BASIC COMMUTATIVE ALGEBRA

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Abstract
The purpose of this work is to build a graded algebra $A = A(q_1, q_2, q_3)$ with three shift parameters, $q_1$, $q_2$, and $q_3$. By introducing a specific filtration connected with the dominance ordering among partitions, we verify the basic features of the algebra $A$, including commutativity and the Poincaré series. The Gordon filtration is a stratification that is characterised by a series of null conditions related with the partitions and the shift parameter $q_i$. The elliptic algebra $[E]$ can be thought of as a smooth limit of our algebra. Specifically, the original algebra is built over an elliptic curve, whereas our algebra $A$ is built over a degenerate CP1.

Keyword: Algebra, Poincaré. Elliptic Algebra

1. Introduction

POLYNOMIAL RING
If $R$ is a ring, the ring of polynomials in $x$ with coefficients in $R$ is denoted $R[x]$. It consists of all formal sums.

$$\sum_{i=0}^{\infty} a_i x^i$$

That is $R[x] = \sum_{i=0}^{\infty} a_i x^i$

QUOTIENT RING
Let $R$ be a ring and $I$ be an ideal of $R$. Then $R/I$ is form a ring. that ring is called quotient ring.

EXAMPLE: Let $K[x]$ is a polynomial ring and its quotient rings are $k[x] = < x^2 + 1 >, k[x] = < x^3 + 2 >$ etc.

PRIME IDEAL
An ideal $P$ is said to be prime ideal if $ab \in P$ then $a \in P$ or $b \in P$.

EXAMPLE: $< x^2 + 1 >$ is an prime ideal of $R[x]$.

NOTE: Let $R$ be a ring and $I$ be an ideal of $R$. Then $R/I$ is said to be integral domain if and only if $I$ is prime ideal.

MAXIMAL IDEAL
Let $R$ be a commutative ring and ideal $M$ of $R$ is said to be maximal ideal of $R$, if there exist an ideal $N$ of $R$ such that $M \subset N \subset R$ then

$M=N \text{ or } N=R$.

EXAMPLE:
Let $K[x]$ be a ring and $< x^2 + 1 >$ its maximal ideal.

NOTE:
1. $R$ be a commutative ring with unity an ideal $M$ of $R$ is maximal ideal of $R$ if and only if $R=M$ is a field.
2. Every maximal ideal is a prime ideal.

NILRADICAL
The set $N$ of all Nilpotent element in a ring $R$ is an ideal and $R/N$ has no Nilpotent element (except zero). The Nilradical of a ring $R$ is the intersection of all prime ideal of $R$. 
EXAMPLE:
Let \( Z \) be a ring and \( pZ \) be its prime ideal (where \( p \) is prime) then Nilradical of this is 0.

JACOBSON RADICAL
The Jacobson radical of \( J \) of a ring \( R \) is defined to be the intersection of all maximal ideals of \( R \).

COLON IDEAL
Let \( R \) be a commutative ring. Let \( S \) be a subset of \( R \) and \( I \) be an ideal of \( R \). Now we define the subset 
\[
(I : S) = \{ a \in R : aS \subseteq I \}
\]
and the \((I:S)\) is an ideal of \( R \). This ideal is called the colon ideal or ideal quotient.

EXAMPLE:
\[
R = \mathbb{K}[x] = \langle x^2 \rangle
\]

EXTENSION AND CONTRACTION
Let \( f : A \rightarrow B \) be a ring homomorphism. If \( I \) be an ideal of \( A \), define the extension \( I^n \) of \( I \) to be \( I^n = \langle f(I) \rangle \) ideal generated in \( B \). That is,
\[
I^n = \left\{ \sum_{i=0}^{n} y_i f(x - i) / n \geq 1, y_i \in B, x - i \forall i \right\}
\]
Let an ideal of \( B \) then \( f^{-1}(J) \) is an ideal of \( A \) called the contraction \( J^c \) of \( J \). That is,
\[
J^c = f^{-1}(J) = \{ x / f(x) \in B \}
\]

MODULES
DEFINITION: Let \( A \) be a commutative ring. An \( A \)-module \( M \) is an abelian group written additively with scalar multiplication and a mapping \( f : A \times M \rightarrow M \) with following properties
\[
\begin{align*}
ax + ay &= a(x+y) \\
(a+b)x &= ax + bx \\
(ab)x &= a(bx) \\
1x &= x
\end{align*}
\]
where \( a; b \in A \), \( x, y \in M \).

