A STUDY ON TOPOLOGICAL CHARACTERIZATIONS OF LINEARITY

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ABSTRACT

The space having uniformities with a totally ordered base are characterized in bigger classes of non-archimedean space and suboderable space, consequentey serveral new matrization results are obtained by example,we show that the condition used in our main theorem cannot be weakened essentially,our example may be interesting else where. In this paper we shall give the most important topological porperties of semi-linear uniform space( )

INTRODUCTION

All the spaces under consideration are supposed to be $T_1$- spaces. If the $x$ is a space, then $X'$ is the set of all cluster points of $X, \varphi X,Xx, cX, dX, wX$, denote the pseudocharacter, character, cellularity, density weight of $X$, respectively. Define ad $X \geq x$ if any intersection of less than $x$ open sets in $X$ is open.

A topological space $X$ is called linearly uniformizable if there is uniformity generating the topology of $X$ and having a linearly ordered base (with respect to inclusion if we regard uniformities as entourages of diagonals or with respect to, refinement if we regard uniformities as covers) obviously, any metrizable topological spaces is linearly uniformizable.

As it is well-known, a metric $d$ on a space $X$ is called a non-archimedean metric if its satisfies the strong triangular inequality $d(x, z) \leq \max^* d(x, y), d(x, y)+$ for all $x, y, z \in X$. It is easy to see that then any two balls $(x), B(y)$ are either disjoint or one contains the other.
Hence a topological spaces $X$ is called a non-archimedean topological space if it has a base $B$ such that any two members of $B$ are either disjoint or comparable by inclusion.

By result of a spaces $X$ is non-archimedeanly metrizable iff it is a metrizable non-archimedean topological space. Proved the metrizable space is non-archimedeanly metrizable iff $\text{Ind} X = 0$.

One can show that non-archimedean topological spaces $X$ are always strongly zero dimensional but, conversely, The sorgenfrey –line $S$ satisfies $\text{Ind} S = 0$ and it is not a non-archimedean topological space. The Michael line is an example of a non-archimedean topological space with is not metrizable .every non-archimedean topological spaces is paracompact hence here ditarily para compact.

Let $E$ be a Banach space and $X$ a $T_1$-space. we shall consider a function $F(x)$ from $X$ into the space $2^E$ of all subsets of $E$ and assume it to be lower semi-continuous in the following sense: the set $* x \in X / F(x) \cap U \forall U$ is open $X$ for every open subset $U$ of $E$.

In this paper we shall give the most important topological properties of uniform space which still valid in semi-linear uniform spaces.
CHAPTER-II

TOPOLOGICAL CHARACTERIZATION OF UNIFORMABLE SPACE

THE MAIN RESULT

A natural problem is to characterize those topological spaces which are linearlyuniformizable. One of the first solution was given by who generalized the metrizability theorem of Nagata – Smirnov to higher cardinals. Proved a lot of further $x$ – matrization theorems yield

"classical" metrization theorems if specialized to $x = \omega$, even though the proofs, in most cases, have to use different methods for uncountable $x$. But, in some sense, this situation is unsatisfying: As we shall indicate by examples, and as it seen by the result below, the theory of non-metrizable linearly uniformizable spaces is not just a direct generalization of metric spaces, it has many autonomous and independent aspects which do not have analogues for metrizable spaces. It is this aspect which we want to emphasize in this paper; and in this sense, our theorems characterization linear uniformizibility seem to be more adept to the whole concept. If one specializes our theorems to the countable case, one does not get general metrizability theorems but one obtains necessary and sufficient conditions for a space to be strongly zerodimensional and metrizable. So, from this point of view, one could consider the theory of linearly uniformizable spaces to be a generalization of strongly zero dimensional metric spaces, but again, this view is not justified completely, too. Again there are characteristic differences between the countable and the non-countable case: in the countable case, these spaces are orderable, whereas this is not true generally in the uncountable case; Baire’s theorem holds for complete metric spaces but not for every complete $\omega_1$-metrizable space.

As we could see above, there are interesting relations between the investigated classes and ordered spaces. We will try to find more about those relations. But at first we want to assemble several properties needed here after:

i) If $X$ is a (strongly) sub orderable space, then $\text{ind} \ X = 0$ iff $\text{Ind} \ X = 0$. It is easy to prove this assertion by modifying a proof that every (sub)-orderable space is normal, or to derive it from results.

ii) If every point of a space $X$ has a monotone base of its neighborhoods and $\underset{x}{(x)} = \underset{\Delta x}{(\Delta x)}$ for every non isolated $x \in X$, then $\text{ad} X = \text{ad} (\Delta X)$.

iii) If $X$ is sub orderable and $\text{ad} X = (\Delta X)$, then $X$ is paracompact.
Perhaps, the last statement needs a proof for non-discrete X: If X is not paracompact then, by X contains a closed homeomorphic copy of a stationary set of an uncountable regular x and since such a set cannot be discrete we have, x > ad X. But the diagonal of no stationary set in X is an intersection of less than x is open set (every such an open set must contain a square of a „tail”). For further information on sub orderable spaces.

