuous. This proves the result when m = 0.

Now, let us suppose that X is an open subset of R^n and that m > 0. As a matter of fact, it suffices to show that gradient of $f: X \to E$ exists, and, is a continuous function, and also, is the limit of gradient of $f: X \to E$, then, obviously, by induction on the order of differentiation easily completes the proof.

We may even suppose that we are dealing with only one variable, so as to simplify the notation. Extension to $n > 1$ variables will be evident. We have, then, for all $e' \in E', \theta \in \mathcal{L}^{\mathrm{m}}(X\,;E)$,

$$(\partial/\partial x_1)\langle e', \theta(x)\rangle = \langle e', \left(\frac{\partial}{\partial x_1}\right)\theta(x)\rangle.$$

Now, by the preceding argument, we know that $(\partial/\partial x_1)\mathcal{F}$ converges uniformly on the compact subsets of X, to a continuous function $f_1(say)$. i.e., to say that $(\partial/\partial x_1)\langle e', \mathcal{F}\rangle$ converges to $\langle e', f_1\rangle$. Thus, we conclude that the complex valued continuous function $\langle e', f_1\rangle$ is the derivative of the function $\langle e', f\rangle$ and then, we find that

$$\langle e', \frac{f(x_1+h)-f(x_1)}{h} - f_1(x_1)\rangle = \frac{1}{h}\int_{x_1}^{x_1+h} \langle e', f_1(t) - f_1(x_1)\rangle dt, \text{ where } h \neq 0.$$

Then, let us assume that U be a convex closed and balanced neighborhood of 0 in E and taking e', which is arbitrary in the polar U° of U. Because of the continuity of f_1 , we may find $|h|$ so small that $f_1(t) - f_1(x_1) \in U$ for every t in the segment joining $x_1\ to\ x_1 + h$; then the integrand, on the right-hand side, and, as a consequence, the left-hand side, have their absolute value less than and equal to 1. Which implies $h^{-1}\{f(x_1+h) - f(x_1)\} \in U + f_1(x_1)$, therefore, we conclude that f_1 is the first derivative of f . Which proves the desired.

Proposition-1 **If X is countable at infinity and E is a Fréchet space, then $\mathcal{L}^m(X\ ;\ E)$ is a Fréchet space**.

Proof. Obviously, by the definition of the topology defined on $\mathcal{L}^m(X\ ;\ E)$, that it is

metrizable whenever X is countable at infinity as it is a countable union of compact subsets and E is metrizable. And, then, theorem-1 completes the required proof.

Proposition-2 **If X is compact and E is a Banach space, then $\mathcal{L}^m(X\ ;\ E)$, equipped with the norm defined as $f \rightsquigarrow \sup_{x\in X}\|f(x)\|$, where $\|\ \ \|$ being the norm on E, is a Banach space.(** Francois Treves; 1967). Evident from above results.

Theorem-2 **If X is a locally compact space, $\mathcal{L}_c^0(X)\otimes E$ is dense in $\mathcal{L}^0(X\ ;\ E)$.**

Proof : When m = 0 , X is locally compact space.

Now, let $f\in\mathcal{L}^0(X\ ;\ E)$, and p be a continuous seminorm on E, K is a compact subset of X. We may find a finite covering $U_1, U_2, \ldots\ldots, U_r$ of K, by relatively compact open subsets of X, such that, for each j = 1, …, r,

$$p\big(f(x)-f(y)\big)<\epsilon;\ \epsilon>0\ and\ for\ each\ pair\ x,y\in U_j\quad .$$

By the general property of compact sets, we can find a continuous partition of unity subordinated to the above covering, i.e., r continuous functions in X, g_j for $1\le j\le r$; such that: (i) for each j, support of $g_j\subset U_j$ and (ii) $\sum_{j=1}^r g_j(x)=1,\ \forall\, x\in K$. In each set U_j, taking a point x_j. Then, we have, for $x\in K, f(x)=\sum_{j=1}^r g_j(x)f(x)$, and, thus,

$$\mathrm{p}\big(f(x)-\textstyle\sum_{j=1}^r g_j(x)f(x)\big)\le\sum_{j=1}^r g_j(x)p(f(x)-f(x_j))\le\epsilon\ .$$

since $p(f(x)-f(x_j))\le\epsilon$ if $x\in$ support of g_j. Hence we find the proof.

On the same , we have

Theorem-3 **If X is an open subset of R^n, $\mathcal{L}_c^m(X)\otimes E$ is dense in $\mathcal{L}^m(X\ ;\ E)$ for any positive integer m, would be positive infinite.**

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