AREA DISTORTION UNDER CERTAIN CLASS OF QUASI CONFORMAL MAPPING

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ABSTRACT

In the paper we study hyperbolic and Euclidean area distortion of measurable sets under classes of K-Quasiconformal mapping, angular and radial Quasiconformal mapping from the upper half plane and the unit disk onto themselves respectively.

INTRODUCTION

Quasiconformal mapping are generalization of conformal mapping. They can be considered not only Riemann surface but also on Riemannian manifolds in all dimension and even an arbitrary metric space quasiconformal mapping occur nationally in various mathematical and often a priori unrelated contexts.

In 1956 Beurling and Ahlfors solved the boundary value problem for quasiconformal mapping. If M ≥ 1 they gave an explicit formula for the extension of an M-quasisymmetric function h:R→R to a quasiconformal mapping f = u + iv from ℏ onto itself, where ℏ denotes the upper half-plane. The mapping f is called the Beurling – Ahlfors extension of h. In particular f satisfies

\[ \frac{1}{\gamma^2} \leq \frac{|f'(z)|}{|z|^2} \leq \gamma^2 \]

Where f'is denoted the Jacobian of f and c = c(K) > 0 depends on K = K(M). Let f:Ω→C be an ACL (absolutely continuous on lines) homeomorphism in a domain Ω⊂C that preserves orientation. If f satisfies, \( D_f = \frac{|f_z|+|f_{\bar{z}}|}{|f_z|-|f_{\bar{z}}|} \leq K \ a.e.\) For some K≥1, then f is K-quasiconformal mapping, where \( f_z = \frac{1}{i}(f_x - if_y) \) and \( f_{\bar{z}} = \frac{1}{i}(f_x + if_y) \)

\( D_f \)is called the dilatation of f. The maximal dilatation of f Thus for each measurable subset E of ℏ is holds that , \( A_H(\frac{f(E)}{c}) \leq A_H(E) \)

Where, \( A_H(\cdot) \) Denotes the hyperbolic area in the half-plane ℏ.

In 1994 Astala provedthat if f is a K-quasiconformal mapping from the unit disk ℰ onto itself normalized by f(0) = 0 and if E is any measurable subset of the unit disk then,

\[ A_e(f(E)) \leq a(K)A_e(E)^\frac{1}{K} \]

Where, \( A_e(\cdot) \) denotes the Euclidean area and a(K)→ 1 when K→ 1⁺.

In 1998 Resends and porter obtained some results about area distortion under quasiconformal mapping on the unit disk ℰ onto itself with respect to the hyperbolic measure. They also showed the existence of explodable sets has bounded hyperbolic area but under a specific quasiconformal mapping it is image has infinite hyperbolic area. In recent year harmonic quasiconformal mapping have been extensively studied and the paper cited therein.
In this paper combining with the knowledge of a harmonic function and its harmonic conjugate function, we get several equivalent condition for a harmonic mapping of \( \mathbb{H} \) onto itself to be a \( K \)-quasiconformal mapping and we use the hyperbolic area distortion under quasiconformal harmonic mapping from the unit disk into itself and analyze the hyperbolic and Euclidean area distortion under quasiconformal mapping.

\[
f(z) = f(x + iy) = u(x, y) + i cy, \text{with } c > 0
\]

**QUASICONFORMAL MAPPINGS**

**DEFINITION 1.1:**

Let \( \Omega \) be an open set in \( \mathbb{C} \) and \( f: \Omega \to \mathbb{C} \) be an oriented preserving homeomorphism we say that \( f: \Omega \to \mathbb{C} \) is quasiconformal mapping.

(i) \( f \) is **absolutely** continuous on lines (ACL)

(ii) For almost every \( z \in \Omega \) we have

An orientation preserving homeomorphism \( f: \Omega \to \mathbb{C} \) is called quasi-conformal.

**DEFINITION 1.2:**

Let \( \Omega \) be an open set in \( \mathbb{C} \) and \( f: \Omega \to \mathbb{C} \) be a continuous map. We say that,

\( f: \Omega \to \mathbb{C} \) is **ACL** (absolutely continuous on line) if for each closed rectangle \( [a, b] \times [c, d] \) contained in \( \Omega \). We have, The following two properties

(i) For almost all \( y \in [c, d] \) the function \( x \to f(x + iy) \) is absolutely continuous on \( [a, b] \).

(ii) For almost all \( x \in [a, b] \) the function \( y \to f(x + iy) \) is absolutely continuous on \([c, d]\).

**DEFINITION 1.3:**

Let \( f: \Omega \to \mathbb{C} \) and let \( u \) and \( v \) be a open subset of \( \mathbb{C} \) take \( K > 1 \) and set \( K = \frac{K - 1}{K + 1} \) so that \( 0 \leq K \leq 1 \). A mapping \( f: U \to V \) is a \( K \)-quasiconformal map if it is homeomorphism whose distributed partial derivatives are in \( L^2_{loc} \) locally in \( L^2 \) and satisfy

\[
\frac{\partial f}{\partial z} = K \frac{\partial f}{\partial \bar{z}} L^2_{loc} \text{ is a.e}
\]

A map quasiconformal it is **K-quasiconformal** for some \( K \).

**DEFINITION 1.4:**

The smallest \( K \) such that \( f \) is \( K \)-quasiconformal is called as a **quasiconformal constant** of \( f \) denoted by \( K(f) \).

**THEOREM 1.1:**

The smallest \( K \) such that \( f \) is \( K \)-quasiconformal is called as a **quasiconformal constant** of \( f \) denoted by \( K(f) \).