Trying to characterize linearly uniformizable spaces X, one usually gets problems with the set of the isolated points x ∈ X. Therefore, we need some special condition for „handing” the isolated points. In fact, we shall need condition guaranteeing that there are not too many isolated points too close by the derived set X′ of X. It turns out that a property, which we call

„property (B)”, is an appropriate one. The background of property (B) is in the theory of generalized metric spaces. In Hodel introduced the following concept: X is said to be a β-space if for every point x ∈ X, there is a family *\( U_n(x)/n \in \omega^+ \) of open neighborhoods of x such that β*\( U_n(x)/n \in \omega^+ \) if *\( U_n(x)/n \in \omega^+ \) then *\( x_n \) has an accumulation point. We shall use here the following generalization:

**DEFINITION: 2.1**

A space X is said to have property B, λ a cardinal, if for every x ∈ X, there is a family *\( U_\alpha(x)/\alpha \in \lambda^+ \) of open neighborhoods of x such that β*\( U_\alpha(x)/\alpha \in \lambda^+ \) if*\( \forall \alpha \in \lambda^+ \) contains a complete accumulation point.

Clearly, any β-space and hence every metrizable space has B_\( \omega \); every x-uniformizable space has B_x; every dense-in-itself space X has B_\( \vartheta \); any λ-stratifiable space has B_\( \lambda \); every space X has B_\( |K| \), and every discrete space has B for any \( \lambda G_0 \).

If X has B and & is an open base for X then we may suppose that the above *\( U_\alpha(x) \) belong to &. The Michel line X is a first-countable space which does not have property B_\( \omega \) as follows from corollary.
THEOREM: 2.2

For any topological space, the following condition are equivalent:

a) X is either linearly uniformizable and non-uniformizable, or X is metrizable, or X is metrizable and strongly zero dimensional.

b) X is a non-archimedean topology space ; \( \varphi(x) = \varphi(\Delta x) \) for every non isolated \( x \in X \); and X has property \( \langle \Delta K \rangle \).

c) X is a non-archimedean topological space; \( \langle x \rangle = \varphi(\Delta x) \) for every non isolated point \( x \in X \); and X is a \( G_{\varphi(\Delta K)} \) set in X.

d) X is strongly sub orderable, zero dimensional; \( \text{ad}(X) = \varphi(\Delta X) \); and \( X' \) is a \( \langle \Delta K \rangle \)-set in X.

(\( X' \) denotes the set of non isolated points of X.)

Proof:
The implication (a) \( \Rightarrow \) (b) follows from the above observations. (b) \( \Rightarrow \) (c).

Suppose that X satisfies (b), \( \Delta X = B^*V_{\alpha} / \alpha \in \lambda^+ \), \( \lambda = \varphi(\Delta X) \), \( V_\alpha \) open in \( X\times X \) and that & is a non-archimedean base for X. We may also assume that the sets \( U_\alpha (x) \) from property \( B \) belongs to& and \( U_\alpha (x) \times U_\alpha (x) < V_\alpha \) and \( U_\alpha (x) = *x+ \) if \( x \) is isolated.

Now, \( B^*U^*U_\alpha (x)/x \in X' / \alpha \in \lambda^+ \); indeed, if the last set contains an isolated point \( x \), say \( x \in B^*U_\alpha (x_\alpha) / \alpha \in \lambda^+ \), \( x_\alpha \in X' \); the \( \{x_\alpha \} \) has a complete accumulation point. Now take \( \alpha \in \beta \in \lambda \) such that \( x \not\in U_\alpha (y) \), \( (U_\alpha (y) \times U_\alpha (y)) - V_\beta G \emptyset \), \( (y) \not\in x_\beta \). Then \( (x_\beta) \cap U_\alpha (y) G \emptyset \), hence \( U_\beta (x_\beta) < U_\alpha (y) \) or \( U_\beta (x_\beta) \geq U_\alpha (y) \). But \( U_\beta (x_\beta) \times U_\beta (x_\beta) < V_\beta \); hence \( x \not\in U_\beta (x_\beta) \) -- a contradiction. Thus (b) \( \Rightarrow \) (c).
(c) \Rightarrow(d). Assume that \(X\) satisfies (c): i.e; let \& be a non-archimedean base of \(X, \phi(\Delta X) = \lambda\) and \(\Delta X = \beta^*U_\alpha/\alpha \in \lambda+, U_\alpha\) open in \(X \times X\) as well as \(X' = \beta^*G_\alpha/\alpha \in \lambda+, G_\alpha\) open in \(X\) by our assumptions. Then, by (ii), \(adX \geq \phi(\Delta X)\), and it remains to prove that \(X\) is strongly sub orderable.

