FUZZY SUPRA STRONGLY HYPER CONNECTED SPACES AND FUZZY SUPRA BAIRES SPACES

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Abstract: In this paper, the concept of fuzzy supra strongly hyperconnected spaces are studied. Finding the relations between fuzzy supra strongly hyperconnected spaces and other fuzzy supra topological spaces. Illustrate the concept with suitable examples.

Keywords - Fuzzy supra nowhere dense set, Fuzzy supra first category, Fuzzy supra P-space, Fuzzy supra submaximal space, Fuzzy supra hyperconnected spaces, Fuzzy supra Baire spaces.

I. INTRODUCTION


II. PRELIMINARIES

Definition 2.1 [8]
A fuzzy set λ in a fuzzy supra topological space (X,T) is called a fuzzy supra dense if there exists no fuzzy supra closed set μ in (X,T) such that λ<μ<1. That is, cl*(λ)=1, in (X,T).

Definition 2.2 [8]
A fuzzy set λ in fuzzy supra topological space (X,T) is called a fuzzy supra nowhere dense set if there exists no non-zero fuzzy supra open set μ in (X,T) such that μ<cl*(λ). That is, int*cl*(λ)=0, in (X,T).

Lemma 2.1 [1]
For a fuzzy set λ of a fuzzy supra topological space (X,T),
(i) 1–int* (λ) = cl* (1–λ),
(ii) 1–cl* (λ) = int* (1–λ).

Definition 2.3 [6]
A fuzzy set λ in a fuzzy supra topological space (X,T) is called a fuzzy supra first category set if λ = ∨i=1∞(λi), where (λi)'s are fuzzy supra nowhere dense set in (X,T). Any other fuzzy set in (X,T) is said to be fuzzy supra second category.

Definition 2.4 [6]
A fuzzy supra topological space (X,T) is called a fuzzy supra P-space if every non zero fuzzy supra Gδ set in (X,T) is fuzzy supra open in (X,T). That is, if λ = ∨i=1∞ (λi) where λi ∈(X,T) for i ∈ I.

Definition 2.5 [7]
A fuzzy set λ in a fuzzy supra topological space (X,T) is called a fuzzy supra Gδ set if λ = ∧i=1∞ (λi), where λi ∈ T for i ∈ I.

Definition 2.6 [7]
A fuzzy supra topological space (X,T) is called a fuzzy supra Baire space if ∨i=1∞(λi)=0, where (λi)'s are fuzzy supra nowhere dense set in (X,T).
III. FUZZY SUPRA HYPERCONNECTED SPACE

Definition 3.1 [6]
A fuzzy supra topological space \((X, T^*)\) is said to be fuzzy supra hyper connected if every non-null fuzzy supra open set of \(X\) is fuzzy supra dense in \((X, T^*)\).

Definition 3.2 [5]
A fuzzy set \(\lambda\) in fuzzy supra topological space \((X, T^*)\) is called a fuzzy supra semiopen if there exists a fuzzy supra open set \(\mu \subseteq T^*\) such that \(\mu \leq \lambda \leq cl^*(\mu)\).

Definition 3.3
In a fuzzy supra topological space \((X, T^*)\) is said to be fuzzy supra connected if \(X\) cannot be represented as the union of two non empty disjoint open fuzzy sets.

Definition 3.4
A fuzzy supra topological space \((X, T^*)\) is called a fuzzy supra extremally disconnected space if \(cl'(\lambda)=\lambda\), where \(\lambda \in T^*\).

Example 3.1:
Let \(X=\{a, b, c\}\). Then the fuzzy sets \(\alpha, \beta\) and \(\gamma\) are defined on \(X\) as follows:
\[
\begin{align*}
\alpha &: X \rightarrow [0, 1] \text{ defined as } \alpha(a) = 0.4, \alpha(b) = 0.6, \alpha(c) = 0.7. \\
\beta &: X \rightarrow [0, 1] \text{ defined as } \beta(a) = 0.3, \beta(b) = 0.2, \beta(c) = 0.7. \\
\gamma &: X \rightarrow [0, 1] \text{ defined as } \gamma(a) = 0.6, \gamma(b) = 0.4, \gamma(c) = 0.3.
\end{align*}
\]
Then, \(T^*=\{0, \alpha, \beta, \alpha \lor \beta, 1\}\) is a fuzzy supra open sets in \((X, T^*)\). Now, we see that \(cl'(\alpha)=\alpha, cl'(\beta)=\gamma\) and \(cl'(\alpha \lor \beta)=\alpha \lor \beta\).