EXAMPLE:
1. An ideal \( I \) of a ring \( A \) is an \( A \)-modules.
2. If \( A \) is a field \( k = \mathbb{R} \), then \( A \)-modules = \( K \) vector space.

HOMOMORPHISM
Let \( M, N \) be \( A \)-module, \( A \) mapping \( f : M \rightarrow N \) is an \( A \)-modules homomorphism if,
\[
\begin{align*}
f(x + y) &= f(x) + f(y) \\
f(ax) &= af(x)
\end{align*}
\]
where for all \( a \in A \) and all \( x, y \in M \).

The set of all \( A \)-module homomorphism from \( M \) to \( N \) is also \( A \)-module follow:

we define, \( f+g \) and \( af \) by the rule
\[
\begin{align*}
(f + g)x &= f(x) + g(x) \\
(af)x &= af(x)
\end{align*}
\]
this is also \( A \)-module and is denoted by \( \text{Hom}_A(M; N) \).

Homomorphism \( u : M' \rightarrow M \) and \( v : N' \rightarrow N \) induce mapping \( u' : \text{Hom} (M,N) \rightarrow \text{Hom}(M',N) \) and \( v' : \text{Hom}(M,N) \rightarrow \text{Hom}(M',N') \)
define as follow,
\[
u'(f) = f_0u, v'(f) = v_0f \] these module are \( A \)-module homomorphism.

SUB-MODULES AND QUOTIENT MODULES
An sub-module \( N \) of \( M \) is a subgroup of \( M \) which is closed under the multiplicaton by element of \( A \). That is \( x:n \in N \) for all \( x \in A \) and \( n \in N \).

EXAMPLE: A be a ring and itself is a \( A \)-modules and its ideal is sub-modules.
The Abelian group $M/N$ gives an $A$-modules structure from $M$ define by $a(x+N)=ax+N$. The module $M/N$ is quotient of $M$ by $N$.

1. The kernel of $f$ is the set $\text{Ker}(f) = x \ 2 \ M : f(x) = 0$ is sub-module of $M$.
2. The image of $f$ is the set $\text{im}(f)=f(M)$ is a sub-module of $N$.
3. The coker of $f$ is $\text{coker}(f)=N/\text{im}(f)$

**ANNIHILATOR**

If $N,P$ are sub-module of $M$, we denote $(N:M)$ to be the set of all a such that $aP \subseteq N$ it is an ideal of $A$. In particular $(0:M)$ is the set of all a $A$ such that $a=0$, this ideal is called the annihilator of $M$ and is also denoted by $\text{Ann}(M)$. An $A$-module is faithful if $\text{Ann}(M)=0$. If $\text{Ann}(M)=a$ then $M$ is faithful as an $A/a$ module.

**DIRECT SUM AND DIRECT PRODUCT**

If $M,N$ are $A$-module their direct sum $M \_ N$ is the set of all pairs $(x,y)$ with $x \in M; \ y \in N$.

$(x_1; \ y_1) + (x_2; \ y_2) = (x_1 + x_2; \ y_1 + y_2)$

$a(x; \ y) = (ax; \ ay)$

If $(M_i)_{i \in I}$ is any family of $A$-module, we can denote the direct sum $(M)$ its element are families $(x_i)_{i \in I}$ such that $(x_i) \ 2 \ (M_i)$ for each $i \in I$ and at most all $(x_i)$ are zero. If we remove on the number of non zero $X$'s we have the direct product.

**CO-MAXIMAL**

Let $R$ be a ring, ideal $A$ and $B$ are said to be co-maximal if $A+B=R$.