Since \(X\) is paracompact and \(\text{Ind } X = 0\), for every \(\alpha \in \lambda\), there is an open decomposition \& of \(X, \&_\alpha < \&\), refining the cover

\[
(\{ \alpha \cap G_\alpha/x \in G_\alpha^+ U \times G_\alpha^+ \};
\]

We may suppose that, for \(\alpha \in \beta \in \lambda, \&_\beta\) refines \& because \(adX = \lambda\) it is almost clear that \(U(\&_\alpha/\alpha \in \lambda^+\) is a base for \(X\) if \(x\) is isolated, then \(\{x\}\ in \&_\alpha\) for some \(\alpha\); if \(x \in X', U \in \&, x \in U\) and \(W \# U\) for every \(W \in U \times \&_\alpha/\alpha \in \lambda^+\) with \(x \in W\), then \(W \rightarrow U\), hence \(\beta(U, x/\alpha \in \lambda^+ \rightarrow U)\).

If \(\lambda \leq \omega\), then \(X\) is metrizable, hence orderable by the result compare also where Herrlich shows that a totally disconnected metric space \(X\) is orderable iff \(\text{Ind } X = 0\).

If \(\lambda > \omega\), we may suppose that \(\beta(V \in \&_\alpha^+ W < W^+) \geq \omega\) for any \(W \in \&_\alpha\) with \(\vert W \vert \geq \omega\). We can order every \& in such a way that for \(\alpha \in \beta \in \lambda\) the canonical “refinement” map \& is an order homomorphism and for \(W \in \&_\alpha, \vert W \vert \geq \omega\), the set \(\{V \in \&_\alpha^+ V < W^+\) is discretely ordered without a first and a last element. If \(x, y \in X\), we define \(x < y\) whenever \(V < W\) for some \(V, W \in \&_\alpha, \alpha \in \lambda\), with \(x \in V, y \in W\), then the order \(<\) induces the topology at every non isolated point of \(X\). Indeed, let \(x \in X^+\), \(W \in \&_\alpha, x \in W\), and take \(T, U, V \in \&_\alpha^+\) in such a way that \(T \cup U \cup V < W, x \in U, T\) is the predecessor of \(U\) in \&\ and then for \(t \in T, r \in V\), we have \(x < t, r < W\). Clearly, every open interval is open set in \(X\).

To finish the proof, we must show (d) \Rightarrow (a). Let \(X\) be satisfy (d), and \((\Delta X) = \lambda\). Hence \(X' = \beta^*G_\alpha/\alpha \in \lambda^+\) and \(\Delta X = \beta^*U_\alpha/\alpha \in \lambda^+\) where each \(G_\alpha\) and \(U_\alpha\) are open. for each \(x \in X'\). There is open interval \(V_x < G_\alpha\ containing x \) and such that \(V_x \times V_x < U_\alpha\); for \(x \in (X - X')\) we put \(V_{x, x} = x^* + x\). If we put \(V_x = \cup \{V_{x, x} \times V_{x, x}/x \in X'\} \cup \Delta X\), then \(V_x < U_\alpha, V_\alpha\) is open in \(X \times X\) and each \(V, x^*\ is a convex set. Since \(\beta(V, x^*/\alpha \in \lambda^+ = x^* + x\) for every \(x \in X\), we see that

\(V_{x, x}/\alpha \in \lambda^+\) is local base at \(x\) (for \(x \in X'\) because all \(V_{x, x}\) are convex; for \(x \in X - X'\) because
\[ V_{\alpha} = *x+ \text{ when ever } x \not\in G_{\alpha}. \]

Since \(X\) is paracompact by(iii), \(\beta\{V_{\alpha}/\alpha \in x+, \ x \in \lambda\}\) generates a uniformity on \(X\) with a linearly ordered base. (In paracompact space, every collection of neighbourhood of \(\Delta X\) is apart of a uniformity on \(X\); we take such a smallest uniformity.)

It remains to remark that \(X\) is metrizable iff \((\Delta X) = \omega\) and then \(\text{Ind } X = \text{dim } X = 0\) by(i) on p.

**REMARK: 2.3**

Notice that we have not used zero dimensional in the main part of (d) \(\Rightarrow\) (a). Thus any strongly sub orderable space \(X\) with \(\text{ad } X = \varphi(\Delta X)\) is linearly uniformizable iff \(X'\) is a \(G_{\varphi(\Delta K)}\)-set.

To get different view of the situation, we indicate another proof for (d) \(\Rightarrow\) (a), too that for any orderable space \(X\), we have \(m(X) = \varphi(\Delta X)\) where \(m(X)\) is the metrizability-degree of the space \(X\). Mutatis mutandis, the same proof works for strongly sub orderable space \(X\) where \(X'\) is a \(G_{\varphi(\Delta K)}\)-set. Now the implication (d) \(\Rightarrow\) (a) follows from the fact space \(X\) is \(x\)-metrizable iff

\[ m(X) = \text{ad } X = x. \]

A further, equivalent, description of space which are either linearly uniformizable and non-metrizable or metrizable and strongly zero dimensional (i.e. condition (a) of our theorem above) will be stated in proposition.