Therefore \((X, T^*)\) is fuzzy supra extremally disconnected space.

Proposition 3.1
Every fuzzy supra hyper connected space \((X, T^*)\) is fuzzy supra extremally disconnected.

Proof.
Suppose that \((X, T^*)\) is fuzzy supra hyper connected space. Then for any fuzzy supra open set \(\beta, cl'(\beta)=1\), which implies that \(cl'(\beta)\) is fuzzy supra dense and as a consequence the space \((X, T^*)\) is fuzzy supra extremally disconnected.

Remark 3.1
The converse of the above proposition need not be true. That is, Every fuzzy supra extremally disconnected space \((X, T^*)\) need not be a fuzzy supra hyper connected space.

For consider the example 3.1, \((X, T^*)\) is fuzzy supra extremally disconnected space. But \(cl'(\alpha)= cl'(\beta)= cl'(\alpha \lor \beta)\neq 1\). Therefore, \(\alpha, \beta, \alpha \lor \beta\) are fuzzy supra open set but not fuzzy supra dense set in \((X, T^*)\). Hence, \((X, T^*)\) is not a fuzzy supra hyper connected space.

Proposition 3.2
Let \((X, T^*)\) be a fuzzy supra topological space. Then the following conditions are equivalent:

(i) \((X, T^*)\) is fuzzy supra hyperconnected spaces,

(ii) Every fuzzy supra preopen set is fuzzy supra dense set.

Proof.
(i) \(\Rightarrow\) (ii)
Let us assume that \(\alpha\) is fuzzy supra pre-open set in \((X, T^*)\), this implies that \(\alpha \leq int^* cl'(\alpha)\). By hypothesis, \((X, T^*)\) is fuzzy supra hyper connected space, we have \(cl'(\alpha)= cl'(int^* cl'(\alpha))=1\). Therefore \(\alpha\) is fuzzy supra dense set.

(ii) \(\Rightarrow\) (i)
Let \(\alpha\) be any fuzzy supra pre-open set, thus \(\alpha \leq int^* cl'(\alpha)\). By hypothesis, given that \(\alpha\) is fuzzy supra dense set. Therefore \(cl'(\alpha)= cl'(int cl(\alpha))=1\). It follows that \((X, T^*)\) is fuzzy supra hyper connected space.

Proposition 3.3
Arbitrary union of fuzzy supra hyper connected set of \(X\) is fuzzy supra hyper connected set.

Proof.
Let \(\{\alpha_{i}/i \in \Lambda\}\) be a collection of fuzzy supra hyper connected sets of \(X\).
Then, for each \(i \in \Lambda\), we have \(\alpha_i\).
\[
\begin{align*}
\alpha_i &\leq cl'(\alpha_i) \lor int cl'(\alpha_i) \\
\forall j \in \Lambda &\leq \lor (cl'(\alpha_i) \lor int cl'(\alpha_i)) \\
= cl'(\lor \alpha_i) \lor int cl'(\lor \alpha_i) &\text{[Since, }\lor \alpha_i \leq cl'(\lor \alpha_i)\text{ and } int(\alpha_i) \leq int'(\lor \alpha_i)]
\end{align*}
\]
Now, \(cl'(\lor \alpha_i) \leq cl'(\lor \alpha_i) \lor int cl'(\lor \alpha_i)\) \[Inequality holds in respect of closure property\]
\[
= [cl'cl'(\lor \alpha_i) \lor int cl'(\lor \alpha_i)]
\]
Therefore, \(cl'(\lor \alpha_i) \lor int cl'(\lor \alpha_i) = 1\).

Hence, \(cl'(\lor \alpha_i) = 1\).