**Proof:**

Let \( 1 \leq K \leq \infty \) & \( \Omega \subset \mathbb{C} \) be a domain. Suppose that, \( f: \Omega \to \mathbb{C} \) is \( K \)-Quasiconformal mapping given by

\[
f(x + iy) = u(x, y) + iv(x, y)
\]

Then, \( f \) satisfies,

\[
\frac{\left| f_x \right|}{f_z} \leq K \text{ a.e.} \quad \frac{\left| f_x \right|^2}{f_z} \leq K^2 \text{ a.e.} \quad (1.1)
\]

Inequality of equation (1.2) is

\[
\frac{1}{K} \frac{\left| u_x - v_y + (v_x + u_y) \right|^2}{\left| u_x + v_y + (v_x - u_y) \right|^2} \leq K^2
\]

\[
(u_x - v_y)^2 + (v_x + u_y)^2 \leq K^2 (u_x + v_y)^2 + (v_x - u_y)^2 \text{is a.e}
\]

\[
u_x^2 + v_y^2 - 2u_x v_y + u_x^2 + v_y^2 + 2v_x u_y \leq K^2 (u_x^2 + v_y^2 + 2u_x v_y + v_x^2 + u_y^2 - 2v_x u_y)
\]

Or equivalently

\[
u_x^2 + u_y^2 + v_y^2 + v_x^2 - \frac{1 + k^2}{1 - k^2} 2u_x v_y + \frac{1 + k^2}{1 - k^2} 2v_x u_y \leq 0 \text{ a.e} \quad (1.3)
\]
We defined $\propto = \propto(K) = \frac{1 + K^2}{1 - K^2} \geq 1$. Then $f$ satisfied inequality (1.2)

$$u_x^2 + u_y^2 + v_x^2 + v_y^2 - 2 \propto u_xv_y + 2 \propto v_xu_y \leq 0 \ a.e$$

(1.4)

From now on each expression that involves partial derivative will be true almost everywhere $\Omega \text{ will denotes a domain of the complex plane } \mathbb{C}$. In this part we focus on $K$-quasiconformal mapping from $\mathbb{H}$ onto itself given by $f(x + iy) = u(x, y) + iv(y)$. In particular $f$ can be extended homeomorphically to $\overline{\mathbb{H}}$, $u(x, y)$ is ACL and $v(y)$ is absolutely continuous. We know that $f$ is satisfied

$$u_x^2 + u_y^2 + v_x^2 + v_y^2 - 2 \propto u_xv_y \leq 0 \ a.e$$

(1.5)

In equation (1.5) we get the form is $f(x + iy) = u(x, y) + iv(y)$. Hence,

The class of $K$-quasiconformal mapping

**THEOREM 1.2:**

Let $f$ be a $K$-quasiconformal mapping from $\mathbb{H}$ onto itself given by $f(x + iy) = u(x, y) + iv(y)$ then $v^+(y) = \lim_{y \to 0^+} v_y(y)$ & $v^-(y) = \lim_{y \to 0^-} v_y(y)$. are finite and the partial derivatives of $f$ satisfy the following inequalities

1. $\frac{1}{K^2} v^+_y(0) \leq v_y(y) \leq K^2 v^-_y(0)$ for almost every $y \in (0, \infty)$

2. $\frac{1}{K^3} v^+_x(0) \leq v_x(x, y) \leq K^3 v^-_y(0)$ for almost every $x + iy \in \mathbb{H}$

**Proof:**

Let $f$ be a $K$-quasiconformal mapping from $\mathbb{H}$ onto itself given by $f(x + iy) = u(x, y) + iv(y)$

Then, $\frac{1}{K^2} v^+_y(0) \leq v_y(y) \leq K^2 v^-_y(0)$

In equation (1.5) we get this inequality defines a circle a.e thus $u_x$ and $v_y$ satisfy in particular

$$\alpha v_y - v_y \sqrt{\alpha^2 - 1} \leq u_x \leq \alpha v_y + v_y \sqrt{\alpha^2 - 1} \ a.e., \ -v_y \sqrt{\alpha^2 - 1} \leq u_y \leq v_y \sqrt{\alpha^2 - 1} \ a.e.$$ 

In fact the circle is a subset of the square described and observes that

$$K = \alpha + \sqrt{\alpha^2 - 1}$$

and,

$$\frac{1}{K} = \alpha - \sqrt{\alpha^2 - 1}$$

(1.6)

Where $K$ is maximal dilatation of $f$.

Let,

$$c = \sqrt{\alpha^2 - 1} \text{ then } c \geq 0$$

$$v_y (\alpha - \sqrt{\alpha^2 - 1}) \leq u_x \leq v_y (\alpha + \sqrt{\alpha^2 - 1})$$

$$v_y \frac{u_x}{K} \leq u_y \leq v_y \alpha$$

$$\text{ and }$$

$$-c v_y \leq u_y \leq v_y c$$

(1.7)

(1.8)

Given $0 \leq x$ are integrate (1.7) on interval $[0, x]$

$$\frac{v_y}{K} [x - 0] \leq u_x(x, y) - u_x(0, y) \leq K v_y(y) x$$

If we choose any fixed $y \in (0, \infty)$ such that $u(x, y)$ is absolutely continuous with respect to $x$ then we obtain

$$\frac{x}{K} v_y(y) + u_x(0, y) \leq u(x, y) \leq K v_y(y) x + u_x(0, y)$$

(1.9)

For each $x \in \mathbb{R}$ and almost every $y \in (0, \infty)$ using left hand side of the last inequality we get

$$\lim_{y \to 0} \sup \frac{v_y(y)x}{K} + u(0, y) \leq \lim_{y \to 0} \sup \ u(x, y)$$

and,

Since, $u(x, y)$ is continuous we obtain

$$\frac{x}{K} \lim_{y \to 0} \sup v_y(y) \leq u(x, 0) - u(0, 0) < \infty$$
For this reason \( \lim_{y \to 0^+} \sup_{y} v_{y}(y) \) exists and consequently \( \lim_{y \to 0^-} \inf_{y} v_{y}(y) \) exists.

We defined, \( v^+(0) = \lim_{y \to 0^+} \sup_{y} v_{y}(y) \), \( v^-(0) = \lim_{y \to 0^-} \inf_{y} v_{y}(y) \)

In this equation in (1.9)

\[
\frac{v^+(y)x}{K} + u(0,0) \leq u(x,0) \leq K v^-(0)x + u(0,0)
\]  

(1.10)

On the other hand we choose any fixed \( x \in [0, \infty] \)

Such that, \( u(x,y) \) is absolutely continuous with respect to \( y \) we integrate (8) on the interval \([0, y]\) so, \( \int_{0}^{y} -cv_{y}(t)dt \leq \int_{0}^{y} u_{y}(x,t)dt \leq \int_{0}^{y} cv_{y}(t)dt \) and since \( v_{y} \) is absolutely continuous we obtain \(-cv(y) + u(x,0) \leq u(x,y) \leq cv(y) + u(x,0)\).

