“MHD HELE-SHAW FLOW OF AN ELASTICOVISCOUS FLUID THROUGH POROUS MEDIA”

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Abstract :-

Purpose of this paper is to study the MHD Hele-Shaw flow of an elasticoviscous fluid through porous media. Here, an attempt is made to solve the unsteady Hele-Shaw flow of viscous-elastic fluid through Porous media, assuming the pressure gradient to the proportional to exp (-mt). the velocity components are obtained and the effect of visco-elasticity is discussed on velocity components. In the end vorticity is also discussed.

Introduction :-

Main aim of this paper is to study the MHD Hele-Shaw flow of an elasticoviscous fluid through Porous media. Many research workers have paid their attention towards the study of Hele-Shaw flow. The steady Hele-Shaw flows have been studied by Buckmaster, Lee and Fung, Thompson and Lamb, Gupta have discussed unsteady Hele-Shaw flow of a non-Newtonian fluid and of a viscoelastic fluid through Porous media.

2. Formulation of the problem:

Here, we have assumed the following notations

\( u, v, w \) = components of velocity

\( \nu \) = Kinetic viscosity

\( \beta \) = Visco-elastic parameter

\( t \) = Time variable

\( \rho \) = density of fluid

\( d \) = Characteristic length

\( u_\phi \) = Velocity
Let us consider the flow of viscous elastic fluid passing through the Porous medium confined between two parallel plates located at \( Z = -d \) and \( Z = d \) (2d is very small quantity) Past a circular cylinder \( x^2 + y^2 = b^2 \), \(-d \leq z \leq d\)

The equations governing the flow are –

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  
(2.1) (continuity equation)

\[
(1-\beta V^2) \frac{\partial u}{\partial t} = \nu V^2 u - \frac{1}{\rho} \frac{\partial p}{\partial x}
\]  
(2.2)

\[
(1-\beta V^2) \frac{\partial v}{\partial t} = \nu V^2 v - \frac{1}{\rho} \frac{\partial p}{\partial y}
\]  
(2.3)

\[
(1-\beta V^2) \frac{\partial w}{\partial t} = \nu V^2 w - \frac{1}{\rho} \frac{\partial p}{\partial z}
\]  
(2.4)

Equations (2.2), (2.3) and (2.4) can be written as

\[
\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} + \beta \frac{\partial^2 (\frac{\partial u}{\partial t})}{\partial z^2} - \frac{1}{\rho} \frac{\partial p}{\partial x}
\]  
(2.5)

\[
\frac{\partial v}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} + \beta \frac{\partial^2 (\frac{\partial v}{\partial t})}{\partial z^2} - \frac{1}{\rho} \frac{\partial p}{\partial y}
\]  
(2.6)

\[
-\frac{1}{\rho} \frac{\partial p}{\partial z} = 0
\]  
(2.7)

Parameter boundary conditions:

\( u = 0, \ v = 0 \) at \( z = \pm d \)

3. **Solution of the problem**:

The non-dimensional variables which are appropriate for fluid transients are

\[
t^* = \frac{tv}{b^2}, \ u^* = \frac{u}{u_0}, \ v^* = \frac{v}{u_0}, \ p^* = \frac{bp}{\nu u_0}, \ \beta^* = \frac{\beta}{b^2}
\]

\[
z^* = \frac{z}{b}, \ x^* = \frac{x}{b}, \ y^* = \frac{y}{b} \quad \text{and} \quad d^* = \frac{d}{b}
\]
Inserting all non-dimensional quantities in (2.1), (2.5), (2.6) and (2.7) and dropping the asterisks, we obtain.

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]  
(3.1)

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial z^2} + \beta \left( \frac{\partial^2 u}{\partial t^2} \right) - \frac{\partial^2 p}{\partial x^2}
\]  
(3.2)

\[
\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial z^2} + \beta \left( \frac{\partial^2 v}{\partial t^2} \right) - \frac{\partial^2 p}{\partial y^2}
\]  
(3.3)

\[
\frac{\partial^2 p}{\partial z^2} = 0
\]  
(3.4)

The boundary conditions are:

\[
Z = \pm d, \quad u = o = v
\]  
(3.5)

By virtue of equations (3.2) and (3.3), we have-

\[
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0
\]  
(3.6)

Remark (3.1): It is noteworthy that the relations (3.4) indicates that \( p \) is independent of \( z \). Therefore, \( p \) is the function of \( x, y \) and \( t \).

Let,

\[
u = F(z, t) \frac{\partial f}{\partial x}
\]  
(3.7)

\[
v = F(z, t) \frac{\partial f}{\partial y}
\]  
(3.8)

Where \( f \) is some function of \( x \) and \( y \). Inserting equation (3.7) and equation (3.8) into equation (3.1), we obtain.