Implies that \((\lor \alpha_i)\) is a fuzzy supra hyperconnected in a fuzzy supra topological spaces \((X, T^*)\).

Proposition 3.4
A fuzzy supra topological spaces \((X, T^*)\) is fuzzy supra hyper connected space, for every non null fuzzy supra open set \(\alpha\) of \(X\), if \(cl'(\alpha) \lor int cl'(\alpha) \leq 1\).

Proof.
Let \(\beta\) be a fuzzy set in \((X, T^*)\). For every non null fuzzy supra open subset \(\alpha \leq cl'(\alpha) \lor int cl'(\alpha)\). Since \(\alpha\) is hyper connected in \((X, T^*)\). Therefore, \(cl'(\alpha)=1\), \(cl'(\alpha) \leq cl'(cl'(\alpha) \lor int cl'(\alpha))\), \(1=cl'(cl'(\alpha) \lor int cl'(\alpha))\). This implies \(cl'(cl'(\alpha) \lor int cl'(\alpha))=1\) \[\alpha\ is finite fuzzy supra open set, \(cl'(\alpha) \geq cl'(\alpha)\]. Hence, \(cl'(\alpha) \lor int cl'(\alpha) \leq 1\).

Proposition 3.5
A fuzzy supra topological spaces \((X, T^*)\) is fuzzy supra hyper connected for non null finite fuzzy supra closed sets \(\beta\) of \(X\) if \(cl'(\beta) \lor int cl'(\beta) \leq 1\).

Proof.
Let \(\beta\) be fuzzy supra closed set. Then by definition, we have \(\beta \geq cl'(\beta) \lor int cl'(\beta), 1 \leq 1\). Therefore, \(int'(1-\beta) \leq int'[1 - (cl'(\beta) \lor int cl'(\beta)), 1 -cl'(\beta) \leq 1 - int(cl'(\beta) \lor int cl'(\beta))\]. Since \(\beta\) is fuzzy supra hyper connected,
cI′(β) = 1 and cI′(β) ⊆ ∧cI′(β). Therefore, 1−1 ≤ 1−cIcI′(β) ∧ cI′cI′(β), 0 ≤ 1−cIcI′(β) ∧ cI′cI′(β). This implies, cI′cI′(β) ∧ cI′cI′(β) ≤ 1.

IV. FUZZY SUPRA STRONGLY HYPERCONNECTED SPACE

Definition 4.1

In a fuzzy supra topological spaces \((X,T)\), a fuzzy set \(λ\) is said to be fuzzy supra strongly nowhere dense set, if \(\lambda \lambda (1−\lambda)\) is a fuzzy supra nowhere dense set in \((X,T)\). That is, \(\text{int}(\text{cl}^* \{ \lambda \wedge (1−\lambda)\}) = 0\), in \((X,T)\).

Definition 4.2

In a fuzzy supra topological spaces \((X,T)\) a fuzzy set \(λ\) is said to be a fuzzy supra strongly first category set if \(\lambda = \bigwedge_{i=1}^{\infty} (\lambda_i)\), where \((\lambda_i)\)’s are fuzzy supra strongly nowhere dense sets in \((X,T)\). Any other fuzzy set in \((X,T)\) is said to be a fuzzy supra strongly second category set in \((X,T)\).

Definition 4.3

A fuzzy supra topological space \((X, T)\) is called a fuzzy supra strongly Baire space if \(\text{cl}(\bigwedge_{i=1}^{\infty} (\lambda_i))=1\), where \((\lambda_i)\)’s are fuzzy strongly nowhere dense sets in \((X,T)\).

Definition 4.4

A fuzzy supra topological space \((X, T)\) is called a fuzzy supra strongly hyper connected space, if the following conditions hold:

(i) if \(λ\) is a fuzzy supra dense set in \((X, T)\), then \(λ\) is a fuzzy supra open set in \((X, T)\) and

(ii) if \(λ\) is a fuzzy supra open set in \((X, T)\), then \(λ\) is a fuzzy supra dense set in \((X, T)\).

Example 4.1:

Let \(X = \{a, b, c\}\). Then the fuzzy sets \(α, β\) and \(γ\) are defined on \(X\) as follows:

\(α: X \rightarrow [0,1] \) defined as \(α(α) = 0.7, α(b) = 0.8, α(c) = 0.9\).