For \( y \in [0, \infty] \) and almost every \( x \in [0, \infty] \) by an argument of continuity of the mapping \( f \) and density we have,

\[
-cv(y) + u(x,0) \leq u(x,y) \leq cv(y) + u(x,0)
\]  

(1.11)

For all \((x,y) \in [0, \infty) \times [0, \infty) \) setting \( x=0 \) in the previous inequality to get

\[
-cv(y) + u(0,0) \leq u(0,y) \leq cv(y) + u(0,0)
\]  

(1.12)

Thus combination (1.9) and (1.12)

For each \( x \in \mathbb{R} \) and almost every \( y \in (0, \infty) \) in some way we use (10) and (11)

\[
\frac{v_{y}(y)x}{K} - cv(y) + u(0,0) \leq u(x,y) \leq Kxv_{y}(y) + cv(y) + u(0,0)
\]  

(1.13)

\[
\frac{v_{y}(y)x}{K} - cv(y) + u(0,0) \leq u(x,y) \leq Kxv_{y}(y) + cv(y) + u(0,0)
\]  

(1.14)

For each \( x \in \mathbb{R} \) and almost every \( y \in (0, \infty) \) since, the left and right hand side of the inequality (1.13) & (1.14) represent linear equation in the variable \( x \), we compare their slopes and the fact that \( x > 0 \) to conclude.

\[
\frac{v_{y}(y)}{K} \leq K v^-y(0) \text{ and } \frac{v_{y}(y)}{K} \leq K v^y(0).
\]

For each \( x \in \mathbb{R} \) and almost every \( y \in (0, \infty) \) \( v(y) \) is absolutely continuous and we integrate the above inequalities on the interval \([0, y]\)

\[
\frac{1}{K^2} \int_{0}^{y} v^+_{y}(0)dt \leq \int_{0}^{y} v_{y}(t)dt \leq K^2 \int_{0}^{y} v^-_{y}(0)dt
\]

To get

\[
\frac{1}{K^2} v^+_{y}(0)(y) \leq v(y) \leq K^2 v^-_{y}(0)y
\]  

(1.15)

In particular \( 0 \leq v^-_{y}(0) \leq v^+_{y}(0) \leq \infty \) \( v \) has the dinis derivatives at 0

Similarly, \( \frac{1}{K^2} v^+_{y}(0)(y) \leq u_x(x,y) \leq K^2 v^-_{y}(0)y \)

**THEOREM 1.3:**

Let \( f \) be a \( K \)-quasiconformal mapping from \( \mathbb{H} \) onto itself given by \( f(x + iy) = u(x,y) + iv(x,y) \) then,

1. There exists \( M > 0 \) such that \( |f_x| - |f_x| \leq M \) and \( |f_y| - |f_y| \leq KM a. e \)
2. The mapping \( f \) is Lipschitz in \( \mathbb{H} \)

3. The mapping \( f \) is hyperbolically Lipschitz in \( \mathbb{H} \)

**Proof:**

1. Since,

\[
|f_z| - |f_\bar{z}| = \frac{1}{2} \sqrt{(u_x + v_y)^2 + u_y^2} - \frac{1}{2} \sqrt{(u_x - v_y)^2 + u_y^2}
\]

\[
|f_z| - |f_\bar{z}| \leq \frac{1}{2} \sqrt{2(u_x^2 + v_y^2 + u_y^2)} \text{ a.e}
\]

We estimate the last expression in above theorem

\[
\frac{1}{2} \sqrt{2(u_x^2 + v_y^2 + u_y^2)} = \frac{\sqrt{2}}{2} \sqrt{(K^2v_\gamma(0))^2 + (K^2v_\gamma(0))^2 + (K^2 - 1)Kv_\gamma(0))^2}
\]

\[
= \frac{K^2v_\gamma(0)}{2K^2} \sqrt{5K^4 + 2K^2 + 1} \text{ a.e}
\]

Thus we choose, \( M = \frac{K^2v_\gamma(0)}{2K^2} \sqrt{5K^4 + 2K^2 + 1} \)

\[
|f_z| - |f_\bar{z}| \leq M \text{ and } |f_z| - |f_\bar{z}| \leq KM \text{ a.e}
\]

2. Let \( z_1, z_2 \in \mathbb{H} \) and \( l \) be the Euclidean segment that joint \( z_1, z_2 \) then,

\[
|f_z| - |f_\bar{z}| \leq \int |df| \\
\int f(l)
\]

\[
\leq \int |f_z| + |f_\bar{z}| dz
\]

\[
|f_z| - |f_\bar{z}| \leq MK|z_1 - z_2|
\]

3. Let \( z_1, z_2 \in \mathbb{H} \) and \( l \) be the hyperbolic segment that joint \( z_1, z_2 \) with \( z_2 \)

\[
d_H(f(z_1), f(z_2)) \leq \int |df| \\
\int_{lmw}
\]

\[
d_H(f(z_1), f(z_2)) \leq LMKd_H(z_1, z_2)
\]

Where,

\[
L = \frac{K^2}{v_\gamma(0)} \text{ and } d_H \text{ denote the hyperbolic metric}
\]

**THEOREM 1.4:**

Let \( f \) be a K-quasiconformal mapping from \( \mathbb{H} \) onto itself given by

\[
f(x + iy) = u(x, y) + iv(x, y)
\]

Then,

1. \( \frac{v^+(y(0))}{K^2} A_0(E) \leq A_0(f(E)) \leq K^2(v^-(y(0))^2 A_0(E)
\]

2. \( \frac{1}{K^2} \left(\frac{v^+(y(0))}{v^-(y(0))} \right)^2 A_0(E) \leq A_0(f(E)) \leq K^2 \left(\frac{v^+(y(0))}{v^-(y(0))} \right)^2 A_0(E)
\]

Since, \( v^-(y(0)) \leq v^+(y(0)) \)

We have

\[
\frac{1}{K^2} A_0(E) \leq A_0(f(E)) \leq K^2 A_0(E) \text{ and these inequalities are asymptotically sharp when } K \rightarrow 1^+
\]