**COROLLARIE: 2.4**

1) Every non archimedean topological space \(X\) with \((x) = (\Delta X)\) for every point \(x \in X\) is linearly uniformizable.

2) Every non-archimedean \(\beta\)-space with a \(G_{\delta}\) –diagonal is metrizable.

3) Every strongly sub orderable space with \(G_{\delta}\)-diagonal and such that the set of all isolated points is \(F_{\sigma}\), is metrizable.

4) Every orderable space \(X\) with \(\text{ad } X = (\Delta X)\) is linearly uniformizable.
5) Every perfect normal non-archimedean space with a $G_δ$ diagonal ismetrizable (van Douwen).

6) Every non-archimedean space $X$ with $φ(x) = dX$ for any non isolated $x ∈ X$ is linearly uniformizable (thus every separable non-archimedean space is metrizable.

7) The Michel-line is first countable but is not a $β$ – space.

PROOF:

Corollaries and follow directly from our theorem 2.4 from the preceding remark, and 7 follows from 2. To prove corollary 4, it suffices to show that in any orderable space $X$ with

$$(ΔX) = ad \ X = x \ \text{the set } X' \text{ is a } G_δ\text{-set, and then to use the preceding remark. Indeed, if}$$

$$ΔX = B^*U_α/α \in x^+ \text{ for an open decreasing family } B^*U_α/α \in X × X \text{, then } iX' = B_1 \{\cup \{V_α \in X)/α \in x\}, \text{ where } V_α \text{ are open intervals such that } x \in V_α \text{, } V_α × V_α < α \in U_α. \text{ If } x \text{ is isolated and condition and contained in the last intersection, then } x \in B^*V_α, α/α \in x^+ \text{ for some sequence } x_α/α \in X'. \text{ Then we may suppose that } x_α < x \text{ for some cofinal set in } x. \text{ Now let } y \text{ be the predecessor of } x, \text{ then } (y, x) \in B^*U_α/α \in x^+, \text{ which is contradiction.}$$

Corollary 4 follows from the following reasoning. Let $&$ be a non-archimedean base of $X$, and denote $d(X) = λ$, one can easily show that $|&| ≤ λ$ (for every $x ∈ X$ the set $&x = *B \in &/x ∈ B$) is a linearly ordered set of clopen sets. Hence $|&| ≤ λ$ and $U*&B/α \in S^+$ where $S$ is a dense set in $X$). Now, for every disjoint pair $B, C \in &$ we denote $V(B, C) = (X × X) \setminus (B × C)$. there are at most $λ$ many such open sets $V(B, C)$ and their intersection is

$$(ΔX) \text{ consequently } (ΔX) < λ \text{ (and, since } φ(X) = λ) \text{ if } X \text{ is not discrete by our assumption one has } φ(ΔX) = λ. \text{ By } d(X) = λ \text{ it follows that } |X - X'| ≤ λ. \text{ Therefore } X' \text{ is an } F_{φ(ΔX)}\text{ set in } X \text{ and it follows from assertion (c) in the above theorem that } X \text{ is linearly uniformizable.}$$

REMARK: 2.4

In different ways, corollaries 3 and 4 generalized Lutzers’ result from , that every orderable space with $G_δ$-diagonal, the set of isolated points is an $F_δ$-set. see e.g the proof of corollary 4).

Corollary which can also be proved differently, answers partly the question posed in, whether every perfectly normal non-archimedean space is metrizable. This question is a Sousline line. Compare also the space described after our corollary.
Corollary generalizes one of the oldest result on non-archimedean spaces proved and also by other authors, e.g., which states that every separable non archimedean space is metrizable.

since \( d(X) = wX \) for archimedean spaces (see the proof above) and since every regular space \( X \) with \( wX \leq ad X \) is linearly uniformizable ,as follows froma result in ,our corollary gives also the following.

**COROLLARY:2.5**

The following conditions for a space \( X \) are equivalent: a) \( X \) is regular ,\( \text{Ind}X=0, wX \leq ad X \).

b) \( X \) is \( wX \)–uniformizable, and \( \text{Ind} X = 0 \) if \( X \) is metrizable.

c) \( X \) is a non-archimedean space with \( (x) = dX \) for every non-isolated \( x \in X \).

**REMARK: 2.6**

Condition (b)says in fact that either \( X \) is a sperable zero dimensional metrizable space (if \( w(X)=w \))or \( X \) is \( w(X) \)-metrizable.

**EXAMPLE:**

It is easy to find examples of spaces which are not linearly uniformizable, but satisfy all the condition of our problem except \( (x) = (\Delta X) \) or \( ad X = (\Delta X) \), respectively; take for instance a disjoint sum of a \( x \)–uniformizable space and a \( \lambda \)-uniformizable space with \( x > \lambda > \omega \). Then the resulting space is non-archimedean, and orderable provided the original spaces are dense-in-itself.