\(β: X \rightarrow [0,1] \) defined as \(β(α) = 0.6, β(b) = 0.9, β(c) = 0.8\).

\(γ: X \rightarrow [0,1] \) defined as \(γ(α) = 0.6, γ(b) = 0.7, γ(c) = 0.8\).

Then, \(T = \{0, α, β, γ, α∧β, α∧β, 1\}\), is a fuzzy supra open sets in \((X,T)\). All the fuzzy supra open sets are fuzzy supra dense sets in \((X,T)\) and all the fuzzy supra dense sets are fuzzy supra open sets in \((X,T)\). Hence \((X,T)\) is a fuzzy supra strongly hyperconnected space.

Definition 4.5

A fuzzy supra topological space \((X, T)\) is called a fuzzy supra submaximal space if for each fuzzy set in \((X,T)\) such that \(\text{cl}(λ) = 1\), then \(λ \text{cl} T \in (X, T)\).

Proposition 4.1:

If \((X,T)\) is a fuzzy supra strongly hyper connected space, then

(i) \((X,T)\) is a fuzzy supra hyper connected space,

(ii) \((X,T)\) is a fuzzy supra submaximal space.

Proof.

(i) Let \(μ\) be a fuzzy supra open set in \((X,T)\). Since \((X,T)\) is a fuzzy supra strongly hyper connected space, the fuzzy supra open set \(μ\) is a fuzzy supra dense set in \((X,T)\) and hence \((X,T)\) is a fuzzy supra hyper connected space.

(ii) Let \(λ\) be a fuzzy supra dense set in \((X,T)\). Since \((X,T)\) is a fuzzy supra strongly hyper connected space, the fuzzy supra dense set \(λ\) is a fuzzy supra open set in \((X,T)\) and hence \((X,T)\) is a fuzzy supra submaximal space.

Proposition 4.2

If \(λ\) is a fuzzy supra dense set in a fuzzy supra strongly hyper connected space, then \(1−λ\) is a fuzzy supra nowhere dense set in \((X,T)\).

Proof.

Let \(λ\) be a fuzzy supra dense set in \((X,T)\). Then, \(\text{cl}(λ) = 1\) in \((X,T)\). Since \((X,T)\) is a fuzzy supra strongly hyper connected space, \(λ\) is a fuzzy supra open set in \((X,T)\) and hence \(1−λ\) is a fuzzy supra closed set in \((X,T)\). Then, \(\text{cl}(1−λ)=1−λ\), in \((X,T)\). This implies that \(\text{int} \text{cl}(1−λ)=\text{int}(1−λ)=1−\text{cl}(λ)=1−1=0\), in \((X,T)\). Hence \(1−λ\) is a fuzzy supra nowhere dense set in \((X,T)\).

Proposition 4.3

If \(λ=\bigwedge_{i=1}^{\infty} (λ_i)\), where \((λ_i)\)’s are fuzzy supra dense sets in a fuzzy supra strongly hyper connected space \((X,T)\), then \(1−λ\) is a fuzzy supra first category set in \((X,T)\).

Proof.

Let \(λ=\bigwedge_{i=1}^{\infty} (λ_i)\), where \((λ_i)\)’s \((i = 1\) to \(\infty)\) are fuzzy supra dense sets in \((X,T)\). Then, by proposition 4.2, \((1−λ)\)’s are fuzzy supra nowhere dense sets in \((X,T)\). Now \(\bigwedge_{i=1}^{\infty}(1−λ_i)=\text{fuzzy supra first category set in } \((X,T)\)\). But \(\bigwedge_{i=1}^{\infty}(1−λ_i)=1−\bigwedge_{i=1}^{\infty}(λ_i)=1−λ\). Thus \(1−λ\) is a fuzzy supra first category set in \((X,T)\).

Proposition 4.4

If \(λ=\bigwedge_{i=1}^{\infty} (λ_i)\), where \((λ_i)\)’s are fuzzy supra dense sets in a fuzzy supra strongly hyper connected space \((X,T)\), then \(λ\) is a fuzzy supra residual set in \((X,T)\).