**Proof:**

Let \( E \subset \mathbb{H} \) be a measurable set the Jacobian of \( f \)

\[
J_f = u_x v_y
\]

By equation 1.1 theorem 1.2

1. We get \( \frac{1}{K^2} v^+(y(0))(y) \leq v(y) \leq K^2 v^-(y(0))y \) a.e

\[
\frac{1}{K^2} v^+(y(0))(y) \leq u_x(x, y) \leq K^3 v^-(y(0))y \text{ a.e}
\]

Comparing above equation we get
By Jacobian \( \int f \leq u_x u_y \)

\[
\frac{(u_x^2 + u_y^2)}{K^2} \leq \frac{f^2}{K^2} \leq K^2(v_x^2 + v_y^2) \text{ a.e.}
\]

The Euclidean area of f(E) is

\[
\int f \leq A_e(f(E))
\]

And in consequence

\[
\frac{(u_x^2 + u_y^2)}{K^2} = A_e(E) \leq A_e(f(E)) \leq K^2(v_x^2 + v_y^2) = A_e(E)
\]

On the other hand by hyperbolic area as same way we get

\[
\frac{1}{K^2} \leq A_e(f(E)) \leq K^2(v_x^2 + v_y^2) = A_e(E)
\]

**THEOREM 1.5:**

Let \( f \) be a \( K \)-quasiconformal mapping from \( \mathbb{H} \) onto itself given by \( f(x+iy) = u(x, y) + iv(y) \) then for each measure set \( E \subseteq \mathbb{H} \).

(i) If \( v \) is differentiable at 0 then, \( \frac{(u_y^2 + v_y^2)}{K^2} \leq A_e(E) \leq K^2(v_1^2 + v_y^2)A_e(E) \)

(ii) If \( v \) is continuously differentiable in a neighborhood of 0 then \( \frac{(u_y^2 + v_y^2)}{K^2} \leq A_e(f(E)) \leq K^2(v_1^2 + v_y^2)A_e(E) \)

Moreover, these inequality are asymptotically sharp when \( K \to 1^+ \)

**Proof:**

If \( v \) is differentiable at 0 then we get in (1.15)

\[
\frac{1}{K^2} v_1^2(y) \leq A_e(E) \leq K^2 v_1^2(y)
\]

If \( v \) is continuously differentiable is neighbourhood of 0 then

\[
v_1^2(y) = v_1^2(y) = v_1^2(y)
\]

We get above theorem results replace the value of \( v_1^2(y) = v_1^2(y) = v_1^2(y) \) hence

\[
\frac{(u_y^2 + v_y^2)}{K^2} = A_e(E) \leq A_e(f(E)) \leq K^2(v_1^2 + v_y^2)A_e(E)
\]

**EXAMPLE:**

Let \( f : \mathbb{H} \to \mathbb{H} \) given by \( f(x + iy) = 2x + \sin(x + y) + iy \) then, \( f \) is a \( \frac{11 + \sqrt{35}}{6} \) quasiconformal mapping with \( v_1(y) = v_1(y) = 1 \)

**2. HARMONIC QUASICONFORMAL MAPPING**

**DEFINITION 2.1:**

A function \( F \) is called Harmonic in a region \( \Omega \) if its laplacian vanishes in \( \Omega \). A topological mapping \( f \) of \( \Omega \) is said to be \( K \)-quasiconformal if it satisfies:

1. \( f \) is ACL in \( \Omega \)
2. \( L^2 \leq KL |f|, K \geq 1 \text{ a.e in } \Omega \)

Where, \( L_f = |f_x| + |f_y|, L_f = |f_x| - |f_y| \)

Then, \( f \) is \( K \)-quasiconformal mapping.

**DEFINITION 2.2:**

Let \( f \) is a harmonic mapping then there exists a holomorphic function \( g : \mathbb{H} \to \mathbb{C} \) such that, \( f(z) = \text{Reg}(z) + ic \)

Thus, \( |f_x| = \frac{1}{2} |g^1(z) + c|, |f_y| = \frac{1}{2} |g^1(z) - c| \). We obtain that \( g^1(z) \) belongs to the circle \( \mathbb{D} \).

**DEFINITION 2.3:**

The hyperbolic density is \( \frac{2dz}{1-|z|^2} \) and \( \frac{|dw|}{ln w} \) for the unit disk \( \mathbb{D} \) and the upper half plane \( \mathbb{H} \). We denote also by \( A_H \) the hyperbolic area in the unit disk \( \mathbb{D} \).

**THEOREM 2.1:**

Let \( f \) be a harmonic mapping of \( \mathbb{H} \) onto itself and continuous up to its boundary with \( f(\infty) = \infty \). If \( f \) is a \( K \)-quasiconformal mapping then \( f \) can be represented by \( f = u + ic \) and the gradient of \( f \) is such that

\[
L_f = |f_x| + |f_y| \leq cK
\]

Where, \( c \) is positive constant.

**Proof:**

By definition of \( K \)-quasiconformal mapping

We have,

\[
L_f^2 \leq KL_f f
\]
This implies that, \( L_f \leq K \ell_f \) \hspace{2cm} (2.1)

We assume that \( f = u + ic \text{Im}(z) \), \( f = \frac{1}{2}[g(z) + c(z)] + \frac{1}{2}[g(z) - c(z)] \)

Hence, \( |f_z| = \frac{1}{2}|g'(z) + c|, \quad |f_{\bar{z}}| = \frac{1}{2}|g'(z) - c| \) \hspace{2cm} (2.2)

Equation (2.1) and (2.2) we have,

\[
|f_z| + |f_{\bar{z}}| \leq K(|f_z| - |f_{\bar{z}}|)
\]

\[
|f_z| + |f_{\bar{z}}| - K|f_z| + K|f_{\bar{z}}| \leq 0
\]

\[
-(K-1)|f_z| + (K+1)|f_{\bar{z}}| \leq 0
\]

\[
(K+1)|f_{\bar{z}}| - (K-1)|f_z| \leq c(K-1)
\]

\[
2|f_z| \leq c(K-1)
\]

\[
|g'(z) - c| \leq c(K-1)
\]

\[
|g'(z)| \leq cK - c + \frac{1}{2}cK + c \leq cK
\]

\[
\therefore L_f = |f_z| + |f_{\bar{z}}| \leq cK
\]