**EXAMPLE:**

A modified Michael-line .We shall show now that in our theorem ,the conditions concerning the isolated points are essential ;to prove that a space \( X \) is linearly uniformizable we must assure that in a certain sense-there are not too many isolated points which are “too close to the subspace \( X \)”.As a first example , take the Michel-line \( X \) which is non-archimedean , strongly
sub orderable and which satisfies $\varphi(x) = \varphi(\Delta X) = ad \ X$ for all non-isolated points x. But the Michel – line is not metrizable. Therefore ,it may serve as a counter example in the countables case. By Barie’s theorem, X’ is not $G_0$ nor has X property $B_\omega$.

The following example concepts concerns the uncountable case, too. Let $Z$ be the ordered set of integers, x a regular infinite cardinal, Y the lexicographically power $Z^+$and X the strongly sub orderable space with the same ordered set as Y in which all points $x \in Y$ for which

$\varphi(x) = \varphi(\Delta X) = ad \ X = x$ for all non-isolated points x and X is not linearly uniformizable.

The facts that X is strongly sub orderable and $(x) = ad \ X = x$ for non-isolated $x \in X$, are clear. Since Y is $x$–uniformizable (the partion’s $\{pr_{-1}x/x \in z^+, \alpha \in x\}$, from a base for the uniformity), one gets that Y- and hence X-is non-archimedean; moreover, we have

$(\Delta X) = (\Delta Y) = x$. To prove that X is not linearly uniformizable, it suffices to show to show that X’ is not a $G_\alpha$-set in Y.

Suppose that $X' = \beta^+u_\alpha / \alpha \in x^+, \ u_\alpha$ open in Y. Since X-X’ in dense in Y, there is $x \in x_0 \in U_0 \ X'$ and a neighbourhood $V_0$ of $x_0$ in Y such that $V_0 \setminus x_0$.

$V_0 = \text{pr}_{-1}x/pr_{-2}x = \text{pr}_{-1}x_0 (\text{for } \alpha \in \beta_0)$ for some $\beta_0 \text{ and } \text{pr}_{-1}x_0 \text{ G 0 for some } \alpha \in \beta_0$. Since X’, and hence $U_1, \text{is dense in Y, there is } x \in x_1 \in U_1 \setminus V_0 \ X$ with a neighborhood $V_1$of $x_1$ in Y such that

$V_1 \setminus u_\alpha = \text{pr}_{-1}x/_\alpha = \text{pr}_{-1}x_1 \alpha \in \beta_1$ for some $\beta_1$ , and $\text{pr}_{-1}x_1 \text{ G 0 for some } \alpha \in \beta_1 \geq 1$. By transfinite induction we get points $x \in U_\alpha \ X'$ and their neighbourhoods

$V_\alpha \setminus u_\alpha$ such that $x \in \beta^+V_\alpha / \alpha \in x^+ \setminus \beta^+U_\alpha / \alpha \ x \in x'^+, \ y \ in \ y_\alpha$ and $x \in \beta^+V_\alpha / \alpha \setminus x^+ \setminus \beta^+U_\alpha / \alpha \ x \in x'^+, \ y \ in \ y_\alpha$ which is contradiction. (Here for each $\alpha \ in x, \text{pr}_{-1}x = \text{pr}_{-1}x_\alpha$ for all $\ y \ in \ y_\alpha$.)

EXAMPLE:

A modified sorgenfrey-line .The next examples shows that one cannot replace

„strongly suborderable” by „suborderable” in condition.

Let $Z$ be a the ordered set of integers,x a regular infinite cardinal, Y the lexicographically power $Z^+$ and X be the set Y with the half open intervals [a, b] as a base for the topology one can easily show that $ad=x=ad \ Y=x$, $\varphi(\Delta X) = \varphi(\Delta Y) = x$ (hence X is represented paracompact by, Ind X=0 and X is dense -in-itself). If $2^x < 2^x$, then XxXis not collectionwise normal: the subspace {$(x, -x)/x \in x^+$ is closed discrete in XxX of cardinally $2^x$, buteX(X X) le d(X X) = dX = x^2 \omega$. If $2^x < 2^x$, then XxX is not normal (use jone classical method for counting continuous functions). Thus, for $x = \omega$, the space X is not metrizable. Suppose now that $x > \omega$. If X would be linearly uniformizable, then X was orderable by a dense order (e.g. the order obtain un the proof (c) $\Rightarrow$ (d) above is dense provided X is density-in-itself). On other hand, X cannot be orderable by a dense order, because the finest order-completion of X would be connected then, which contradicts corollary.
Therefore, we have: for every regular cardinal \( x \) there is a paracompact sub orderable 0-dimensional space \( X \) without isolated points with \( \text{ad} \ X = \varphi(\Delta X) = x \), \( (\varphi(x), \text{for any } x) \), which is not linearly uniformizable (even not non-archimedean).