Proof.

Let \(λ=\bigwedge_{i=1}^{\infty} (λ_i)\), where \((λ_i)\)’s \((λ)\)’s are fuzzy supra dense sets in \((X,T)\). Since \((X,T)\) is a fuzzy supra strongly hyper connected space, by proposition 4.3, \(1−λ\) is a fuzzy supra first category set and hence \(λ\) is a fuzzy supra residual set in \((X,T)\).

Proposition 4.5

If \(λ\) and \(μ\) are any two non-zero fuzzy open sets in a fuzzy supra strongly hyper connected space \((X,T)\), then \(λ ∧ μ ≠ 0\), in \((X,T)\).

Proof.

Suppose that \(λ ∧ μ = 0\), where \(λ, μ ∈ T\). Then, \(λ ≤ 1−μ\) in \((X,T)\). This implies that \(cI′(μ) ≤ cI′(1−μ)\), in \((X,T)\). Since \((X,T)\) is a fuzzy supra strongly hyper connected space, the fuzzy supra open set \(λ\) is a fuzzy supra dense set in \((X,T)\). Then \(1 ≤ cI′(1−μ)\) in \((X,T)\). That is, \(cI′(1−μ)=1\) and then by lemma 2.1, \(1−\text{int}(μ)=1\) in \((X,T)\). This implies that \(\text{int}(μ)=0\), a contradiction to \(μ ∈ T\) for which \(\text{int}(μ)=μ ≠ 0\). Hence, \(λ ∧ μ ≠ 0\), in \((X,T)\).
Proposition 4.6
If int(λ) = 0 and int(µ) = 0 for any two non-zero fuzzy sets defined on X in a fuzzy supra strongly hyperconnected space (X, T'), then int(λ ∨ µ) = 0, in (X, T').

Proof.
Let λ and µ be two non-zero fuzzy sets defined on X such that int(λ) = 0 and int(µ) = 0 in (X, T'). We want to prove that int(λ ∨ µ) = 0. Assume the contrary and suppose that int(λ ∨ µ) ≠ 0 in (X, T'). Then, there exists a fuzzy supra open set δ in(X, T') such that δ ≤ λ ∨ µ. Then cl(δ) ≤ cl(λ ∨ µ) in (X, T'). Since (X, T') is a fuzzy supra strongly hyper connected space, for the fuzzy supra open set δ in (X, T'), cl(δ) = 1, in (X, T'). This implies that 1 ≤ cl(λ ∨ µ) in (X, T'). That is, cl(λ ∨ µ) = 1, in (X, T'). If cl(λ) = 1, in the fuzzy strongly hyper connected space (X, T') then λ will be a fuzzy supra open set in (X, T') for which int(λ) = λ and this will contradict the fact that int(λ) = 0. Hence cl(λ) ≠ 1, in (X, T'). Similarly, cl(µ) ≠ 1, in (X, T'). Now 1− cl(λ) ≠ 0 and 1− cl(µ) ≠ 0 in (X, T') and [1−cl(λ)]∩[1−cl(µ)] = 1− cl(λ ∨ cl(µ)) = 1− cl(λ ∨ µ) = 1−1 = 0. Thus, 1− cl(λ) and 1− cl(µ) are non-zero fuzzy supra open sets in (X, T') with [1− cl(λ)] ∩ [1− cl(µ)] = 0. But this is a contradiction [by proposition 4.5]. Hence int(λ ∨ µ) = 0, in (X, T').

V. FUZZY SUPRA HYPERCONNECTED SPACE AND FUZZY SUPRA BAIRE SPACE

Proposition 5.1
If cl(λ [˅ i=1 µ]) = 1, where (λi)'s are fuzzy supra dense set in a fuzzy supra submaximal space (X, T'), then (X, T') is a fuzzy supra Baire space.