**LEMMA 2.1:**

Assume that \( f \) is a harmonic mapping of \( \mathbb{H} \) onto itself and continuous on \( \mathbb{H} \cup \mathbb{R} \) with \( f(\infty) = \infty \), if \( f \) is K-quasiconformal mapping then, \( f(z) = \frac{1}{2}(g(z) + c(z) + g(z) - c(z)) \)

Where, \( g \) is holomorphic function in \( \mathbb{H} \) and \( c \) is positive constant and the Jacobian of \( f \) is such that \( J_f = |f_z| - |f_{\bar{z}}| \leq c^2K \)

**Proof:**

According to the definition of K-quasiconformal mapping and the inequality

\[
|f_z| = \frac{1}{2}|g'(z) + c|, \quad |f_{\bar{z}}| = \frac{1}{2}|g'(z) - c|
\]

\[
J_f = |f_z| - |f_{\bar{z}}| \leq c^2K
\]

**THEOREM 2.2:**

Let \( f = u + iv \) be a harmonic mapping of \( \mathbb{H} \) onto itself and continuous up to its boundary with \( f(\infty) = \infty \), if \( f \) is a harmonic mapping then for any measurable subset \( E \subset \mathbb{H} \)

\[
A_{euc}(f(z)) \leq c^2K A_{euc}(E)
\]

Where, \( A_{euc}(\cdot) \) Denotes the Euclidean area and \( c \) is positive constant.

**THEOREM 2.3:**

Let \( f \) be a harmonic mapping of \( \mathbb{H} \) onto itself and continuous up to its boundary with \( f(\infty) = \infty \), if \( f \) is a K-quasiconformal mapping then for any measurable subset \( E \subset \mathbb{H} \) we get

\[
A_{hyp}(f(z)) \leq c^2K A_{hyp}(E)
\]

Where, \( A_{hyp} \) denotes the hyperbolic area and \( c \) is positive constant.

**3. ANGULAR AND RADIAL QUASICONFORMAL MAPPING**

**Proposition 3.1:**

Let \( f: \Omega \to \mathbb{C} \) be an ACL mapping. If \( f(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta}) \) then for a.e \( \Omega \)

**Proof:**

Let \( f: \Omega \to \mathbb{C} \) be an ACL mapping. Then, \( f(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta}) \) is a.e

We know that, \( |f_{\theta}| = \frac{1}{2}|f_x - if_y|, |f_{\bar{\theta}}| = \frac{1}{2}|f_x + if_y| \)

Taking square on above equation,

\[
4|f_z|^2 = \left[(u_x + v_y) + i(v_x - u_y)\right]^2
\]

\[
4|f_{\bar{z}}|^2 = \left[(u_x - v_y) + i(v_x + u_y)\right]^2, \quad x = r, \quad \frac{\theta}{r} = y
\]

\[
4|f_z|^2 = \left[(u_x + v_y) + i(v_x - u_y)\right]^2
\]

\[
4|f_{\bar{z}}|^2 = \left[(u_x + v_y)^2 + (v_x - u_y)^2\right]
\]

\[
4|f_{\bar{z}}|^2 = \left[(u_x + v_y)^2 + (v_x + u_y)^2\right]
\]

And the Jacobian of \( f \) is \( J_f = |f_z|^2 - |f_{\bar{z}}|^2 \)

\[
J_f = \frac{1}{4} \left[4u_xv_y - 4v_xu_y\right]
\]

**DEFINITION 3.1:**

A mapping \( f: \mathbb{H} \to \mathbb{H} \) is said to be radial at \( x \in \mathbb{R} \) if \( f \) leaves invariant all Euclidean rays in \( \mathbb{H} \) that meet at \( x \). Let \( f: \mathbb{H} \to \mathbb{H} \) be a radial mapping at \( x \). Since, hyperbolic area is invariant under horizontal translations, we can assume that, the point \( x \in \mathbb{R} \). Where, the Euclidean rays meet is \( x = 0 \) if \( f \) is a radial mapping then \( f \) can be written in polar-coordinates \((r, \theta)\) as
THEOREM 3.1:
Let \( f \) be a ACL mapping from \( \mathbb{H} \) onto itself suppose that \( f \) is a radial mapping at 0, then its Jacobian mapping at 0 is \( J_f = \frac{\rho r}{r} \ a.e \) if \( f \) preserves orientation then \( \rho _r > 0 \ a.e \).

Proof:
Since, \( f(z) = f(re^{i\theta}) = \rho(r, \theta)e^{i\theta} \), \( f(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta}) \)
Then, \( (r, \theta) = (\rho(r, \theta)\cos \theta, \nu(r, \theta) = \rho(r, \theta)\sin \theta) \)
By Jacobian of \( f \) is 
\[
J_f = \frac{\rho r}{r}[\sin \theta - \cos \theta]
\]
\[\theta = \pi J_f = \frac{\rho r}{r}[0+1] \quad \therefore \quad J_f = \frac{\rho r}{r} \text{ if } \rho _r > 0
\]
We get,
\[
J_f = \frac{\rho r}{r} \ a.e
\]

PROPOSITION 3.2:
Let \( f \) be a K-quasiconformal mapping from \( \mathbb{H} \) onto itself. Suppose that \( f \) is a radial mapping at 0. Then the function \( \rho \) satisfies the following:
1. For \( 1 \leq r < \infty \) \( \frac{\rho (r, \theta)}{r} \leq K \)
2. For \( 0 < r < 1 \) \( \frac{\rho (r, \theta)}{r} \leq r^K \)

Proof:
We first prove that the function \( (0, \infty) \ni r \mapsto \ln(\rho(r, \theta)) \) is absolutely continuous for almost every \( \theta \in (0, \pi) \). It is enough to prove that for every \( M > 1 \),
The function \( \left[ \frac{1}{M} M \right] \ni r \mapsto \ln(\rho(r, \theta)) \) is absolutely continuous for almost every \( \theta \in (0, \pi) \).