**EXAMPLE:**

Our next step is to show that one cannot weaken the condition “\( \text{ad} \ X = (\Delta X) \)” in (d) to “\( \text{ad} \ X = \varphi(x) \), for all non-isolated \( x \in X \)”.

Let the reals and \( I \) be the subspace of the irrational and let \( X = \omega(r, s) \) either \( r \in I \), or \( s = \sqrt{2}+ \). Take the lexicographically order on \( X \) and the order topology \( r \), then Michael-line \( M \) is embeddable into \( X \). Let \( \&_1 \) be a non-archimedean base of \( M \) and \( \&_2 \) be a non-archimedean base of the irrational \( I \). Then

\[
\& = \star x \times B \mid x \in I, B \in \&_2 + U \times (B \times I) \cap X \mid B \in \&_1
\]

Is an non-archimedean base of \( (X, r) \) . Therefore, \( X \) is a non-archimedean space without isolated points; it satisfies \( \varphi(x) = \text{ad} \ X = \omega \) for every \( x \in X \), but \( (X, r) \) is not metrizable since it contains the contains the Michael-line as a base subspace. Therefore \( \varphi(\Delta X) = \omega \sc{**}

**REMARK: 2.7**

The construction described above can be generalized to arbitrary regular \( x \) if one replaces the Michael-line by the space of our example and the irrational by \( Z^* \) (compare example).

Therefore, for every regular cardinal \( x \) there is a paracompact zero dimensional orderable space \( X \) without isolated points, with \( \text{ad}X = \varphi(x) = x \) for every \( x \in X \), which is not linearly uniformizable.

**REMARK: 2.8**

The space constructed above was in fact non-archimedean. Here is an example of a paracompact zero dimensional space which serves the same purpose as above, but is not non-archimedean: For some infinite regular cardinal \( x \), let \( D \) be an ordered discrete space of cardinality \( x \) and cofinality \( x \). Define \( X = Z^* \times (D \times Z^* \times (D \times Z^*) \cap X \), where all the products and powers are lexicographical, and \( \infty \) is a “last point” added to \( D \times Z^* \). Then \( \text{ad} \ X = \varphi(x) = x \) for every \( x \in X \), but \( X \) is not linearly uniformizable, because its subspace \( Z^* \times (\infty) \) is homeomorphic with the space from our example.

**EXAMPLE:**

We will show now that we cannot weaken the condition “\( \text{ad} \ X = \varphi(\Delta X) \)” in (d) to “\( \varphi(x) = \varphi(\Delta X) \) for all \( x \)”, and at the same time that we cannot weaken the non-archimedean property in (b), (c) to “hereditarily paracompact 0-dimensional”, even if the space in question is dense—in-itself.

The example is a modification of the last of the example: \( X = (D \times Z^*) \cap X + \infty \), where again the product, the powers and the sum are lexicographical, and \( \infty \) is a “last point” added to \( D \times Z^* \). Then \( X \) is hereditarily paracompact, \( (x) = x \) for all \( x \), \( X \) cannot be linearly uniformizable.
for $x > \omega$ because $\text{ad } X = \omega$ (coinitiality of $\mathcal{A}^*$ is $\omega$). It remains to show that $(\Delta X) = x$: we know that both spaces $(D \times \mathcal{A})^*_{x+}$, $\mathcal{A}^*$ are $x$-uniformizable, hence, their diagonals are intersection of monotone open families $*U_{\alpha}/\alpha \in x+$ and $*V_{\alpha}/\alpha \in x+$, respectively. If $*y_{\alpha}/\alpha \in x+$ is cofinal in $D \times Z^*$, $*x_{n}/n \in \omega+$ coinitial in $\mathcal{A}^*$, and the map $\varphi: x \rightarrow \omega$ given by the formula $\alpha = x. \omega + f \alpha$, then

$$\mathcal{B}\{ W_{\alpha}/\alpha \in x+ = \Delta X, \text{ where}$$

$$W_{\alpha} = \mathcal{U}_{\alpha} \mathcal{V}_{\alpha} \cup ( -y_{\alpha}, x_{\varphi}, \alpha - y_{\alpha}, x_{\varphi}, \alpha ) \}.$$  

Our result is: For every uncountable regular $x$ there is a hereditarily paracompact 0-dimensinal orderable space without isolated points, with $(x) = \varphi(\Delta X) = x$ for all $x \in X$, which is not linearly uniformizable (even not non-archimedean).

For $x = \omega$ such a space cannot exist because of Lutzer”s result that orderable spaces with $aG_\delta$-diagonal are metrizable. If in this case, we replace „orderability” by „suborderability”, then the sorgenfrey-line as a analogue of the example above.

**REMARK: 2.9**

The space $X$ constructed above has metrizability degree $(x) = x$, but $X$ is not $x$-metrizable.