**EXAMPLE:** Let $Z$ be a ring and $I=Z \times Z$ and $J=Z \times Z$ be two co-maximal ideal.

2. **Chinese Remainder Theorem**

**Theorem 1.** Let $A_1; A_2; A_3; \ldots; A_k$ be an ideals in $R$. The mapping $R \to R/A_1 \times R/A_2 \times \ldots \times R/A_k$ define by,

$r \to (r \mod A_1, \ r \mod A_2, \ldots, r \mod A_k)$ is a ring homomorphism with kernel $A_1 \cap A_2 \cap \ldots \cap A_k$. If for each $i; j \in \{1; 2; 3; \ldots; k\}$ with $i \neq j$ the ideals $A_i$ and $A_j$ are co-maximal, then this map is Surjective and $A_1 \cap A_2 \cap \ldots \cap A_k = A_1 A_2 \ldots A_k$, so $R/(A_1 A_2 \ldots A_k) \cong R/A_1 \times R/A_2 \times \ldots \times R/A_k$.

**Proof.** For $k=2$

We first prove this for $k=2$; the general case will follow by induction.

Let $A = A_1$ and $B = A_2$.

Consider the map $f : R \to R/A \times R/B$.

defined by $f(r) = (r \mod A; \ r \mod B)$, where mod $A$ means the class in $R/A$ containing $r$ (that is, $r+ A$).

when $A$ and $B$ are co-maximal, $f$ is surjective and $A \cap B = AB$.

Since $A + B = R$, there are elements $x \in A$ and $y \in B$ s.t. $x+y = 1$. This equation show that $f(x) = (0,1)$ and $f(y) = (1,0)$.

since, for example, $x$ is an element of $A$ and $x = 1- y \in 1 + B$. If now $(r \mod A; \ r \mod B)$ is an arbitrary element in $R/A \times R/B$, then element $r x + r y$ maps to this to element.

Since $f(r x + r y) = f(r) f(x) + f(r) f(y)$

$=(r \mod A; \ r \mod B)(0; 1) + (r \mod A; \ r \mod B)(1; 0)$

$=(0; \ r \mod B) + (r \mod A; 0)$

$=(r \mod A; \ r \mod B)$.

This shows that $f$ is indeed surjective. Finally, the ideal $AB$ is always contained in $A \cap B$. If $A$ and $B$ are co maximal and $x$ and $y$ are as above, then for any $c \in A \cap B ; \ c = c_1 = cx + cy \in AB$.

The general case follows easily by induction from the case of two ideals using $A = A_1$ and $B = A_2 \ldots A_k$ once we show that $A_1$ and $A_2 \ldots A_k$ are co-maximal. By hypothesis for each $i \in \{2, 3, 4 \ldots k\}$ there are elements $x_i \in A_i$ and $y_i \in A_i$ s.t.

$x + y_i = 1$. Since $x+y_i \equiv y_i \mod A_i$, it follows that $1 = (x_2 + y_2) \ldots (x_k + y_k)$

is an element in $A_1 + (A_2 \ldots A_k)$.


3. Hilbert Basis Theorem
We first describe some general finiteness condition. Let \(R\) be a ring and let \(M\) be a left \(R\)-module.

**Definition:**
1. The left \(R\)-module \(M\) is said to be a Noetherian \(R\)-module or to satisfy the ascending chain condition on submodules (or A.C.C on submodules) if there are no infinite increasing chain of submodules is an increasing chain of submodules of \(M\), then there is a positive integer \(m\) such that for all \(k \geq m\); \(M_k = M_m\) (of the chain becomes stationary at stage \(m\); \(M_m = M_{m+1} = M_{m+2} = \ldots\)).
2. The ring \(R\) is said to be Noetherian if it is Noetherian as a left module over itself, i.e. if there is no infinite increasing chains of left ideals in \(R\).

**Example:**
Any field is a Noetherian ring.

Any Principal Ideal Domain is also a Noetherian ring. So, the integers, considered as a module over the ring of integers, is a Noetherian module.