Let \( \Omega = \{ z = x + i\theta \mid \exists (x, \theta) \in (-\infty, \infty) \times (0, \pi) \} \)
Then,
The mapping \( \ln f \circ \exp : \Omega \to \Omega \) is K-quasiconformal mapping

Thus,
The function \( (-\infty, \infty) \ni r \mapsto \ln(\rho(r, \theta)) \) is absolutely continuous for almost every \( \theta \in (0, \pi) \).
Let \( \epsilon > 0 \) there exists \( \delta > 0 \) such that, for every finite collection of disjoint intervals \( (\alpha_j, \beta_j) \in \mathbb{R} \), \( j = 1, 2, ..., n \) with \( \sum_{j=1}^{\infty} (\beta_j - \alpha_j) < \delta \)
Then, \( \sum_{j=1}^{\infty} \left( \ln(\rho(\beta_j, \theta)) - \ln(\rho(\alpha_j, \theta)) \right) < \epsilon \)

Since, \( r \) is absolutely continuous on \( [\frac{1}{M}, \frac{1}{M}] \) there exists \( \delta > 0 \) such that, for every finite collection of disjoint intervals \( (c_l, d_l) \in \left[ \frac{1}{M}, \frac{1}{M} \right], l = 1, 2, ..., m \) with,
\[
\sum_{l=1}^{m} (d_l - c_l) < \delta
\]
Then, \( \sum_{l=1}^{m} (\ln(d_l) - \ln(c_l)) < \delta \)
And by the inequality \( \sum_{l=1}^{m} (\ln(\rho(d_l, \theta)) - \ln(\rho(c_l, \theta)) < \epsilon \)
If, \( f(z) = \rho(r, \theta)e^{i\theta} \) from (1) and (2)
\[
|f(z)|^2 = \frac{1}{4} \left( \rho_r^2 + 2\rho r + \rho^2 \right) \ a.e
\]
\[
|f(z)|^2 = \frac{1}{4} \left( \rho_r^2 - 2\rho r + \rho^2 \right) \ a.e
\]
By (4.4) and (4.5) in equation \( |f(z)|^2 \leq K^2 \leq |f|z|^2 \)
\[
\frac{1}{4} \left( \rho_r - \rho_r^2 + \rho^2 \right) \leq K^2 \frac{1}{4} \left( \rho_r + \rho_r^2 + \rho^2 \right) \ a.e
\]
Or equivalently,

\[
\frac{2(r^2 \rho_r^2 + \rho^2 + \rho^2)}{4r^2 \rho_r} \leq \frac{K^2 + 1}{1 - K^2} = \alpha a.e
\]
Then,
\[
\frac{1}{2} \left( \frac{\rho r}{r} + \frac{\rho^2}{r^2} \right) \leq \alpha a.e
\]
And thus, \( \frac{(\rho r)^2}{r} - 2a \left( \frac{\rho r^2}{r^2} \right) + 1 \leq 0 a.e
\]
\[
\frac{\rho r}{r} \left( \frac{2}{r} \rho r - 2ar \right) + 1 \leq 0
\]
\[
\frac{1}{K^r} \leq \frac{\rho r}{r} \leq \frac{K}{r} a.e
\]
Or equivalently,
\[
\frac{1}{K^r} \leq \frac{\rho}{r} \ln \rho \leq \frac{K}{r} a.e
\]
We choose any fixed $\theta \in (0, \pi)$ such that $\ln \rho(r, \theta)$ is absolutely continuous on $r$ and then, the inequality holds
\[
\int_1^R \frac{1}{Kr} dr \leq \int_1^R \frac{\partial}{\partial r} \ln \rho dr \leq \int_1^R \frac{K}{r} dr.
\]
Thus, $\frac{1}{K} \ln r|_1^R \leq \ln \rho(r, \theta) \leq K|\ln r|_1^R$ for almost every $\theta \in (0, \pi)$ and $R \in (1, \infty)$.

By an argument of continuity of $\ln \rho$ and $\ln |\ln r|$ we finally obtain, $R^2 \leq \frac{\rho(r, \theta)}{\rho(1, \theta)} \leq R^2$ for all $(R, \theta) \in (1, \infty) \times (0, \pi)$. In a similar way, if we suppose that $0 < R < 1$ then,
\[
R^2 \leq \frac{\rho(r, \theta)}{\rho(1, \theta)} \leq R^2 \text{ for all } (R, \theta) \in (0, 1) \times (0, \pi).
\]

**THEOREM 3.2:**

Let $f$ be a $K$-quasiconformal mapping from $\mathbb{H}$ onto itself that leaves invariant each ray in $\mathbb{H}$ that meets a real base point of $E \subset \mathbb{H}$ a measurable set then,
\[
\frac{1}{K} A_h(E) \leq A_h(f(E)) \leq K A_h(E).
\]

These inequalities are asymptotically sharp when $K \to 1^+$

**DEFINITION 3.2:**

A mapping $f: \mathbb{D} \to \mathbb{D}$ is said to be angular at 0 if $f$ leaves each ray invariant where, $\varphi: [0, 1) \times [0, 2\pi) \to \mathbb{R}$.

**LEMMA 3.1:**

Let $f$ be a ACL mapping from $\mathbb{D}$ onto itself such that $f$ is angular at 0 then its Jacobian is $J_f = \varphi_0$. If $f$ preserves orientation then, $\varphi_0 > 0$ a.e.

**Proof:**

Since, $f(z) = f(re^{i\theta}) = re^{i\varphi(r, \theta)}$. Then, $f(re^{i\theta}) = u(re^{i\theta}) + iv(re^{i\theta}).$ $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$\n
We know that $J_f = \frac{1}{r}[u_r, v_r]$. $J_f = \frac{1}{r}[r \sin \varphi(r, \theta) - r \cos \varphi(r, \theta)].$ We interval $[0, 1] \times [0, 2\pi]$ to get $J_f = \varphi_0$ is angular at 0 then its Jacobian is $J_f = \varphi_0$.