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0
\]  
(3.9)

Inserting equations (3.7) and (3.8) into (3.2) and (3.3) and integrating, we obtain

\[
p = \left( \frac{\partial^2 F}{\partial z^2} + \beta \frac{\partial^2 F}{\partial z^2 \partial t} - \frac{\partial F}{\partial t} \right)f + A
\]  
(3.10)
Where $A$ is some arbitrary function of time. Let pressure gradient be proportional to $\exp(-mt)$. In equation (3.10), we assume that

\[
\frac{\partial^2 F}{\partial z^2} + \beta \frac{\partial^3 F}{\partial z^2 \partial t} - \frac{\partial F}{\partial t} = -Ce^{-mt}
\]

(3.11)

Where $m$ is positive integer and $C$ is given constant. To solve (3.11), we try assuming $F(Z,t) = e^{-mt}\Phi(Z)$, and we have

\[
\Phi(Z) = -\frac{C}{m} \left(1 - \frac{\cos b_d Z}{\cos b_d d}\right)
\]

Where $b_d = \frac{m}{1 - Bm}$

(3.12)

\[
F(Z,t) = -\frac{Ce^{-mt}}{m} \left(1 - \frac{\cos b_d Z}{\cos b_d d}\right)
\]

(3.13)

Thus,

Now, the function $f(x, y)$ can be evaluated by (3.9) subject to the condition.

\[
u \cos \theta + v \sin \theta = 0, \text{ When } r = b
\]

(3.14) or

\[
\frac{\partial u}{\partial r} = 0
\]

(3.15) We have,

\[
x = r \cos \theta
\]

(3.16) and

\[
y = r \sin \theta
\]

and

\[
\frac{\partial f}{\partial n} \rightarrow 1, \frac{\partial f}{\partial y} = 0
\]

squaring equations (3.15) and (3.16) and adding, then we get

\[
r^2 = x^2 + y^2
\]

(3.17)

Differentiation partially with respect to “$x$”, then

\[
\frac{\partial r}{\partial x} = \frac{x}{r}
\]

(3.18)

\[
\frac{\partial r}{\partial x} = \cos \theta
\]

Again, differentiation partially with respect to “$y$”
\[ \frac{\partial r}{\partial y} = \sin \theta \]  

(3.19) i.e.  \[ \frac{\partial y}{\partial y} \]

Now, dividing equation (3.16) by equation (3.15), then

\[ \theta = \tan^{-1} \left( \frac{y}{x} \right) \]

(3.20)

Differentiation equation (3.20) partially with respect to “x” and “y”

\[ \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{\sin \theta}{r} \]

and

\[ \frac{\partial \theta}{\partial y} = -\frac{x}{x^2 + y^2} = \frac{\cos \theta}{r} \]

Since \(|x/y| \rightarrow \infty\)

Therefore,

\[ : f(x, y) = \left( r + \frac{1}{r} \right) \cos \theta \]

(3.21)

Differentiating partially with respect to “x”

\[ \frac{\partial f}{\partial x} = \left( r + \frac{1}{r} \right) \sin^2 \theta + \cos^2 \left( \frac{1}{r^2} \frac{\partial r}{\partial x} \right) \]

(3.22)

Again, differentiating equation (3.21) partially with respect to “y”

\[ \frac{\partial f}{\partial y} = -\left( r + \frac{1}{r} \right) \sin \theta \frac{\partial \theta}{\partial y} + \cos \left( \frac{\partial r}{\partial y} - \frac{1}{r^2} \frac{\partial r}{\partial y} \right) \]

\[ \frac{\partial f}{\partial y} = -\left( r + \frac{1}{r} \right) \frac{\sin \theta \cos \theta}{r} + \cos \left( \sin \theta - \frac{\sin \theta}{r^2} \right) \]

\[ \frac{\partial f}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2} \]

(3.23)

Putting the value of \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) from equations (3.22) and (3.23) into equations (3.7) and (3.8), then we get

\[ u = A_r e^{-\beta m} \left( 1 - \frac{\cos \beta Z}{\cos \beta d} \right) \left[ 1 - \frac{(x^2 - y^2)}{(x^2 + y^2)^2} \right] \]

(3.24)
\[ v = A_i e^{-\eta t} \left( 1 - \frac{\cos b_i Z}{\cos b_i d} \left[ \frac{-2xy}{(x^2 + y^2)^2} \right] \right) \]
(3.25) and

Now introducing a new function as vorticity function \( \xi \) and given by

\[ \xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]

Putting the value of \( \frac{\partial v}{\partial x} \) and \( \frac{\partial u}{\partial y} \) from equations (3.25) and (3.24), then we get

\[ \xi = 4A_i e^{-\eta t} \left( 1 - \frac{\cos b_i Z}{\cos b_i d} \left[ \frac{y}{(x^2 + y^2)^2} \right] \right) \]

References:-