**EXAMPLE:**

If we look at the last three examples, we can see that the constructed spaces are dense-in-itself. For such space, our theorem has the following form:

**COROLLARY: 2.10**

If $X$ is dense-in-itself then the following assertion are equivalent:

1) $X$ is linearly uniformizable and non-metrizable or metizable and $\dim X = 0$.

2) $X$ is non-archimedean and $(x) = (\Delta X)$ for every $x \in X$. 

3) $X$ is orderable, 0-dimensional and $ad X = (∆X)$.

REMARKS: 2.11

In the preceding examples we have shown that the conditions in (2) and (3) cannot be weakened; e.g. the non-archimedean property in (2) to orderability. Here, orderability of $X$ would be weaker than the no-archimedean property, since if $φ(ΔX) = ω$, $X$ is orderable since it is metrizable and $Ind X=0$, and if $φ(ΔX) > ω$, orderability of $X$ follows from (theorem).

Further, we showed that orderability in (3) cannot be weakened to suborderability. We also showed that the condition “$ad X=φ(x)$” in (3) cannot be replaced by “$ad X = φ(x)$ for every non-isolated $x ∈ X$” and moreover (in example ) that the condition “$φ(x) = φ(ΔX)$” in (2) cannot be weakened to “$φ(x) = ad X$”.

Let $S$ be a souslin-tree (which exists if a souslin-line exists; e.g. Moreover, we can assume that every element $s ∈ S$ has infinitely many immediate successors.

Now let $X$ be the space described as follows: points $x ∈ X$ are maximal chains of $S$; a basic neighborhood of $x$ will consist of those maximal chains of $y ∈ X$ which coincide at a fixed element $α$ of $x$ (and hence at all elements $β ≤ α$). Then it follows from the construction that $X$ is a non-archimedean topological space, $X$ is first countable and dense-in-itself. Moreover, because of our assumption concerning the souslin-tree $S$, $X$ is linearly orderable (which follows analogously as in the proof (c) ⇒(d) of our theorem above. More or less, this is a standard construction. On the other side, $X$ cannot be metrizable, since $c(X) = ω$ but obviously $d(X) > ω$. Hence $(ΔX) > ω$. Further, $X$ is perfectly normal because $X$ is orderable, dense-in-itself and $(x) = ω$.

Using $x^+$ souslin-trees, the situation can be generalized to higher cardinals.

APPLICATION, PROBLEMS AND FURTHER RESULTS:

At the end of our paper we want to show several, results where the preceding theorems are used as tools.

Recall that a Hausdorff-space $(X, r)$ is linearly stratifiable sapces if there is a cardinal $λ$ anda map $S: (λ × r) → r$ having the following properties:

i) $β ∈ λ, r ⇒ (∆, U) < S(β, U) < U$.

ii) $U ∈ r ⇒ U = U * (∆, U)α ∈ λ+$.

iii) $V ∈ r, U < V, α ∈ λ ⇒ (∆, U) < (α, V)$. 

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A linearly stratifiable space is called $\lambda$-stratifiable, if $\lambda$ is the smallest cardinal which can be used in the above definition. Every linearly stratifiable space is hereditarily paracompact and every linearly uniformizable space is linearly stratifiable. Indeed, if $*_{a/\alpha} \in \lambda+$ is linearly ordered base of a (covering) uniformity on $X$, and we put $S(\alpha, U) = U \backslash \overline{s(\alpha)}$ we obtain a

$\lambda$-stratification of $X$. Every $\lambda$-stratifiable space has property $\beta$. Therefore, the following proposition show how can weaken the other condition in our theorem if we replace property (\( )\) by $\lambda$-stratifiability. That a sub orderable space $X$ is linearly uniformizable if $X$ is linearly stratifiable.

Therefore, by combining this with our results, we obtain:

**PROPOSITION: 2.12**

If $X$ is linearly stratifiable and non-metrizable, then the following are equivalent:

1) $X$ is strongly sub orderable

2) $X$ is sub orderable.

And consequently, we get.

**PROPOSITION: 2.13**

For any space $X$, the following are equivalent:

1) $X$ is either linearly uniformizable and non-metrizable, or $X$ is metrizable and $dimX = 0$.

2) $X$ is zerodimensional, suborderable and linearly stratifiable.

Another application concerns metrizability of orderable spaces. We state the next three proposition for orderable spaces, which may be more convenient for the reader; but it follows
from our theorem that all these proposition are valid for strongly suborderable space X whose set of isolated points is an $F_x$--set where $x = ad X$.

**PROPOSITION: 2.14**

An orderable space X is metrizable iff it is a k-space and $(\Delta X) \leq ad X$.

**PROOF:**

Suppose that an orderable space X is a nondiscrete k-space and $\omega_1 \leq (\Delta X) \leq ad X$. Then every compact subspace of X is finite (since it is $\omega_1$-additive), thus X is discrete a contradiction. Hence $(\Delta X) = \omega$ and everything follows from Lutzer’s result that any orderabele space with a $G_{\delta}$-diagonal is metrizable.