**PROPOSITION 3.3:**

Let $f$ be a $K$-quasiconformal mapping from $\mathbb{D}$ onto itself which is angular at 0 then, $\frac{1}{K} \leq \varphi_0 \leq K$ a.e in $[0, 1) \times [0, 2\pi]

**Proof:**

If $f(z) = f(re^{i\theta}) = re^{i\varphi(r, \theta)}$ From equation 1 & 2
\[
\frac{4|f_z|^2}{|f_r|^2} \leq (1 + \varphi_0)^2 + r^2 \varphi_r^2 \text{ a.e.} \quad \frac{4|f_r|^2}{|f_z|^2} \leq (1 - \varphi_0)^2 + r^2 \varphi_r^2 \text{ a.e.}
\]

Since, $\frac{4|f_z|^2}{|f_r|^2} \leq K^2 \leq K^2 \leq K^2$ a.e

Taking square on both side $\frac{1 - \varphi_0}{1 + \varphi_0} \leq \frac{|f_z|}{|f_r|} \leq K$

\[
\frac{1}{K} \leq \varphi_0 \leq K \text{ a.e.} \quad (3.7)
\]

**THEOREM 3.3:**

Let $f$ be a $K$-quasiconformal mapping from $\mathbb{D}$ onto itself which is angular at 0. If $E \subset \mathbb{H}$ is a measurable set then,
\[
\frac{1}{K} A_h(E) \leq A_h(f(E)) \leq K A_h(E).
\]

These inequalities are asymptotically sharp when $K \to 1^+$
4. PARTIAL DERIVATIVES OF K-QUASICONFORMAL MAPPING

THEOREM 4.1:

Let \( 1 \leq K < \infty \) if \( f : \Omega \to \mathbb{C} \) is a K-quasiconformal mapping by \( f(x + iy) = u(x) + iv(y) \).

Then its partial derivative belong to one of the angular region defined by

\[
u^2 + v^2 - 2\alpha u_x v_y \leq 0 \ a.e
\]

Proof:

Then by \( f(x + iy) = u(x, y) + iv(x, y)f \) Satisfies inequality if and only if

\[
u^2 + u^2 + v^2 - 2\alpha u_x v_y + 2 \alpha v_x u_y \leq 0\ a.e \quad (4.1)
\]

In equation (1) its partial derivatives satisfy the inequality

\[
u^2 + v^2 - 2\alpha u_x v_y \leq 0 \ a.e \quad (4.2)
\]

Since, \( \alpha \geq 1 \) The discriminant of \( \nu^2 + v^2 - 2\alpha u_x v_y \) is non-negative and equation (4.2) defines the interior of an angular region with the identification \( u_x \sim x - \alpha x \) and \( v_y \sim y - \alpha x \). That the Jacobian of ‘f’ is \( f = u_x v_y \) Is always positive.

GENERAL CASE:

If \( f \) is a K-quasiconformal mapping given by \( f(x + iy) = u(x, y) + iv(y) \).

Then reduces to, \( u^2 + u^2 + v^2 - 2\alpha u_x v_y \leq 0 \ a.e \). Inequality suggests studying the quadratic from \( Q(x, y, w) = x^2 + y^2 + w^2 - 2\alpha xw \). Whose, associated symmetric matrix is

\[
N = \begin{pmatrix}
    x^2 & xy & xw \\
    xy & y^2 & yw \\
    xw & yw & z^2
\end{pmatrix}
\]

PROPOSITION 4.1:

There exists an invertible matrix \( P \) such that \( P^{-1}NP = D \). Where \( D = \begin{pmatrix}
    1 - \alpha & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1 + \alpha
\end{pmatrix}
\]

Proof:

The matrix \( N \) is \( N = \begin{pmatrix}
    1 & 0 & -\alpha \\
    0 & 1 & 0 \\
    -\alpha & 0 & 1
\end{pmatrix}
\)

To find Eigen values of matrix \( N \), \( \text{Det}(N - \lambda I) = 0 \)

\[
I = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad \lambda I = \begin{pmatrix}
    \lambda & 0 & 0 \\
    0 & \lambda & 0 \\
    0 & 0 & \lambda
\end{pmatrix}
\]

\[
N - \lambda I = \begin{pmatrix}
    1 - \lambda & 0 & -\alpha \\
    0 & 1 - \lambda & 0 \\
    -\alpha & 0 & 1 - \lambda
\end{pmatrix}
\]

\[
\text{det}(N - \lambda I) = \begin{vmatrix}
    1 - \lambda & 0 & -\alpha \\
    0 & 1 - \lambda & 0 \\
    -\alpha & 0 & 1 - \lambda
\end{vmatrix}
\]

\[
\text{det}(N - \lambda I) = (1 - \lambda)(1 - \lambda^2 - \alpha^2), \text{det}(N - \lambda I) = 0
\]

\[
\lambda = 1, \lambda = (1 \pm \alpha).
\]

The Eigen values of \( N \) are \( \lambda_1 = 1 - \alpha, \lambda_2 = 1, \lambda_3 = 1 + \alpha \)

The Eigen vector is \( (N - \lambda I)x = 0, \lambda_1 = 1 - \alpha \)

\[
\begin{pmatrix}
    -\alpha & 0 & -\alpha \\
    0 & -\alpha & 0 \\
    -\alpha & 0 & -\alpha
\end{pmatrix}
\]

By Cramer rule: \( \frac{x}{-\alpha^2} = \frac{y}{-\alpha} = \frac{w}{\alpha^2} \Rightarrow (x, y, w) = (1, 0, -1) \)

Eigen vector is \( (1, 0, -1) \). The norm of Eigen vector is \( ||v_1|| = \sqrt{1 + 0 + 1} = \sqrt{2} \)
\[
\frac{1}{\sqrt{2}} \|v_1\| = \frac{1}{\sqrt{2}} (1, 0, -1)
\]

\[
= \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)
\]

Eigen vector \((1, 0, -1)\) with normalized \((\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\)

By Cramer rule we get \((x, y, w) = (0, 1, 0)\). The norm of \((0,1,0)\) is \((0, 1, 0)\).