**Theorem 2.**
Let \(R\) be a ring and let \(M\) be a left \(R\)-module. Then the following are equivalent,
1. \(M\) is a Noetherian \(R\)-module.
2. Every nonempty set of submodules of \(M\) contain a maximal element under inclusion.
3. Every submodule of \(M\) is finitely generated.

**Proof.** First we prove that \(1 \Rightarrow 2\)

Assume \(M\) is Noetherian and let \(P\) be any nonempty collection of submodules of \(M\). Choose any \(M_1 \in \Sigma\).
If \(M_1\) is a maximal element of \(\Sigma\), so assume \(M_1\) is not maximal. Then there is some \(M_2 \in \Sigma\) such that \(M_1 \subsetneq M_2\).

And if \(M_2\) is maximal in \(\Sigma\), so we may assume there is an \(M_3 \in \Sigma\) properly containing \(M_2\). Proceeding in this way one sees that if \(2\) fails we can produce by axiom of choice an infinite strictly increasing chain of elements of \(\Sigma\), contrary to \(1\).

Now we prove \(2 \Rightarrow 3\)
Assume \(2\) holds and let \(N\) be any submodule of \(M\).
Let \(\Sigma\) be the collection of all finitely generated submodules of \(N\). Since \(0 \in \Sigma\), this collection is nonempty.
By \(2\), \(\Sigma\) contains a maximal element \(N'\).
If \(N = N'\), let \(x \in N \setminus N'\), since \(N' \in \Sigma\), the submodule \(N'\) is finitely generated by assumption, hence also the submodule generated \(N'\) and \(x\) is finitely generated.
This contradicts the maximality of \(N'\), so \(N = N'\) is finitely generated.

Now we prove that \(3 \Rightarrow 1\)
Assume \(3\) holds and let \(M_1, M_2, \ldots M_3, \ldots\) be a chain of submodules of \(M\).
Let \(N = \bigcup_{i=1}^{\infty} M_i\) and note that \(N\) is a submodule.
By \(3\) \(N\) is finitely generated by, say \(a_1, a_2, \ldots, a_n\). Since \(a_i \in N\) for all \(i\), each \(a_i\) lies in one of the submodule in the chain, say \(M_{j_i}\).
Let \(m = \max\{j_1, j_2, j_3, \ldots, j_n\}\).
Then \(a_i \in M_m\) for all \(i\) so the module they generate is contained in \(M_m\) i.e. \(N \subsetneq M_m\).
This implies \(M_m = N = M_k\) for all \(k \geq m\), which proves \(1\).

**Corollary 1.** \(R\) is a Noetherian ring if and only if every ideal of \(R\) is finitely generated.

**Corollary 2.** If \(R\) is a P.I.D, then every nonempty set of ideal of \(R\) has a maximal element and \(R\) is a Noetherian ring.

**Proof.** The P.I.D \(R\) satisfies condition of the above theorem with \(M = R\).
Recall that even if M itself is a finitely generated R-module, submodule of M need not be finitely generated, so the condition that M be a noetherian R-module is in general stronger than the condition that M be a finitely generated R-module.

**Proposition** Let R be an integral domain and let M be a free R-module of rank n < 1. Then any n+1 element of M are R-linearly dependent i.e for any y₁, y₂, ..., yₙ+1 ∈ M there are elements r₁, r₂, ..., rₙ+1 ∈ R not all zero, such that r₁y₁ + r₂y₂ + ... + rₙ+1yₙ+1 = 0

**Proof.** The quickest way of proving this is to embed R in its quotient field F. since R is an integral domain and observe that since M ≅ R ⊕ R ⊕ ... ⊕ R (n times), the latter is an n-dimensional vector space over F, so any n+1 element of M are F-linearly dependent.

By clearing the denominators of the scalar, we obtain an R-linear dependence relation among the n+1 elements of M.