**PROPOSITION: 2.15**

If X is an orderable space with $cX = \omega$, $(\Delta X) \leq ad X$, then X is metrizable.

**PROOF:**

If X is nonmetrizable, then $(\Delta X) > \omega$, and X has a strongly decreasing chain of clopen sets length $\omega_1$, so that $cX \geq \omega_1$; a contradiction.

**REMARK: 2.16**

In proposition , the condition “$cX = \omega$” is essential, as the following example shows.

If X is a one-point Lindelöfication of a discrete space of cardinality $\omega_1$, then X is orderable. Nonmetrizable, $= \omega_1$, $(\Delta X) = ad X = \omega_1$.

**REMARK: 2.17**

Examples of such a space must be scattered, at least under $\sim CH$, as follows from the following proposition:

**PROPOSITION: 2.18**
If X is orderable, not scattered, $X < 2^\omega$, and $\varphi(\Delta X) \leq ad X$, then X is metrizable.

PROOF:

Take a subspace $Y$ of X without isolated points. Then since X is orderable $cY < 2^\omega$, $(\Delta X) \leq ad Y$, and we can construct a cantor tree of length $x = ad Y$ composed of open subsets of Y such that each $\alpha$-level of this tree would consist of $2^\omega$ open sets. Hence $(\Delta X) \leq ad X \leq ad Y = \omega$, and X must be metrizable.

REMARK: 2.19

Since the same argument holds for every $\alpha$-level of the tree, we in fact have shown that for dense-in-itself spaces X, one has $cX \geq 2^{2dK}$.

On the other hand $X = Z^{2\omega}$, orderable lexicographically, satisfies all the conditions of proposition 4 except “$cX < 2^\omega$”, and X is not metrizable.

The Michael-line X as, a typical example of a non-archimedean topological space, originates from the space of the result turning some points (the irrational) into isolated points and leaving the neighborhoods of all other points as they are. “X is a discretization of the space by the subset \".

It was shown that every non-archimedean topological space is a discretization of a certain space Y which can be gotten from X as an inverse limit of some quotient space of $X^6$. By the examples above, we know that a non-archimedean topological space X with $\varphi(x) = \varphi(\Delta X)$ for every non-isolated point $x \in X$ need not be linearly uniformizable. In fact, to obtain linear uniformizability we need some further condition assuring that, there are not “not many” isolated points which are “too close” to the subspace $X^6$ (property $B_{\varphi(\Delta K)}$ of condition (b) in our theorem above can serve as a typical example).

Therefore, it may not be surprising that we can prove the following theorem:

PROPOSITION: 2.20
A non-archimedean topological space $X$ is homeomorphic with a discretization of a linearly uniformizable space $Y$ if and only if $(x) = (\Delta X)$ for every non-isolated point $\in X$.

**COROLLARY: 2.21**

Any non-archimedean space $X$ with a $G_\delta$-diagonal is homeomorphic with a discretization of a metric space $Y$.

**PROOF:**

We have to show that only sufficient of the condition. To show that, we adapt the notation of the proof of $(b) \implies (c)$ of our main theorem. For every point $x \in X$ and for every $\alpha \in \lambda$, choose a basis set $B_x, \in &$ such that $(B_x, \times B_x) < V_\alpha$, and consider the covering $= *B_x/x \in X$. Then, for every $x \in X, B^*st(x, \alpha) \in \lambda = *x^+, since BV_\alpha = \Delta X$.

Since every non-archimedean space is paracompact and strongly zero-dimensional, we can choose partition $\&_a < &_a$ of $X$ refining $u_\alpha$ for each $\alpha \in \lambda$. Moreover, we may suppose that the covering $\&_a, \alpha \in \lambda$, are linearly ordered by refinement, because $ad X \geq \varphi(\Delta X)$; see (ii) on p.

Therefore, $*\&_a/\alpha \in \lambda^+$ defines a uniformity $G$ on the underlying set of $X$ with a linearly ordered base such that for every non-isolated point $x \in X, *st(x, &_a) / \alpha \in \lambda^+$ is local base of the topology of $X$.

**CHAPTER-VAPPLICATION**

Topology being somehow very recent in nature but has got tremendous applications over almost all other fields. Theoretical or fundamental topology is a bit dry but the application part is what drives crazy once we get used. In this paper we discuss some application in various fields of science and Technology, like application to Biology, robotics, GIS, Engineering, computer science. Topology though being a part of mathematics but it has influenced the whole with so strong effects and incredible applications.

**CONCLUSION**

The topological space linearly uniformizable if there if there is uniformity generating The topology of having a linearly orderd base. Obviously any metrizable topological. Space is linearly uniformizable.
By a result it is a metrizable non-archimedean topological spaces every non-archimedean topological space is para compact hence here ditarily para compact the most important topological properties of uniform space which still valid in semi-linear uniform space.

BIBILOGRAPHY