By Cramer rule we get \((x, y, w) = (1,0,1)\). The norm of \((1,0,1)\) is \((\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\)

\[
P = \begin{bmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}
\end{bmatrix} = p^{-1}
\]

\[
P^{-1}N = \begin{bmatrix}
1 - \alpha & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 + \alpha
\end{bmatrix}
\]

PROPOSITION 4.2:

Let \(1 \leq K < \infty\), if \(f: \Omega \rightarrow \mathbb{C}\) is a \(K\)-quasiconformal mapping by \(f(x + iy) = u(x) + iv(y)\) then its partial derivative \(u_x, u_y, v_y\) belong to one branch of the elliptic cone

\[u_x^2 + u_y^2 + v_y^2 - 2au_xv_y \leq 0 \text{ a.e}\]

Proof:

As we saw \(f\) is \(K\)-quasiconformal mapping iff \(u_x, u_y, v_y\) satisfy \(Q(u_x, u_y, v_y) \leq 0\)

That describes the solid cone \(u_x^2 + u_y^2 + v_y^2 - 2au_xv_y \leq 0\). As \(f\) preserves orientation then,

\[f = u_xv_y > 0 \text{ a.e}\]

Since, \(v_y > 0\) then necessarily \(u_x > 0 \text{ a.e}\)

GENERAL CASE:

That \(f\) is a quasiconformal mapping \(f: \Omega \subset \mathbb{C} \rightarrow \mathbb{C}\) given by \(f(x + iy) = u(x, y) + iv(x, y)\) then by \(u_x^2 + u_y^2 + v_y^2 - 2au_xv_y + 2au_xv_x \leq 0\) a.e.

In the case we study the quadratic form \(Q(x, y, z, w) = x^2 + y^2 + z^2 + w^2 - 2axw + 2ayz\)

With the associated symmetric matrix \(N = \begin{bmatrix} 1 & 0 & 0 & -\alpha \\ 0 & 1 & \alpha & 0 \\ 0 & \alpha & 1 & 0 \\ -\alpha & 0 & 0 & 1 \end{bmatrix}\)

PROPOSITION 4.3:

There exists an invertible matrix \(P\) such that \(P^{-1}NP = D\) where,

\[
D = \begin{bmatrix}
1 - \alpha & 0 & 0 & 0 \\
0 & 1 - \alpha & 0 & 0 \\
0 & 0 & 1 + \alpha & 0 \\
0 & 0 & 0 & 1 + \alpha
\end{bmatrix}
\]

Proof:

The characteristic polynomial of the matrix \(N\) is \(\text{det}(N - \lambda I) = 0\)
\[
\begin{pmatrix}
1 - \lambda & 0 & 0 & -\alpha \\
0 & 1 - \lambda & \alpha & 0 \\
0 & \alpha & 1 - \lambda & 0 \\
-\alpha & 0 & 0 & 1 - \lambda \\
\end{pmatrix} = 0
\]

The characteristic polynomial is \((1-\lambda)^4 - \alpha^2(1-\lambda)^2 - \alpha^2(1-\lambda)^2 + \alpha^4 = 0\) with Eigen value \(\lambda_1 = 1 + \alpha, \lambda_2 = 1 - \alpha\) and both with multiplicity two. The Eigen vector is \((N - \lambda I) = 0\) put the value \(\lambda_1 = 1 + \alpha, \lambda_2 = 1 - \alpha\). The Eigen vectors are

\(\lambda_1(1,0,0,-1)\) and \((0,1,1,0)\) and for \(\lambda_2\) are \((1,0,0,1)\) and \((0,1,-1,0)\). After normalization we obtain the matrix.

\[
P = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0
\end{pmatrix}
\]

with inverse \(P^{-1} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0
\end{pmatrix}\)

Thus \(P^{-1}NP = \begin{pmatrix}
1 - \alpha & 0 & 0 & 0 \\
0 & 1 - \alpha & 0 & 0 \\
0 & 0 & 1 + \alpha & 0 \\
0 & 0 & 0 & 1 + \alpha
\end{pmatrix}\)

\(\therefore P^{-1}NP = D\)

**THEOREM 4.2:**

The quadratic form

\[Q(\hat{x}, \hat{y}, \hat{z}, \hat{w}) = (1 - \alpha)\hat{x}^2 + (1 - \alpha)\hat{y}^2 + (1 + \alpha)\hat{z}^2 + (1 + \alpha)\hat{w}^2\]

Represents the quadratic form \(Q(x, y, z, w) = x^2 + y^2 + z^2 + w^2 - 2axw + 2ayz\) in the basis

\[c = \left\{ \left( \frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}} \right), \left( 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), \left( \frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}} \right), \left( 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right) \right\}\]

Where,

\[
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\hat{x} \\
\hat{y} \\
\hat{z} \\
\hat{w}
\end{pmatrix}
\]

In particular \(Q(x, y, z, w) \leq 0\) if \& only if \(Q(\hat{x}, \hat{y}, \hat{z}, \hat{w}) \leq 0\)

**Proof:**

We have the relations \(\begin{pmatrix}
\hat{x} \\
\hat{y} \\
\hat{z} \\
\hat{w}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{2}}(\hat{x} + \hat{z}) \\
\frac{1}{\sqrt{2}}(\hat{y} + \hat{w}) \\
\frac{1}{\sqrt{2}}(\hat{w} - \hat{y}) \\
\frac{1}{\sqrt{2}}(\hat{x} - \hat{z})
\end{pmatrix}\)

Thus, \(Q(x, y, z, w) = x^2 + y^2 + z^2 + w^2 - 2axw + 2ayz\)

\[= (1 - \alpha)\hat{x}^2 + (1 - \alpha)\hat{y}^2 + (1 + \alpha)\hat{z}^2 + (1 + \alpha)\hat{w}^2\]

\(Q(x, y, z, w) = \hat{Q}(\hat{x}, \hat{y}, \hat{z}, \hat{w})\)
And this means that \( Q(x, y, z, w) \leq 0 \) if and only if \( \hat{Q}(\hat{x}, \hat{y}, \hat{z}, \hat{w}) \leq 0 \).

**CONCLUSION**

The classes of mapping introduced in this paper have precise geometrical mapping in particular the class of quasiconformal mapping \( z = u(x, y) + iv(y) \). We have obtained left and right asymptotic bound for the hyperbolic or Euclidean area distortion. Moreover the example showed that the different classes of mapping defined in the paper are not empty and coincides or not with the region of variation of the partial derivatives of quasiconformal mapping \( f(z) = u(x, y) + iv(y) \).

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