Proof.
Let (λi)'s be fuzzy supra dense sets in a fuzzy supra submaximal space. Then λi cl(λ) in (X, T'). Now cl(λ) = 1 and int(λ) = λi implies that cl int(λ) = 1 Then we have 1− cl int(λ) = 0. This implies that int cl(1− λ) = 0. Hence (1− λi)'s are fuzzy supra nowhere dense sets in (X, T'). Now cl(1− λ) = 1, implies that 1− cl(1− λ) = 0, then int(1− λ) = 0, where (1− λi)'s are fuzzy supra nowhere dense sets in (X, T'). Hence, (X, T') is a fuzzy supra Baire space.

Proposition 5.2
If cl(λ [˄ i=1 µ]) = 1, where (λi)'s are fuzzy supra dense set in a fuzzy supra strongly hyperconnected space (X, T'), then (X, T') is a fuzzy supra Baire space.

Proof.
Let (λi)'s be fuzzy supra dense sets in a fuzzy supra strongly hyperconnected space (X, T'). By proposition 4.1, Every fuzzy supra strongly hyperconnected space is fuzzy supra submaximal space. Therefore (λi)'s be fuzzy supra dense sets in a fuzzy supra submaximal space with cl(λ [˄ i=1 µ]) = 1. Hence by proposition 5.1, (X, T') is a fuzzy supra Baire space.

Remark 5.1
The converse of the above preposition is need not be true. That is, A fuzzy supra topological spaces (X, T*) is a fuzzy supra Baire space then (X, T*) need not be a fuzzy supra strongly hyperconnected space. For consider the example.

Example 5.1
Let X = {a,b,c}. Then the fuzzy sets λ, µ and γ are defined on X as follows:
λ: X → [0,1] defined as λ(a) = 1, λ(b) = 0, λ(c) = 0.3.
µ: X → [0,1] defined as µ(a) = 0.6, µ(b) = 1, µ(c) = 0.7.
γ: X → [0,1] defined as γ(a) = 0.8, γ(b) = 0, γ(c) = 1.
Then, T = {0, λ, µ, γ, λ ∨ µ, µ ∨ γ, λ ∨ µ ∨ γ, λ ∨ µ ∨ γ, λ ∨ µ ∨ γ, 1}, is a fuzzy supra open sets in (X, T). The fuzzy sets 1− λ, 1− µ, 1− γ, 1− (λ ∨ µ), 1− (µ ∨ γ), 1− (λ ∨ γ), 1− (λ ∨ µ ∨ γ), 1− (µ ∨ λ ∨ γ), 1− (λ ∨ µ ∨ γ) are fuzzy supra nowhere dense sets in (X, T). Also, int*[1− λ] ∩ (1− µ) ∩ 1− (λ ∨ µ) ∩ 1− (µ ∨ γ) ∩ 1− (λ ∨ γ) ∩ 1− (λ ∨ µ ∨ γ) ∩ 1− (µ ∨ λ ∨ γ)] ≠ 0. Therefore (X, T') is a fuzzy supra Baire space. But the fuzzy supra open sets {λ, µ, γ, λ ∨ µ, µ ∨ γ, λ ∨ µ ∨ γ, λ ∨ µ ∨ γ, λ ∨ µ ∨ γ} are not a fuzzy supra dense sets in (X, T'). Therefore (X, T') is not a fuzzy supra strongly hyperconnected space.

Proposition 5.3
If a fuzzy supra P-space in a fuzzy supra hyper connected space (X, T') is a fuzzy supra Baire space.

Proof.
Let α be a fuzzy supra Gδ set in fuzzy supra P-space in (X, T'). Since, (X, T') is fuzzy supra hyper connected space. Therefore fuzzy supra open set α is a fuzzy supra Gδ set in (X, T'). That is, cl(α) = 1. Since α is fuzzy supra dense set in a fuzzy supra P-space. Therefore, (1− α) is a fuzzy supra first category set in (X, T'). Therefore, (1− α) = ∪ i=1 (αi), where (αi)'s are fuzzy supra nowhere dense sets in (X, T'). Then int(∪ i=1 (αi)) = int(1− α) = 1− cl(α) = 1−1 = 0. Hence, int(∪ i=1 (αi)) = 0. Where (αi)'s are fuzzy supra nowhere dense set in (X, T'). Then (X, T') is fuzzy supra Baire space.
REFERENCES