If R is any integral domain and M is any R-module recall that

Torsion (M) = {x ∈ M | rx = 0 for some nonzero r ∈ R}

**Theorem.** R is a Noetherian if and only if every prime ideal is finite generated.

**Proof.** Assume R is Noetherian. Let I = {I is an ideal of R which is NOT finitely generated}

By assumption T ≠ Ø

Now we used zorn’s lemma to show that T has a maximal element.

Let {lα} be a chain, Let I=U lα.
Then I is an ideal.
we claim I is not Finite generated because if it is check, then

I = Rx₁ + Rx₂ + ... + Rxₙ for xᵢ ∈ l αᵢ

Set r = max αᵢ, where j=1,2,....,n

where {α₁, ......., αⱼ} = totally ordered finite set.

xᵢ ∈ lᵢ for all i

By zorn’s lemma T as a maximal element say P. we will show P is prime which will be a contradiction. suppose ∃x ∈ R/P; y ∈ R/P with xy ∈ P.

Look at (x, P) is not proper superset of P and P:x is not proper super set of P.

By maximality of P, (P, x), (P : x) are finitely generated

=> (P; x) = Rx₁ + Rx₂ + ... + Rxₙ + (P : x)x

Assume x₁; x₂; ....... ∈ P.

But P = Rx₁ + Rx₂ + ... + Rxₙ + (P : x)x

because if z P∈

=> Z ∈ (P; x) = Rx₁ + Rx₂ + ... + Rxₙ + Rx

=> z =λ₁x₁ + λ₂x₂ + .......λₙxₙ + λx

=> λx ∈ P

=> λ ∈ (P : x).

4. **Hilbert Basis Theorem**

If R is a Noetherian Ring, then the polynomial ring R[x₁; x₂; ...... xₙ] is Noetherian.

**Proof.** we may assume that n=1, we show that R is Noetherian.

=> R[x] is Noetherian.

I ⊆ R[x] be an R[x]-ideal.

Set

Iₙ = {a ∈ R | ∃f ∈ (x) / I with degree n and leading coefficient a} U {0}.
Then \( I_n \) is an \( R \)-ideal.

(For all \( a; b \in I_n \) two polynomial \( f; g \) of order \( n \) such that the leading coefficient of \( f; g \) are \( a; b \) respectively .

\[ f - g \] is also a polynomial of degree \( n \) and leading coefficient of \( f - g \) in \( a - b \)

\[ \Rightarrow a - b \in I_n \]
Therefor(6 \( \neq 0 \)) \( \in R; a \in I_n \), \( rf \) is a polynomial of degree \( n \) and leading coefficient is \( ra \).

\[ \Rightarrow ra \in I_n \]
\( I_n \) is an \( R \)-ideal.)

(\( \forall a \in I \) \( \exists \) a polynomial \( f \) of degree \( j \) such that leading coefficient of \( f \) is \( a \).Now consider \( xf \). This is a polynomial of degree \( j + 1 \) and leading coefficient is \( a \). This implies \( a \in I_{j+1} \)

Since \( R \) is Noetherian.

Let \( f_n; 1 \leq i \leq l_n; 0 \leq n \leq r \) be polynomial such that \( \deg f_n = n \) and leading coefficient of \( f_n; 1 \leq i \leq l_n \) generated \( I_n \).

claim:{ \( f_n; 1 \leq i \leq l_n; 0 \leq n \leq r \} = W \) generates \( I \)

By definition the leading coefficient of \( f_n \) generates \( I_n \); \( \forall i \).

Now let \( g(x) \in I; g \neq 0 \). By induction on \( \deg g \),

We show \( g \in (W)R[x] \).

If degree of \( g \) equals 0 then \( g \in I \cup R, Rf_0 = Rf_0 + \_\_\_ + Rf_0 \)

Assume by induction that the result is true for all degree \( n - 1 \).

Now let \( \deg g = n \). Since the leading coefficient of \( g \) is \( I_n; \exists I_i \in R \\
\text{with } \deg(g - \Sigma I_i F_n - 1) \leq n - 1 \\
\text{By induction we know } F_n - 1 \in (W)T[x]. \\
\text{REFERENCE} \\
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